Citations | From References: 4 | From Reviews: 0

**MR4202003** 17B66 18N10 58H05
**Berwick-Evans, Daniel** (1-IL); **Lerman, Eugene** [**Lerman, Eugene M.**] (1-IL)
**Lie 2-algebras of vector fields. (English summary)**
*Pacific J. Math.* **309** (2020), *no. 1,* 1–34.

It was shown in [R. A. Hepworth, Theory Appl. Categ. **22** (2009), 542–587; MR2591949] that vector fields on a Lie groupoid form a category. The first main result of this paper (Theorem 3.4) is that the category of vector fields on a geometric stack has the structure of a Lie 2-algebra, answering a conjecture of Hepworth. The second main result of the paper (Theorem 4.1) is that the map

$$G \mapsto \mathbb{X}(G),$$

assigning to each Lie groupoid its category of vector fields, extends to a functor

$$\mathbb{X}\colon \mathsf{Bi}_{\rm iso} \to \mathsf{Lie2Alg},$$

where $\mathsf{Lie2Alg}_{\rm strict}$ denotes the strict 2-category of Lie 2-algebras, internal functors and internal natural transformations, being localized at the essential equivalence to yield a bicategory $\mathsf{Lie2Alg}$, while $\mathsf{Bi}$ denotes the localization of the strict 2-category of Lie groupoids, internal functors and internal natural transformations at the class of functors that are fully faithful and essentially surjective, $\mathsf{Bi}_{\rm iso}$ being the sub-bicategory of $\mathsf{Bi}$ consisting of Lie groupoids as objects, weakly invertible bibundles (i.e., Morita equivalences) as 1-morphisms and isomorphisms of bibundles as 2-morphisms. The third main result of the paper (Theorem 6.1) is 2-commutativity of the diagram

$$\begin{tikzcd}
& \mathsf{Grp} & \\
\mathsf{GeomStack}_{\rm iso} \arrow[ur, "\mathsf{Vect}'"] \arrow[rr, phantom, "\Longleftarrow_{\gamma}"] & & \mathsf{Lie2Alg}, \arrow[ul, "u"'] \\
& \mathsf{Bi}_{\rm iso} \arrow[ul, "\mathbb{B}"] \arrow[ur, "\mathbb{X}"'] &
\end{tikzcd}$$

where:

(1) $\mathsf{Grp}$ denotes the $(2,1)$-category of groupoids, functors and natural isomorphisms.
(2) $\mathsf{GeomStack}_{\rm iso}$ is the $(2,1)$-category of geometric stacks, isomorphisms of stacks (i.e., weakly invertible 1-morphisms of stacks) and 2-morphisms.
(3) To every Lie groupoid $G$ there corresponds the stack $\mathbb{B}G$ of principal $G$-bundles, and Morita equivalent Lie groupoids $G$ and $H$ correspond to isomorphic stacks $\mathbb{B}G$ and $\mathbb{B}H$. The assignment $G \mapsto \mathbb{B}G$ extends to a fully faithful functor

$$\mathbb{B}\colon \mathsf{Bi} \to \mathsf{Stack}$$

whose essential image is the 2-category $\mathsf{GeomStack}$ of geometric stacks. Restricting the functor to the bicategory of groupoids and Morita equivalences gives an equivalence of bicategories

$$\mathbb{B}\colon \mathsf{Bi}_{\rm iso} \to \mathsf{GeomStack}_{\rm iso}.$$

(4) The functor

$$u\colon \mathsf{Lie2Alg} \to \mathsf{Grp}$$

assigns to each Lie 2-algebra its underlying groupoid.

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*Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*



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