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Isbell conjugacy and the reflexive completion. (English) Zbl 07377650

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This paper is concerned with *Isbell conjugacy* [*J. R. Isbell*, Ill. J. Math. 4, 541–552 (1960; Zbl 0104.01704)]. Given a small category $\mathcal{A}$, any functor $X:\mathcal{A}^{\mathrm{op}}\to\boldsymbol{Set}$ gives rise to a functor $X^{\vee}:\mathcal{A}\to\boldsymbol{Set}$, its *Isbell conjugate*, defined by

$$X^{\vee}(a)=[\mathcal{A}^{\mathrm{op}},\boldsymbol{Set}](\mathcal{A}(-,a),X)$$

The same operation with $\mathcal{A}$ in place of $\mathcal{A}^{\mathrm{op}}$ produces from $X^{\vee}$ a further functor $X^{\vee\vee}:\mathcal{A}^{\mathrm{op}}\to\boldsymbol{Set}$, and so on, giving an infinite sequence $X,X^{\vee},X^{\vee\vee},\ldots$ of functors $\mathcal{A}$ on with alternating variances. The conjugacy operations define an adjunction between $[\mathcal{A}^{\mathrm{op}},\boldsymbol{Set}]$ and $[\mathcal{A},\boldsymbol{Set}]^{\mathrm{op}}$, so that we have

$$[\mathcal{A}^{\mathrm{op}},\boldsymbol{Set}](X,Y^{\vee})\cong[\mathcal{A},\boldsymbol{Set}](Y,X^{\vee})$$

naturally in $X:\mathcal{A}^{\mathrm{op}}\to\boldsymbol{Set}$ and $Y:\mathcal{A}\to\boldsymbol{Set}$, the unit and counit of the adjunction being canonical maps $X\to X^{\vee\vee}$ and $Y\to Y^{\vee\vee}$. A covariant or contravariant functor on $\mathcal{A}$ is said to be *reflexive* if the canonical map to its double conjugate is an isomorphism. The *reflexive completion* $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ is the category of reflexive functors on $\mathcal{A}$ (covariant or contravariant), being the invariant part of the conjugacy adjunction.

The main text of the paper is divided into two parts, the first part consisting of §§2–6 while the second part consisting of §§7–12. A synopsis of the paper goes as follows.

- The paper begins with the definition of conjugacy on small categories, giving several characterizations of the conjugacy operations with a raft of examples (§2 and §3).
- To define conjugacy on an arbitrary category, the notion of small functor is exploited (§4).
- §5 states the definition of the reflexive completion, while §6 gives a slew of examples.
- §7 collects necessary results on dense and adequate functors, which are used to give a unique characterization of the reflexive completion sharpening a result of Isbell's in §8.
- Reflexive completion is functorial, but only with respect to a very limited class of functors, namely, small-adequate ones, as is shown in §9.differs from many other completions in that it is only functorial in a very restricted sense. It is often the case that the functor $\mathcal{R}(F):\mathcal{R}(\mathcal{A})\to\mathcal{R}(\mathcal{B})$ induced by a functor $F:\mathcal{A}\to\mathcal{B}$ is an equivalence, which is the case, by way of example, if $\mathcal{B}$ is either small or both complete and cocomplete (Corollary 9.8).
- The relationship between reflexive completion and Cauchy completion is discussed in §10.
- A category is called *reflexively complete* if it is the reflexive completion of some category. A reflexively complete category has absolute (co)limits, and if it is the reflexive completion of a small category, then it has initial and terminal objects too, but these are all the limits and colimits that it generally has, as is discussed in §11.
- Informally speaking, one can understand $\mathcal{R}(\mathcal{A})$ as the intersection of the free completions of $\mathcal{A}$ under small colimiits and small limits. §12 makes this idea precise.

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MSC:

18A99 General theory of categories and functors
18A35 Categories admitting limits (complete categories), functors preserving limits, completions
18A40 Adjoint functors (universal constructions, reflective subcategories, Kan extensions, etc.)

Keywords:

Isbell conjugacy; reflexive completion; small functor; Cauchy completion; Isbell envelope

Full Text: Link

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