Kvinge, Henry; **Oğuz, Can Ozan**; **Reeks, Michael**

The center of the twisted Heisenberg category, factorial Schur Q-functions, and transition functions on the Schur graph. (English) Zbl 1468.18017

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M. Khovanov [Fundam. Math. 225, 169–210 (2014; Zbl 1304.18019)] described a linear monoidal category $\mathcal{H}$ conjecturally categorifying the Heisenberg algebra, its morphisms being governed by a graphical calculus of planar diagrams. The trace of $\mathcal{H}$, being defined diagrammatically as the algebra of the diagrams on the annulus, was shown to be isomorphic to the W-algebra $W_{1+\infty}$ at level one in [*S. Cautis* et al., J. Inst. Math. Jussieu 17, No. 5, 981–1017 (2018; Zbl 1405.81045)], while the center of $\mathcal{H}$, being the algebra $\mathrm{End}_{\mathcal{H}}(\mathbf{1})$ of endomorphisms of the monoidal identity, was shown in [*H. Kvinge* et al., Algebr. Comb. 2, No. 1, 49–74 (2019; Zbl 1405.05188)] to be isomorphic to the algebra of shifted symmetric functions Λ^* of *A. Yu. Okun'kov* and *G. Ol'shanskij* [St. Petersbg. Math. J. 9, No. 2, 1 (1997; Zbl 0894.05053); translation from Algebra Anal. 9, No. 2, 73–146 (1997)].

The twisted Heisenberg category $\mathcal{H}_{tw}$, being a $\mathbb{C}$-linear additive monoidal category with an additional $\mathbb{Z}/2\mathbb{Z}$-grading, was introduced in [*S. Cautis* and *J. Sussan*, Commun. Math. Phys. 336, No. 2, 649–669 (2015; Zbl 1327.17009)], conjecturally categorifying the twisted Heisenberg algebra. The center $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbf{1})$ of $\mathcal{H}_{tw}$ was investigated in [*H. Kvinge* et al., Sémin. Lothar. Comb. 80B, 80B.76, 12 p. (2018; Zbl 1411.05265); J. Algebr. Comb. 52, No. 4, 469–504 (2020; Zbl 07339568)].

The principal objective in this paper is to study the combinatorial and representation-theoretic properties of $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbf{1})$. The principal result (Theorem 5.2) establishes an isomorphism

$$\varphi : \mathrm{End}_{\mathcal{H}_{tw}}(\mathbf{1}) \to \Gamma$$

where $\Gamma = \mathbb{C}[p_1, p_3, p_5, \dots]$ is a subalgebra of the algebra of symmetric functions, being also known as the algebra of supersymmetric [*V. N. Ivanov*, J. Math. Sci., New York 121, No. 3, 2330–2344 (2001; Zbl 1069.60025); translation from Zap. Nauchn. Semin. POMI 283, 73–97 (2001)] or doubly symmetric [*L. Petrov*, J. Math. Sci., New York 168, No. 3, 437–463 (2010; Zbl 1288.60056); translation from Zap. Nauchn. Semin. POMI 373, 226–272 (2009)] functions. The construction of φ relies heavily on the fact that there are embeddings of both $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbf{1})$ and Γ into the algebra of functions on strict partitions $\mathrm{Fun}(\mathcal{SP}, \mathbb{C})$.

One interesting feature of the center of the non-twisted Heisenberg category $\mathcal{H}$ is that, as shifted symmetric functions, the curl generators are best understood in terms of moments of Kerov's transion and cotransition measures on Young diagrams, which is shown in this paper to extend to the twisted Heisenberg category.

There is a natural action of the trace of a category on its center, which is to be diagrammatically defined as gluing annular diagrams around planar ones. In case of Khovanov's Heisenberg category, the results in [*S. Cautis* et al., J. Inst. Math. Jussieu 17, No. 5, 981–1017 (2018; Zbl 1405.81045); *H. Kvinge* et al., Algebr. Comb. 2, No. 1, 49–74 (2019; Zbl 1405.05188)] give rise to an action of $W_{1+\infty}$ on the algebra of shifted symmetric functions, which is described in terms of symmetric group representation theory [*A. Lascoux* and *J. Y. Thibon*, J. Math. Sci., New York 121, No. 3, 2380–2392 (2001; Zbl 1078.20011)]. The trace of $\mathcal{H}_{tw}$ was shown in [*C. O. Oğuz* and *M. Reeks*, Commun. Math. Phys. 356, No. 3, 1117–1154 (2017; Zbl 1388.18009)] to be isomorphic to a classical-type subalgebra of $W_{1+\infty}$ discovered by *V. G. Kac* et al. called W^- [Adv. Math. 139, No. 1, 56–140 (1998; Zbl 0938.17018)], which, along with Theorem 5.2, gives a representation of W^- on Γ, being described as a twisted version of the representation in [*A. Lascoux* and *J. Y. Thibon*, J. Math. Sci., New York 121, No. 3, 2380–2392 (2001; Zbl 1078.20011)].

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MSC:

18M05 Monoidal categories, symmetric monoidal categories
20C30 Representations of finite symmetric groups
20C08 Hecke algebras and their representations
05E05 Symmetric functions and generalizations
20C25 Projective representations and multipliers
05E10 Combinatorial aspects of representation theory
05E16 Combinatorial aspects of groups and algebras

Keywords:

Hecke algebras; spin representation theory; Schur Q-functions; Schur graph

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