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On the ternary commutator. I: Exact Mal'tsev categories. (English) Zbl 07377653

Theory Appl. Categ. 36, 379-422 (2021)

A. Bulatov [Contrib. Gen. Algebra 13, 41–54 (2001; Zbl 0986.08003)] introduced a *higher-order n-ary commutator operator* of congruences in Mal'tsev algebras, which extends the binary smith commutator [*J. D. H. Smith*, Mal'cev varieties. Berlin-Heidelberg-New York: Springer-Verlag (1976; Zbl 0344.08002)] and is based on a generalization of the *term condition*. *E. Aichinger* and *N. Mudrinski* [Algebra Univers. 63, No. 4, 367–403 (2010; Zbl 1206.08003)] have developed further the higher-order commutator theory in the context of Mal'tsev varieties, where they found analogues for the binary Smith commutator such as monotonicity, stability with respect to joins, stability with respect to restriction, and so on. *J. Opršal* [Algebra Univers. 76, No. 3, 367–383 (2016; Zbl 1357.08002)] introduced a relational description of the higher-order commutator in Mal'tsev varieties, studying the connection between the term condition and a certain n-fold relation, called the *algebra of 2^n-matrices*, which can be seen as a higher-order version of the double relation $\Delta(R,S)$.

The theory of higher commutators has recently been extended beyond Mal'tsev varieties. *A. Moorhead* [J. Algebra 513, 133–158 (2018; Zbl 1414.08003)] used the term condition as a basis for higher-order commutator theory in congruence modular varieties, while he ["Some notes on the ternary modular commutator", Preprint, `arXiv:1808.01407`] introduced a concept of higher centrality based on matrix constructions leading to a connection with the term condition in congruence modular varieties and a characterization of the ternary commutator in terms of a three-fold relation $\Delta(R,S,T)$. *A. Wires* [Algebra Univers. 80, No. 1, Paper No. 1, 37 p. (2019; Zbl 1439.08007)] has developed further higher commutator properties outside congruence modular varieties.

The principal objective in this paper is to give a categorical description of the Bulatov commutator in the context of exact Mal'tsev categories, extending [*M. C. Pedicchio*, J. Algebra 177, No. 3, 647–657 (1995; Zbl 0843.08004); *D. Bourn* and *M. Gran*, Algebra Univers. 48, No. 3, 309–331 (2002; Zbl 1061.18006); *F. Borceux* and *D. Bourn*, Mal'cev, protomodular, homological and semi-abelian categories. Dordrecht: Kluwer Academic Publishers (2004; Zbl 1061.18001)]. It is shown to have many of the convenient properties of the universal-algebraic counterparts. After Moorhead's work, the authors introduce a construction of a three-fold equivalence relation $\Delta(R,S,T)$ based on Pedicchio's $\Delta(R,S)$, which in turn develops into *3-fold Δ-equivalence relations*, the concept of a *3 -dimensional connector*, and a *ternary Bulatov commutator* $[R,S,T]^{\mathrm{B}}$. It is quite clear that the results in this paper can in principle be extended to higher orders.

The authors' future work goes as follows.

- In a forthcoming article, the authors restrict the context to algebraically coherent [*A. S. Cigoli* et al., Theory Appl. Categ. 30, 1864–1905 (2015; Zbl 1366.18005)], semi-abelian [*G. Janelidze* et al., J. Pure Appl. Algebra 168, No. 2–3, 367–386 (2002; Zbl 0993.18008)] categories, establishing that the commutator introduced in this paper corresponds to the ternary Higgins commutator in [*M. Hartl* and *T. Van der Linden*, Adv. Math. 232, No. 1, 571–607 (2013; Zbl 1258.18007)].
- The authors are currently struggling to generalize the binary Smith-Pedicchio commutator to a higher-order version, which may be exploited to characterize higher central extensions in the sense of [*T. Everaert* et al., Adv. Math. 217, No. 5, 2231–2267 (2008; Zbl 1140.18012)].

Some open problems remain.

- The availability of the commutator in congruence modular varieties [*A. Moorhead*, J. Algebra 513, 133–158 (2018; Zbl 1414.08003)] suggests that the categorical counterpart might be extended beyond the exact Mal'tsev context.
- The relationship between 2-nilpotency defined in terms of the commutator considerd in this paper and the 2-folded objects in [*C. Berger* and *D. Bourn*, J. Homotopy Relat. Struct. 12, No. 4, 765–835 (2017; Zbl 1397.18021)] remains open, being related to the main question in [*A. Moorhead*, Trans. Am. Math. Soc. 374, No. 2, 1229–1276 (2021; Zbl 07291897)] concerning the relationship between

so-called supernilpotency (defined in terms of higher-order commutators) and nilpotency (defined in terms of binary commutators).

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