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Categorical extensions of conformal nets. (English) Zbl 07333693
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A systematic study of the relation between vertex operator algebras (VOAs) and conformal nets, two major mathematical formulations of chiral conformal field theory, was initiated in [*S. Carpi* et al., From vertex operator algebras to conformal nets and back. Providence, RI: American Mathematical Society (AMS) (2018; Zbl 1434.17001)]. The principal objective in this paper is to establish the equivalence between the ribbon categories of unitary VOAs and thoase of conformal nets for many familiar examples. The basic idea is to construct categorical extensions. This paper assumes that a VOA V is regular [*C. Dong* et al., Adv. Math. 132, No. 1, 148–166 (1997; Zbl 0902.17014)], so that there exists a modular tensor categorical structure on the category $\mathrm{Rep}^{\mathrm{u}}(V)$ of unitary V-modules [*Y.-Z. Huang*, Commun. Contemp. Math. 10, 871–911 (2008; Zbl 1169.17019)].

This paper consists of 6 chapters. A synopsis of the paper goes as follows.

- Chapter 2, consisting of 6 sections, is concerned with Connes fusion products [*A. Connes*, J. Funct. Anal. 35, 153–164 (1980; Zbl 0443.46042)]. §2.1 reviews some of the basic facts about conformal nets and their representations. §2.2 defines the notion of path continuation, which plays a centrally important role in this paper. §2.3 uses path continuations to define the action of a conformal net $\mathcal{A}$ on the Connes fusion $\mathcal{H}_i \boxtimes \mathcal{H}_j$ of A-modules $\mathcal{H}_i$ and $\mathcal{H}_j$. §2.4 describes the conformal structure of $\mathcal{H}_i \boxtimes \mathcal{H}_j$ in terms of $\mathcal{H}_i$ and $\mathcal{H}_j$. §2.5 discusses Connes fusions of three or more representations. §2.6 defines the C^*-tensor categorical structure on $\mathrm{Rep}(\mathcal{A})$ using the theory of Connes fusions, defining braiding, which is shown to be identical with the one in [*A. Wassermann*, Invent. Math. 133, No. 3, 467–538 (1998; Zbl 0944.46059), §33].
- Chapter 3, consisting of 6 sections, is concerned with Connes fusions and categorical extensions. §3.1 defines categorical extensions of conformal nets. §3.2 uses Connes fusions to construct what are called Connes categorical extensions, by which the Hexagon axioms for $\mathrm{Rep}(\mathcal{A})$ are established in §3.3. The succeeding two sections are devoted to the uniqueness of categorical extensions. §3.4 shows that if $\mathcal{E}$ is a categorical extension of $\mathcal{A}$ over a braided C^*-tensor category $\mathcal{C}$ which is a full subcategory of $\mathrm{Rep}(\mathcal{A})$, then $\mathcal{C}$ is equivalent to the corresponding braided C^*-tensor category defined by Connes fusions. §3.5 shows that $\mathcal{E}$ is to be extended to a unique maximal categorical extension $\overline{\mathcal{E}}$ defined also over $\mathcal{C}$ called the closure of $\mathcal{E}$, which is naturally equivalent to a Connes categorical extension. §3.6 shows that a categorical local extension $\mathcal{E}^{\mathrm{loc}}$ generates a categorical extension $\mathcal{E}$, demonstrating that if A (resp. B) commutes with the right (resp. left) action of $\mathcal{F}$ on $\mathcal{C}$, then A and B commute adjointly (Theorem 3.17).
- The aim of Chapter 4 is to construct vertex categorical extensions by using smeared intertwining operators. §4.1 reviews Huang-Lepowsky construction of ribbon categories for VOA-modules [*Y.-Z. Huang* and *J. Lepowsky*, in: Differential geometric methods in theoretical physics. Proceedings of th 20th international conference, June 3-7, 1991, New York City, NY, USA. Vol. 1-2. Singapore: World Scientific. 344–354 (1992; Zbl 0829.17025); Sel. Math., New Ser. 1, No. 4, 699–756 (1995; Zbl 0854.17032); Sel. Math., New Ser. 1, No. 4, 757–786 (1995; Zbl 0854.17033); J. Pure Appl. Algebra 100, No. 1–3, 141–171 (1995; Zbl 0841.17014); J. Pure Appl. Algebra 100, No. 1–3, 173–216 (1995; Zbl 0841.17015); Prog. Math. 123, 349–383 (1994; Zbl 0848.17031)]. §4.2 gives the construction of the intertwining operators $\mathcal{L}_i$ and $\mathcal{R}_i$, which are closely related to the left and right actions L and R in categorical extensions.§4.3 reviews unitary structures on these tensor categories [*B. Gui*, Commun. Math. Phys. 366, No. 1, 333–396 (2019; Zbl 1425.17039); Commun. Math. Phys. 372, No. 3, 893–950 (2019; Zbl 1425.17040)]. §4.4 reviews energy bounds condtions and smeared intertwining operators, establishing the adjoint commutativity of the smeared $\mathcal{L}$ and $\mathcal{R}$. §4.5 discusses constructions of conformal nets and their representationf from their VOA-modules. §4.6 constructs vertex categorical extensions by using these smeared intertwining operators.
- Chapter 5 is concerned with applications. §5.1 shows that if V is a $c < 1$ unitary Virasoro VOA, or a unitary affine VOA of type A, C, G_2, then the category $\mathrm{Rep}^{\mathrm{ss}}(\mathcal{A}_V)$ of semisimple $\mathcal{A}_V$-modules

is equivalent to $\mathrm{Rep}^{\mathrm{u}}(V)$ as unitary modular tensor categories, demonstrating that if V is an affine VOA of type B or D, then the monoidal subcategory $\mathcal{C}$ of $\mathrm{Rep}^{\mathrm{u}}(V)$ tensor-generated by the smallest non-vacuum irreducible V-module is equivalent to $\mathfrak{F}(\mathcal{C})$ as unitary ribbon fusion categories (The braided tensor categorical structure on $\mathfrak{F}(\mathcal{C})$ is defined using Connes fusions). §5.2 establishes the equivalence of the ribbon fusion categories $\mathrm{Rep}^{\mathrm{u}}(V)$ and $\mathfrak{F}(\mathrm{Rep}^{\mathrm{u}}(V))$ when V is a unitary Heisenberg VOA (In this case $\mathrm{Rep}^{\mathrm{u}}(V)$ is defined to be the tensor category of semisimple unitary V-modules), demonstrating the strong intertwining and braiding properties for all intertwining operators of unitary Heisenberg VOAs, which is used in §5.3 to establish the strong intertwining and braiding properties of an even lattice VOA V.

- In the literature of conformal nets, the braided tensor categories are often defined using Doplicher-Haag-Roberts (DHR) superselection theory [*K. Fredenhagen* et al., Commun. Math. Phys. 125, No. 2, 201–226 (1989; Zbl 0682.46051); Rev. Math. Phys. Spec. Issue, 113–157 (1992; Zbl 0774.46041); Comm. Math. Phys. 23, 199–230 (1971); ibid. 35, 49–85 (1974)]. It is well known tha, as far as conformal nets are strongly additive, Connes fusions and DHR theory define the same monoidal structures, though it is not clear that these two theories give the same braidings. Chapter 6 clarifies the relation between the two theories, showing that the braided C^*-tensor categories defined by them are equivalent. As an important application, we can conclude that the Reshetikhin-Turaev 3d topological quantum field theories [*N. Reshetikhin* and *V. G. Turaev*, Invent. Math. 103, No. 3, 547–597 (1991; Zbl 0725.57007); Quantum invariants of knots and 3-manifolds. Berlin: Walter de Gruyter (1994; Zbl 0812.57003)] constructed from $\mathrm{Rep}^{\mathrm{u}}(V)$ and from $\mathrm{Rep}^{\mathrm{ss}}(\mathcal{A}_V)$ are the same.

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MSC:

18M20 Fusion categories, modular tensor categories, modular functors
17B69 Vertex operators; vertex operator algebras and related structures
46L60 Applications of selfadjoint operator algebras to physics
46L87 Noncommutative differential geometry
81T40 Two-dimensional field theories, conformal field theories, etc. in quantum mechanics

Full Text: DOI

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