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On the computation and verification of π using BBP-type formulas

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Abstract In this paper, we propose two Bailey–Borwein–Plouffe (BBP)-type formulas for π. We show that computation and verification of π using the two different BBP-type formulas require 20% fewer terms than verification by shifting the starting position of a few hexadecimal digits of π using Huvent's formula, which is known as the BBP-type formula with the least number of terms.

Keywords BBP-type formulas · arctangent relations · Taylor series

Mathematics Subject Classification (2000) 65D20 · 11Y16 · 65Y20

1 Introduction

The Bailey–Borwein–Plouffe (BBP) formula for π was discovered by Plouffe in 1995 and published in 1997 [4], and is named after the paper's authors. The BBP formula was found experimentally using the PSLQ algorithm [11, 7]. This formula enables a specific bit in π to be computed without computing all the previous bits. Several BBP-type formulas for π have since been proposed [1, 2, 9, 13, 10, 3, 6]. Using these BBP-type formulas, a few hexadecimal digits of π starting at a position n can be verified by computing the digit at position $n-1$ (or $n+1$) [5]. Such computations have been carried out for a large starting position [4, 8, 16, 20, 14, 15, 21].

Another way to verify a calculation of π is to make a comparison of results calculated using two different formulas. For example, Shanks and Wrench [17]

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computed π to 100,000 decimal digits by using Størmer's formula [18]:

$$\pi = 24 \arctan \frac{1}{8} + 8 \arctan \frac{1}{57} + 4 \arctan \frac{1}{239}. \quad (1)$$

They then verified the result by using Gauss's formula [12]:

$$\pi = 48 \arctan \frac{1}{18} + 32 \arctan \frac{1}{57} - 20 \arctan \frac{1}{239}. \quad (2)$$

Between (1) and (2), the terms $\arctan(1/57)$ and $\arctan(1/239)$ are common. Thus, the common terms calculated in (1) can be reused for the verification using (2). When there exist different BBP-type formulas that have common terms, it is possible to reduce the number of terms for the verification of a π calculation using these formulas.

In this paper, we propose two BBP-type formulas for π that have common terms. These BBP-type formulas are derived from the two-term arctangent relations for π [22].

The remainder of this paper is organized as follows. Section 2 describes the conventional BBP-type formulas for π. In Section 3, we propose two BBP-type formulas for π. Finally, Section 4 presents some concluding remarks.

2 Conventional BBP-type formulas for π

The BBP formula [4] is as follows:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \quad (3)$$

Consider computing a few hexadecimal digits of π starting at position $n+1$ for a positive integer n. Note that this is equivalent to computing $\{16^n \pi\}$, where $\{\cdot\}$ denotes the fractional part [5]. From (3), we have

$$\{16^n \pi\} = \{4\{16^n S(1)\} - 2\{16^n S(4)\} - \{16^n S(5)\} - \{16^n S(6)\}\}, \quad (4)$$

where

$$S(j) = \sum_{k=0}^{\infty} \frac{1}{16^k (8k+j)}. \quad (5)$$

We note that

$$\begin{aligned} \{16^n S(j)\} &= \left\{ \left\{ \sum_{k=0}^{n} \frac{16^{n-k}}{8k+j} \right\} + \sum_{k=n+1}^{\infty} \frac{16^{n-k}}{8k+j} \right\} \\ &= \left\{ \left\{ \sum_{k=0}^{n} \frac{16^{n-k} \bmod (8k+j)}{8k+j} \right\} + \sum_{k=n+1}^{\infty} \frac{16^{n-k}}{8k+j} \right\}. \quad (6) \end{aligned}$$

The reason for $\bmod (8k+j)$ appearing in the numerator of the first summation in (6) is that we only need to compute the fractional part.

The numerator of the first summation in (6), namely, $16^{n-k} \bmod (8k+j)$, can be computed efficiently using the binary method for modular exponentiation. Since the exponent of 16 is negative in the numerator of the second summation in (6), it is only necessary to compute until the remaining terms are less than the machine epsilon of the floating-point arithmetic being used [4]. The number of terms for the second summation in (6) is negligible when n is sufficiently large. For simplicity, we do not consider the second summation in this paper. In this case, the BBP formula requires $4n$ terms for the computation of a few hexadecimal digits of π starting at position n. By converting the final result to hexadecimal representation, a few hexadecimal digits of π starting at position $n+1$ can be obtained.

Similar to the BBP formula, a few hexadecimal digits of π starting at position $n+1$ can be computed using the BBP-type formulas described below. Adamchik and Wagon [1,2] proposed the following simple BBP-type formula:

$$\pi = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \left(\frac{2}{4k+1} + \frac{2}{4k+2} + \frac{1}{4k+3} \right). \tag{7}$$

The following BBP-type formula [13,10] requires 20% fewer terms than the BBP formula:

$$\begin{aligned} \pi = {} & \frac{1}{2^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{6k}} \left(\frac{2^3}{4k+1} + \frac{2^2}{4k+2} + \frac{1}{4k+3} \right) \\ & + \frac{1}{2^6} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{10k}} \left(\frac{2^5}{4k+1} + \frac{2^3}{4k+2} + \frac{1}{4k+3} \right). \end{aligned} \tag{8}$$

Bellard's formula [9], which requires 30% fewer terms than the BBP formula, is as follows:

$$\begin{aligned} \pi = {} & \frac{1}{2^6} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{10k}} \left(-\frac{2^5}{4k+1} - \frac{1}{4k+3} + \frac{2^8}{10k+1} - \frac{2^6}{10k+3} - \frac{2^2}{10k+5} \right. \\ & \left. - \frac{2^2}{10k+7} + \frac{1}{10k+9} \right). \end{aligned} \tag{9}$$

The following formula by Huvent [13] requires approximately 33% fewer terms than the BBP formula:

$$\begin{aligned} \pi = {} & \frac{1}{128} \sum_{k=0}^{\infty} \frac{1}{2^{12k}} \left(\frac{768}{24k+3} + \frac{512}{24k+4} + \frac{128}{24k+6} - \frac{16}{24k+12} - \frac{16}{24k+14} \right. \\ & \left. - \frac{12}{24k+15} + \frac{2}{24k+20} - \frac{1}{24k+22} \right). \end{aligned} \tag{10}$$

BBP-type formulas require a bit complexity of $O(n \log n M(\log n))$ where $M(d)$ is the complexity of multiplying d-bit integers [4].

3 Two proposed BBP-type formulas for π

The two-term arctangent relations for π found by Machin in 1706 [22] are as follows:

$$\pi = 4 \arctan \frac{1}{2} + 4 \arctan \frac{1}{3}, \tag{11}$$

$$\pi = 8 \arctan \frac{1}{2} - 4 \arctan \frac{1}{7}, \tag{12}$$

$$\pi = 8 \arctan \frac{1}{3} + 4 \arctan \frac{1}{7}, \tag{13}$$

$$\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}. \tag{14}$$

The Taylor series of $\arctan x$ is given by

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}. \tag{15}$$

With $x = 1/2$ in (15), we have

$$\arctan \frac{1}{2} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(2k+1)}. \tag{16}$$

Bellard presented the following relation [9]:

$$\begin{aligned} \arctan \frac{1}{a-1} &= \mathrm{Im}\left[-\log \left(1 - \frac{1+i}{a} \right) \right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}}{a^{4k+3}} \left(\frac{a^2}{4k+1} + \frac{2a}{4k+2} + \frac{2}{4k+3} \right). \end{aligned} \tag{17}$$

Using $a = 2$ in (17) yields Adamchik and Wagon's formula in (7). With $a = 4$ in (17), we obtain

$$\arctan \frac{1}{3} = \frac{1}{2^5} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{6k}} \left(\frac{2^3}{4k+1} + \frac{2^2}{4k+2} + \frac{1}{4k+3} \right). \tag{18}$$

Furthermore, with $a = 8$ in (17), we obtain

$$\arctan \frac{1}{7} = \frac{1}{2^8} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{10k}} \left(\frac{2^5}{4k+1} + \frac{2^3}{4k+2} + \frac{1}{4k+3} \right). \tag{19}$$

From (13), (18), and (19), we have the BBP-type formula in (8). By expanding (16) 15 times with respect to k, we have

$$\begin{aligned}\arctan\frac{1}{2} = \frac{1}{2^{29}}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{30k}}\Bigg(&\frac{2^{28}}{30k+1} - \frac{2^{26}}{30k+3} + \frac{2^{24}}{30k+5} - \frac{2^{22}}{30k+7}\\ &+ \frac{2^{20}}{30k+9} - \frac{2^{18}}{30k+11} + \frac{2^{16}}{30k+13} - \frac{2^{14}}{30k+15} + \frac{2^{12}}{30k+17}\\ &- \frac{2^{10}}{30k+19} + \frac{2^{8}}{30k+21} - \frac{2^{6}}{30k+23} + \frac{2^{4}}{30k+25} - \frac{2^{2}}{30k+27}\\ &+ \frac{1}{30k+29}\Bigg). \end{aligned} \tag{20}$$

By expanding (18) five times with respect to k, we have

$$\begin{aligned}\arctan\frac{1}{3} = \frac{1}{2^{29}}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{30k}}\Bigg(&\frac{2^{25}}{10k+1} - \frac{2^{19}}{10k+3} + \frac{2^{13}}{10k+5} - \frac{2^{7}}{10k+7}\\ &+ \frac{2^{1}}{10k+9} + \frac{2^{27}}{20k+1} + \frac{2^{24}}{20k+3} - \frac{2^{21}}{20k+5} - \frac{2^{18}}{20k+7}\\ &+ \frac{2^{15}}{20k+9} + \frac{2^{12}}{20k+11} - \frac{2^{9}}{20k+13} - \frac{2^{6}}{20k+15} + \frac{2^{3}}{20k+17}\\ &+ \frac{1}{20k+19}\Bigg). \end{aligned} \tag{21}$$

By expanding (19) three times with respect to k, we have

$$\begin{aligned}\arctan\frac{1}{7} = \frac{1}{2^{28}}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{30k}}\Bigg(&\frac{2^{22}}{6k+1} - \frac{2^{12}}{6k+3} + \frac{2^{2}}{6k+5} + \frac{2^{25}}{12k+1}\\ &+ \frac{2^{20}}{12k+3} - \frac{2^{15}}{12k+5} - \frac{2^{10}}{12k+7} + \frac{2^{5}}{12k+9} + \frac{1}{12k+11}\Bigg). \end{aligned} \tag{22}$$

Consider the computation and verification of π using BBP-type formulas derived from (11), (12), and (20)–(22). From (12), the subtraction between the terms with denominators of $30k+5j$ for $j=1,3,5$ in (20) and the terms with denominators of $6k+j$ for $j=1,3,5$ in (22) can be reduced to terms with common denominators of $30k+5j$ for $j=1,3,5$. Thus, we obtain the following BBP-type formula from (12), (20), and (22):

$$
\begin{aligned}
\pi = \frac{1}{2^{26}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{30k}} \biggl(& -\frac{2^{25}}{12k+1} - \frac{2^{20}}{12k+3} + \frac{2^{15}}{12k+5} + \frac{2^{10}}{12k+7} - \frac{2^{5}}{12k+9} \\
& - \frac{1}{12k+11} + \frac{2^{28}}{30k+1} - \frac{2^{26}}{30k+3} - \frac{2^{22}}{30k+5} - \frac{2^{22}}{30k+7} + \frac{2^{20}}{30k+9} \\
& - \frac{2^{18}}{30k+11} + \frac{2^{16}}{30k+13} + \frac{2^{12}}{30k+15} + \frac{2^{12}}{30k+17} - \frac{2^{10}}{30k+19} \\
& + \frac{2^{8}}{30k+21} - \frac{2^{6}}{30k+23} - \frac{2^{2}}{30k+25} - \frac{2^{2}}{30k+27} + \frac{1}{30k+29} \biggr). \qquad (23)
\end{aligned}
$$

The identity (23) can also be obtained by expanding Bellard's formula in (9) three times with respect to k. The identity (23) can be expressed as follows:

$$
\begin{aligned}
\pi = & -2^{-1}S(12,1) - 2^{-6}S(12,3) + 2^{-11}S(12,5) + 2^{-16}S(12,7) - 2^{-21}S(12,9) \\
& - 2^{-26}S(12,11) + 2^{2}S(30,1) - S(30,3) - 2^{-4}S(30,5) - 2^{-4}S(30,7) \\
& + 2^{-6}S(30,9) - 2^{-8}S(30,11) + 2^{-10}S(30,13) + 2^{-14}S(30,15) \\
& + 2^{-14}S(30,17) - 2^{-16}S(30,19) + 2^{-18}S(30,21) - 2^{-20}S(30,23) \\
& - 2^{-24}S(30,25) - 2^{-24}S(30,27) + 2^{-26}S(30,29), \qquad (24)
\end{aligned}
$$

where

$$
S(m,j) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{30k}(mk+j)}. \qquad (25)
$$

On the other hand, from (11), the subtraction between the terms with denominators of $30k+3j$ for $j=1,3,7,9$ in (20) and the terms with denominators of $10k+j$ for $j=1,3,7,9$ in (21) can be reduced to terms with common denominators of $30k+3j$ for $j=1,3,7,9$. Thus, we obtain the following BBP-type formula from (11), (20), and (21):

$$
\begin{aligned}
\pi = \frac{1}{2^{27}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{30k}} \biggl(& \frac{2^{13}}{10k+5} + \frac{2^{27}}{20k+1} + \frac{2^{24}}{20k+3} - \frac{2^{21}}{20k+5} - \frac{2^{18}}{20k+7} \\
& + \frac{2^{15}}{20k+9} + \frac{2^{12}}{20k+11} - \frac{2^{9}}{20k+13} - \frac{2^{6}}{20k+15} + \frac{2^{3}}{20k+17} \\
& + \frac{1}{20k+19} + \frac{2^{28}}{30k+1} + \frac{2^{25}}{30k+3} + \frac{2^{24}}{30k+5} - \frac{2^{22}}{30k+7} - \frac{2^{19}}{30k+9} \\
& - \frac{2^{18}}{30k+11} + \frac{2^{16}}{30k+13} - \frac{2^{14}}{30k+15} + \frac{2^{12}}{30k+17} - \frac{2^{10}}{30k+19} \\
& - \frac{2^{7}}{30k+21} - \frac{2^{6}}{30k+23} + \frac{2^{4}}{30k+25} + \frac{2^{1}}{30k+27} + \frac{1}{30k+29} \biggr). \qquad (26)
\end{aligned}
$$

Table 1 Number of terms for the computation and verification of hexadecimal digits of π starting at position n. Lower-order terms were not computed.

Formula	Number of terms for computation	Number of terms for verification	Total number of terms	Ratio
Adamchik–Wagon [1,2]	$6n$	$6n$	$12n$	1.500
Bailey–Borwein–Plouffe [4]	$4n$	$4n$	$8n$	1.000
Bellard [9]	$(14/5)n$	$(14/5)n$	$(28/5)n$	0.700
Huvent [13]	$(8/3)n$	$(8/3)n$	$(16/3)n$	0.667
Proposed (23)	$(14/5)n$	$(14/5)n$	$(28/5)n$	0.700
Proposed (26)	$(52/15)n$	$(52/15)n$	$(104/15)n$	0.867
Proposed (23) + (26)	$(14/5)n$	$(22/15)n$	$(64/15)n$	0.533

The identity (26) can be expressed as follows:

$$\begin{aligned}\pi = {} & 2^{-14}S(10,5) + S(20,1) + 2^{-3}S(20,3) - 2^{-6}S(20,5) - 2^{-9}S(20,7)\\ & + 2^{-12}S(20,9) + 2^{-15}S(20,11) - 2^{-18}S(20,13) - 2^{-21}S(20,15)\\ & + 2^{-24}S(20,17) + 2^{-27}S(20,19) + 2^{1}S(30,1) + 2^{-2}S(30,3)\\ & + 2^{-3}S(30,5) - 2^{-5}S(30,7) - 2^{-8}S(30,9) - 2^{-9}S(30,11)\\ & + 2^{-11}S(30,13) - 2^{-13}S(30,15) + 2^{-15}S(30,17) - 2^{-17}S(30,19)\\ & - 2^{-20}S(30,21) - 2^{-21}S(30,23) + 2^{-23}S(30,25) + 2^{-26}S(30,27)\\ & + 2^{-27}S(30,29),\end{aligned} \tag{27}$$

where

$$S(m,j) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{30k}(mk+j)}. \tag{28}$$

In (27), the subtraction between $2^{-14}S(10,5)$ and $2^{-13}S(30,15)$ can also be reduced to $2^{-14}S(30,15)$. However, $2^{-14}S(30,15)$ also exists in (24). In this case, if an error occurs in the calculation of $S(30,15)$, that error cannot be detected. Thus, the reduction between $2^{-14}S(10,5)$ and $2^{-13}S(30,15)$ should not be performed.

Between (24) and (27), the terms $S(30,j)$ for $j = 1,3,5,\ldots,29$ are common. Thus, the common terms calculated in (24) can be reused for the verification using (27). That is, it is sufficient to compute only the terms $S(10,5)$ and $S(20,j)$ for $j = 1,3,5,\ldots,19$.

Performance evaluation of three BBP-type formulas, the original BBP formula, Bellard's formula, and the Adamchik–Wagon formula, has previously been presented [19]. For comparison, Table 1 lists the number of terms for the computation and verification of hexadecimal digits of π starting at position n. We assume that the results are verified by shifting the starting position of a few hexadecimal digits of π, except when combining the two proposed BBP-type formulas (23) and (26). In the proposed two BBP-type formulas and their combination, the calculation result of (23) is verified using (26). As shown in Table 1, the proposed BBP-type formula in (23) requires $(14/5)n$ terms for a given starting position n, which is the same as Bellard's formula in (9). On the

Table 2 Execution time to compute the 10^{11}-th hexadecimal digit of π on an Intel Xeon Phi 7250.

	Computation time (sec)	Verification time (sec)	Total time (sec)	Ratio
Adamchik–Wagon	761.96	761.92	1523.88	1.534
Bailey–Borwein–Plouffe	496.58	496.55	993.13	1.000
Bellard	355.57	355.57	711.14	0.716
Huvent	331.25	331.26	662.51	0.667
Proposed (23)	353.93	353.92	707.85	0.713
Proposed (26)	438.18	438.18	876.36	0.882
Proposed (23) + (26)	353.93	185.39	539.32	0.543

other hand, the proposed BBP-type formula in (26) requires $(52/15)n$ terms for a given starting position n. However, computation and verification using the two proposed BBP-type formulas require 20% fewer terms than those of Huvent's formula in (7). This is because the common terms calculated in (24) can be reused for the verification using (27).

Table 2 shows the execution time required to compute the 10^{11}-th hexadecimal digit of π on an Intel Xeon Phi 7250. The programs used for the execution are implemented based on [21]. It can be seen that the execution time of each formula in Table 2 is approximately proportional to the number of terms shown in Table 1. As shown in Table 2, computation and verification using the two proposed BBP-type formulas require approximately 19% less execution time than those using Huvent's formula.

4 Conclusion

In this paper, we have proposed two BBP-type formulas for π. We have shown that computation and verification of π using these two formulas require 20% fewer terms than verification by shifting the starting position of a few hexadecimal digits of π using Huvent's formula, which is known as the BBP-type formula with the least number of terms.

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