Safronov, Pavel
A categorical approach to quantum moment maps. (English) Zbl 07377655
Theory Appl. Categ. 37, 818-862 (2021)

The principal objective in this paper is to provide a comprehensive study of moments maps on the quantum level within the context of quantum Manin pairs and quantum Manin triples, showing that they recover Alekseev-Kosmann-Schwarzbach notion after classical degeneration. This paper is heavily inspired by the theory of shifted symplectic structures [*T. Pantev* et al., Publ. Math., Inst. Hautes Étud. Sci. 117, 271–328 (2013; Zbl 1328.14027)] and shifted Poisson structures [*D. Calaque* et al., J. Topol. 10, No. 2, 483–584 (2017; Zbl 1428.14006)]. As was constructed in [*L. Amorim* and *O. Ben-Bassat*, Adv. Theor. Math. Phys. 21, No. 2, 289–381 (2017; Zbl 1394.53079)], one may organize n-shifted symplectic stacks into the symmetric monoidal 2-category $\mathrm{LagrCorr}_n$, which is extended to an (∞, m)-category for any m in the upcoming work of Calaque, Haugseng and Scheimbauer.

The main results are the following two theorems.

(Theorem 3.17) Suppose $(D, H, H^\vee)$ is a triple of Hopf algebras giving rise to a quantum Manin triple $(\mathrm{CoMod}_D, \mathrm{CoMod}_H, \mathrm{CoMod}_{H^\vee})$. Then an algebra map $\mu : \mathcal{F} \to A$ in H-comodules is a quantum moment map iff it satisfies

$$\mu\left(h\right)a = \left(h_{(1)} \triangleright a\right)\mu\left(h_{(2)}\right)$$

for every $a \in A$ and h$\in \mathcal{F}$, where

$$\Delta\left(h\right) = \in h_{(1)} \otimes h_{(2)} \in H^\vee \otimes \mathcal{F}$$

is the $H^\vee$-coaction on $\mathcal{F}$.

(Theorem 4.30) Suppose (D, G) is a group pair and $(\mathrm{Rep}_\hbar D, \mathrm{Rep}_\hbar G)$ is a quantum Manin pair quantizing it. If an algebra map $\mu_\hbar : \mathcal{F} \to A_\hbar$ in $\mathrm{Rep}_\hbar G$ is a quantum moment map, then its value $\mu : \mathcal{O}\left(D/G\right) \to A_G$ at $\hbar = 0$ is a classical moment map.

Combining the above two theorems, one can conclude that a classical degeneration of Varagnolo-Vasserot moment maps [*M. Varagnolo* and *E. Vasserot*, Represent. Theory 14, 510–600 (2010; Zbl 1280.20005)] gives rise to Alekseev-Kosmann-Schwarzbach moment maps [*A. Alekseev* and *Y. Kosmann-Schwarzbach*, J. Differ. Geom. 56, No. 1, 133–165 (2000; Zbl 1046.53055)].

A synopsis of the paper, consisting of four sections, goes as follows.

[1] §1 recalls the necessary facts about monoidal categories [*A. Brochier* et al., Compos. Math. 157, No. 3, 435–483 (2021; Zbl 1461.18014); *C. L. Douglas* et al., Kyoto J. Math. 59, No. 1, 167–179 (2019; Zbl 07081625)]. The author works in the 2-category Pr^{L} of locally presentable categories, which admits a natural symmetric monoidal structure. The notion of a $\mathcal{C}$-monoidal category is defined, namely, a monoidal category $\mathcal{D}$ with a compatible action of a braided monoidal category $\mathcal{C}$. The pair $(\mathcal{C}, \mathcal{D})$ is to be thought of as an algebra in Pr^{L} over the two-dimensional Swiss-cheese operad, reminiscent of $\mathbb{E}_2$-algebras in Pr^{L} in terms of braided monoidal categories. The author shows in particular that the relative tensor product $\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}$ of $\mathcal{C}$-monoidal categories $\mathcal{E}$ and $\mathcal{D}$ carries a natural monoidal structure, describing its universal property (Proposition 1.25).

[2] §2 gives the main definitions of the paper, defining and studying quantum Manin pairs and quantum Manin triples. A *quantum Manin pair* is a pair $(\mathcal{C}, \mathcal{D})$ consisting of a braided monoidal category $\mathcal{C}$ acting in a compatible way on a monoidal category $\mathcal{D}$ via a monoidal functor $T : \mathcal{C} \to \mathcal{D}$. A *quantum Manin triple* consists of a braided monoidal category $\mathcal{C}$, a pair of monoidal categories $\mathcal{D}, \mathcal{E}$ with quantum Manin pairs $(\mathcal{C}, \mathcal{D})$ and $(\mathcal{C}, \mathcal{E})$ as well as a monoidal functor $\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \to \mathrm{Mod}_k$. A quantum Manin triple is shown to encode a wealth of information, namely, an important algebra $\mathcal{F} = TT\left(1\right) \in \mathcal{D}$, a monoidal category $\mathcal{HC} = \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}$, a pair of bialgebras $FF^R\left(k\right)$ and $\widetilde{F}\widetilde{F}^R\left(k\right)$, an algebra map $F\left(\mathcal{F}\right) \to \widetilde{F}\widetilde{F}^R\left(k\right)$ and a skew-Hopf pairing

$$\mathrm{ev} : \widetilde{F}\widetilde{F}^R\left(k\right) \otimes FF^R\left(k\right) \to k$$

which turns $FF^R(k)$-comodules into $\widetilde{F}\widetilde{F}^R(k)$-modules, giving a functor

$$\mathrm{CoMod}_{FF^R(k)} \to \mathrm{LMod}_{\widetilde{F}\widetilde{F}^R(k)}$$

[3] Given a quantum Manin pair, §3 introduces a definition of quantum moment map (Definition 3.2), giving several ways to describe them (Proposition 3.15). It is observed that the data of a quantum moment map allows of extending a $\mathcal{D}$-module structure to an $\mathcal{HD}$-module structure. The author describes a procedure of fusion of algebras with quantum moment map, remarking that on the level of categories, given two $\mathcal{HD}$-module categories $\mathcal{M}_1$ and $\mathcal{M}_2$, it is simply given by the relative tensor product $\mathcal{M}_1 \otimes_{\mathcal{D}} \mathcal{M}_2$.

[4] §4 recalls the definitions of quasi-Poisson groups and quasi-Poisson spaces, where the author provides a definition of moment maps (Definition 4.15), which is a slight variant of the definition of *A. Alekseev* and *Y. Kosmann-Schwarzbach* [J. Differ. Geom. 56, No. 1, 133–165 (2000; Zbl 1046.53055)]. Both definitions are shown to be equivalent (Proposition 4.20). It is shown in Lemma Lemma 4.24 that for moment maps factoring such as

$$X \to G^* \to D/G$$

the definition reduces to *J.-H. Lu*'s one [Commun. Math. Phys. 157, No. 2, 389–404 (1993; Zbl 0801.17019)]. The author shows in §4.26 that the classical degeneration of quantum moment maps recovers classical moment maps.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18M15 Braided monoidal categories and ribbon categories

Keywords:

moment maps; Manin triples; quantum groups

Full Text: Link

References:

[1] Th´eorie des topos et cohomologie ´etale des sch´emas. Tome 2, Lecture Notes in Mathematics, Vol. 270, Springer-Verlag, Berlin-New York, 1972, S´eminaire de G´eom´etrie Alg´ebrique du Bois-Marie 1963-1964 (SGA 4), Dirig´e par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] J. Ad´amek and J. Rosick´y,Locally presentable and accessible categories, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994. · Zbl 0795.18007

[3] A. Alekseev, Y. Kosmann-Schwarzbach, and E. Meinrenken,Quasi-Poisson manifolds, Canad. J. Math.54(2002), no. 1, 3-29. · Zbl 1006.53072

[4] A. Alekseev and Y. Kosmann-Schwarzbach,Manin pairs and moment maps, J. Differential Geom.56(2000), no. 1, 133-165. · Zbl 1046.53055

[5] A. Alekseev, A. Malkin, and E. Meinrenken,Lie group valued moment maps, J. Differential Geom.48(1998), no. 3, 445-495. · Zbl 0948.53045

[6] L. Amorim and O. Ben-Bassat,Perversely categorified Lagrangian correspondences, Adv. Theor. Math. Phys.21(2017), no. 2, 289-381.

[7] M. Balagovic and D. Jordan,The Harish-Chandra isomorphism for quantumGL2, J. Noncommut. Geom.12(2018), no. 3, 1161-1197. · Zbl 1428.17012

[8] D. Ben-Zvi, A. Brochier, and D. Jordan,Integrating quantum groups over surfaces, J. Topol.11(2018), no. 4, 873-916. · Zbl 1409.14028

[9] D. Ben-Zvi, A. Brochier, and D. Jordan,Quantum character varieties and braided module categories, Selecta Math. (N.S.)24(2018), no. 5, 4711-4748. · Zbl 1456.17010

[10] J. N. Bernstein and S. I. Gelfand,Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras, Compositio Math.41(1980), no. 2, 245- 285. · Zbl 0445.17006

[11] R. Bezrukavnikov,On tensor categories attached to cells in affine Weyl groups, Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., vol. 40, Math. Soc. Japan, Tokyo, 2004, pp. 69-90. · Zbl 1078.20044

[12] R. Bezrukavnikov, M. Finkelberg, and V. Ostrik,CharacterD-modules via Drinfeld center of Harish-Chandra bimodules, Invent. Math.188(2012), no. 3, 589-620. · Zbl 1267.20058

[13] M. Brandenburg, A. Chirvasitu, and T. Johnson-Freyd,Reflexivity and dualizability in categorified linear algebra, Theory Appl. Categ.30(2015), Paper No. 23, 808-835. · Zbl 1374.18003

[14] A. Brochier, D. Jordan, and N. Snyder,On dualizability of braided tensor categories, Compos. Math.157(2021), no. 3, 435-483. · Zbl 1461.18014

[15] A. Brugui'eres and A. Virelizier,Hopf monads, Adv. Math.215(2007), no. 2, 679-733. · Zbl 1168.18002

[16] H. Bursztyn and M. Crainic,Dirac geometry, quasi-Poisson actions andD/G-valued moment maps, J. Differential Geom.82(2009), no. 3, 501-566. · Zbl 1177.53072

[17] 860PAVEL SAFRONOV

[18] D. Calaque,Lagrangian structures on mapping stacks and semi-classical TFTs, Stacks and categories in geometry, topology, and algebra, Contemp. Math., vol. 643, Amer. Math. Soc., Providence, RI, 2015, pp. 1-23. · Zbl 1349.14005

[19] D. Calaque, T. Pantev, B. To¨en, M. Vaqui´e, and G. Vezzosi,Shifted Poisson Structures and Deformation Quantization, J. Topol.10(2017), no. 2, 483-584. · Zbl 1428.14006

[20] V. Chari and A. Pressley,A guide to quantum groups, Cambridge University Press, Cambridge, 1995, Corrected reprint of the 1994 original. · Zbl 0839.17010

[21] A. Davydov, M. M¨uger, Dmitri Nikshych, and Victor Ostrik,The Witt group of nondegenerate braided fusion categories, J. Reine Angew. Math.677(2013), 135-177. · Zbl 1271.18008

[22] C. L. Douglas, C. Schommer-Pries, and N. Snyder,The balanced tensor product of module categories, Kyoto J. Math.59(2019), no. 1, 167-179. · Zbl 07081625

[23] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik,On braided fusion categories. I, Selecta Math. (N.S.)16(2010), no. 1, 1-119. · Zbl 1201.18005

[24] P. Etingof,Calogero-Moser systems and representation theory, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Z¨urich, 2007. · Zbl 1331.53002

[25] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik,Tensor categories, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015. · Zbl 1365.18001

[26] P. Etingof, D. Nikshych, and V. Ostrik,Fusion categories and homotopy theory, Quantum Topol.1(2010), no. 3, 209-273, With an appendix by Ehud Meir. · Zbl 1214.18007

[27] P. Etingof and O. Schiffmann,Lectures on quantum groups, Lectures in Mathematical Physics, International Press, Somerville, MA, 2002. · Zbl 1106.17015

[28] D. Gaitsgory,A conjectural extension of the Kazhdan-Lusztig equivalence, (2018).

[29] N. Gambino,Homotopy limits for 2-categories, Math. Proc. Cambridge Philos. Soc.145 (2008), no. 1, 43-63. · Zbl 1145.55019

[30] R. Haugseng,The higher Morita category ofEn-algebras, Geom. Topol.21(2017), no. 3, 1631-1730. · Zbl 1395.18011

[31] T. Johnson-Freyd and C. Scheimbauer,(Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories, Adv. Math.307(2017), 147-223. · Zbl 1375.18043

[32] D. Jordan,Quantized multiplicative quiver varieties, Adv. Math.250(2014), 420-466. · Zbl 1337.16012

[33] A. Klimyk and K. Schm¨udgen,Quantum groups and their representations, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997.

[34] S. Kolb and J. V. Stokman,Reflection equation algebras, coideal subalgebras, and their centres, Selecta Math. (N.S.)15(2009), no. 4, 621-664. · Zbl 1236.17023

[35] P. P. Kulish and E. K. Sklyanin,Algebraic structures related to reflection equations, J. Phys. A25(1992), no. 22, 5963-5975. · Zbl 0774.17019

[36] R. Laugwitz,The relative monoidal center and tensor products of monoidal categories, Commun. Contemp. Math.22(2020), no. 8, 1950068, 53. · Zbl 1451.18033

[37] D. Li-Bland and P. ˇSevera,Quasi-Hamiltonian groupoids and multiplicative Manin pairs, Int. Math. Res. Not. IMRN (2011), no. 10, 2295-2350. · Zbl 1218.53083

[38] J.-H. Lu,Momentum mappings and reduction of Poisson actions, Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 20, Springer, New York, 1991, pp. 209-226.

[39] J.-H. Lu,Moment maps at the quantum level, Comm. Math. Phys.157(1993), no. 2, 389-404. · Zbl 0801.17019

[40] J.-H. Lu and A. Weinstein,Poisson Lie groups, dressing transformations, and Bruhat decompositions, J. Differential Geom.31(1990), no. 2, 501-526. · Zbl 0673.58018

[41] F. Lucatelli Nunes,Pseudo-Kan extensions and descent theory, Theory Appl. Categ.33 (2018), Paper No. 15, 390-444. · Zbl 1405.18002

[42] J. Lurie,Higher algebra, 2017.

[43] G.Lusztig,Introductiontoquantumgroups,ModernBirkh¨auserClassics, Birkh¨auser/Springer, New York, 2010, Reprint of the 1994 edition.

[44] S. MacLane,Categories for the working mathematician, Springer-Verlag, New York-Berlin, 1971, Graduate Texts in Mathematics, Vol. 5. · Zbl 0705.18001

[45] S. Majid,Braided groups, J. Pure Appl. Algebra86(1993), no. 2, 187-221. · Zbl 0797.17004

[46] J. E. Marsden and T. S. Ratiu,Introduction to mechanics and symmetry, Texts in Applied Mathematics, vol. 17, Springer-

Verlag, New York, 1994, A basic exposition of classical mechanical systems. · Zbl 0811.70002

[47] A. Nyman and S. P. Smith,A generalization of Watts's theorem: right exact functors on module categories, Comm. Algebra44(2016), no. 7, 3160-3170. · Zbl 1353.18016

[48] T. Pantev, B. To¨en, M. Vaqui´e, and G. Vezzosi,Shifted symplectic structures, Publ. Math. Inst. Hautes ´Etudes Sci.117(2013), 271-328.

[49] P. Safronov,Quasi-Hamiltonian reduction via classical Chern-Simons theory, Adv. Math. 287(2016), 733-773. · Zbl 1440.53096

[50] 862PAVEL SAFRONOV

[51] P. Safronov,Poisson-Lie structures as shifted Poisson structures, Adv. Math.381(2021), 107633, 68. · Zbl 07319262

[52] C. Scheimbauer,Factorization homology as a fully extended topological field theory, Ph.D. thesis, 2014.

[53] M. Varagnolo and E. Vasserot,Double affine Hecke algebras at roots of unity, Represent. Theory14(2010), 510-600. · Zbl 1280.20005

[54] P. ˇSevera,Left and right centers in quasi-Poisson geometry of moduli spaces, Adv. Math. 279(2015), 263-290. · Zbl 1319.53096

[55] M. B. Wischnewsky,On linear representations of affine groups. I, Pacific J. Math.61 (1975), no. 2, 551-572. · Zbl 0324.18006

[56] Institut f¨ur Mathematik, Universit¨at Z¨urich, Zurich, Switzerland

[57] Email:p.safronov@ed.ac.uk

[58] This article may be accessed athttp://www.tac.mta.ca/tac/

[59] THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles that significantly advance

[60] the study of categorical algebra or methods, or that make significant new contributions to mathematical

[61] science using categorical methods. The scope of the journal includes: all areas of pure category theory,

[62] including higher dimensional categories; applications of category theory to algebra, geometry and topology

[63] and other areas of mathematics; applications of category theory to computer science, physics and other

[64] mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

[65] Articles appearing in the journal have been carefully and critically refereed under the responsibility of

[66] for publication.

[67] Subscription informationIndividual subscribers receive abstracts of articles by e-mail as they

[68] are published. To subscribe, send e-mail totac@mta.caincluding a full name and postal address. Full

[69] text of the journal is freely available athttp://www.tac.mta.ca/tac/.

[70] Information for authorsLATEX2e is required. Articles may be submitted in PDF by email

[71] directly to a Transmitting Editor following the author instructions at

[72] http://www.tac.mta.ca/tac/authinfo.html.

[73] Managing editor.Geoff Cruttwell, Mount Allison University:gcruttwell@mta.ca

[74] TEXnical editor.Michael Barr, McGill University:michael.barr@mcgill.ca

[75] Assistant TEX editor.Gavin Seal, Ecole Polytechnique F´ed´erale de Lausanne:

[76] gavin seal@fastmail.fm

[77] Transmitting editors.

[78] Clemens Berger, Universit´e de Nice-Sophia Antipolis:cberger@math.unice.fr

[79] Julie Bergner, University of Virginia:jeb2md (at) virginia.edu

[80] Richard Blute, Universit´e d' Ottawa:rblute@uottawa.ca

[81] Gabriella B¨ohm, Wigner Research Centre for Physics:bohm.gabriella (at) wigner.mta.hu

[82] Valeria de Paiva: Nuance Communications Inc:valeria.depaiva@gmail.com

[83] Richard Garner, Macquarie University:richard.garner@mq.edu.au

[84] Ezra Getzler, Northwestern University:getzler (at) northwestern(dot)edu

[85] Kathryn Hess, Ecole Polytechnique F´ed´erale de Lausanne:kathryn.hess@epfl.ch

[86] Dirk Hofmann, Universidade de Aveiro:dirk@ua.pt

[87] Pieter Hofstra, Universit´e d' Ottawa:phofstra (at) uottawa.ca

[88] Anders Kock, University of Aarhus:kock@math.au.dk

[89] Joachim Kock, Universitat Aut'onoma de Barcelona:kock (at) mat.uab.cat

[90] Stephen Lack, Macquarie University:steve.lack@mq.edu.au

[91] Tom Leinster, University of Edinburgh:Tom.Leinster@ed.ac.uk

[92] Matias Menni, Conicet and Universidad Nacional de La Plata, Argentina:matias.menni@gmail.com

[93] Ieke Moerdijk, Utrecht University:i.moerdijk@uu.nl

[94] Susan Niefield, Union College:niefiels@union.edu

[95] Kate Ponto, University of Kentucky:kate.ponto (at) uky.edu

[96] Robert Rosebrugh, Mount Allison University:rrosebrugh@mta.ca

[97] Jiri Rosicky, Masaryk University:rosicky@math.muni.cz

[98] Giuseppe Rosolini, Universit‘a di Genova:rosolini@disi.unige.it

[99] Alex Simpson, University of Ljubljana:Alex.Simpson@fmf.uni-lj.si

[100] James Stasheff, University of North Carolina:jds@math.upenn.edu

[101] Ross Street, Macquarie University:ross.street@mq.edu.au

[102] Tim Van der Linden, Universit´e catholique de Louvain:tim.vanderlinden@uclouvain