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From constructive mathematics to computable analysis via the realizability interpretation.
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Starting with realizability models over a typed version of partial combinatory algebras, this dissertation shows that the ensuing models provide the features necessary in order to interpreto impredicative logics and type theories iff the underlying typed partial combinatory algebra is equivalent to an untyped one. The author exhibits realizability models to demonstrate that the implication between continuity principles are strict. The author concentrates *K. Weihrauch*'s [Computable analysis. An introduction. Berlin: Springer (2000; Zbl 0956.68056)] approach to computable analysis among others.

A synopsis of the dissertation, consisting of four chapters, goes as follows.

Chapter 1 is a review of several schools of constructive mathematics, among which *E. A. Bishop*'s constructivism [Foundations of constructive analysis. Maidenhead, Berksh.: McGraw-Hill Publishing Company, Ltd. (1967; Zbl 0183.01503)] and *M. Dummett*'s intuitionism [Elements of intuitionism. Oxford: Oxford University Press (1977; Zbl 0358.02032)] play a significant role in this dissertation.

Chapter 2 is concerned with realizability interpretations. §2.1 is concerned with formalized realizability interpretation, reviewing *S. C. Kleene*'s numerical realizability [J. Symb. Log. 10, 109–124 (1945; Zbl 0063.03260)] and *S. C. Kleene*'s function realizability [Acta Philos. Fenn. 18, 71–80 (1965; Zbl 0133.25201); Formalized recursive functionals and formalized realizability. Providence, RI: American Mathematical Society (AMS) (1969; Zbl 0184.02004)]. The author also examines classes of formulas, whose realizability can be proved classically. §2.2 is concerned with categorical realizability semantics. In place of giving realizability interpretations for E-HA$^\omega$ and higher order arithmetic in the same style as for HA and EL, the author takes a different route, defining categories of sets with a realizability structure, which have good properties allowing of the interpretation of constructive arithmetic. The categorical model is defined first, observing which features and which formal system find an interpretation in the model. The author makes an abstraction as to the notion of realizers, using the notions of partial combinatory algebra and typed partial combinatory algebra [https://homepages.inf.ed.ac.uk/jrl/Research/unifying.txt]. §2.3 demonstrate that all of the implications are strict in the diagram on the dependency between the continuity principles discussed in the previous chapter

$$\begin{array}{ccccc} \mathrm{WC}-\mathrm{N} & \Rightarrow & \mathrm{WC}_{\mathrm{cp}}\!-\!\mathrm{N} & \Rightarrow & \mathrm{WC}_{\mathrm{seq}}\!-\!\mathrm{N} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathrm{CP}\,(\mathbb{N}\to\mathbb{N},\mathbb{N}) & \Rightarrow & \mathrm{CP}\,(\mathbb{N}\to 2,\mathbb{N}) & \Rightarrow & \mathrm{CP}\,(\mathbb{N}^+,\mathbb{N}) \end{array}$$

Chapter 3 is concerned with computable analysis. §3.1 recalls some basic notions and facts of computable analysis such as numbering, Baire space and admissibility [*M. Schröder*, Theor. Comput. Sci. 284, No. 2, 519–538 (2002; Zbl 1042.68050)]. §3.2 is engaged in category-theoretic approach to representations, showing that standard constructions known in computable analysis can be rediscovered through their universal properties, giving an abstract proof of *M. Schröder*'s theorem [Admissible representations for continuous computations. Hagen: Fernuniv.-GHS Hagen, Fachbereich Informatik (2002; Zbl 1020.18005), Theorem 3.2.4] and presenting a logical characterization of admissible representations. §3.3 interprets constructive mathematics in categorical realizability models, systematically obtaining representations of spaces with desirable properties by interpretation of an appropriate constructive description of the underlying set of the space in a suitable realizability category.

Chapter 4 is concerned with semantic strong normalization. In order to get a semantic normalization proof for a type theory to work as proposed in [*J. M. E. Hyland* and *C. H. L. Ong*, Lect. Notes Comput. Sci. 664, 179–194 (1993; Zbl 0793.03079); *C. H. L. Ong* and *E. Ritter*, Lect. Notes Comput. Sci. 832, 261–279 (1994; Zbl 0953.03526)], it is necessary to build a model of the respective type theory based on a right-absorptive partial combinatory algebra (SN_*, Θ) consisting of strongly normalizing λ-terms with a distinguished constant $*$. There are fibrations built upon the notion of right-absorptive conditionally partial combinatory algebra that do have a generic family. *J. M. E. Hyland* and *C. H. L. Ong* [Lect. Notes

Comput. Sci. 664, 179–194 (1993; Zbl 0793.03079)] introduced the fibration of PER-extension pairs over $\boldsymbol{Set}$, which is a complete fibred CCC with a generic family, giving rise to a semantic normalization proof for system F. *C. H. L. Ong* and *E. Ritter* [Lect. Notes Comput. Sci. 832, 261–279 (1994; Zbl 0953.03526)] extended this fibration to the fibration of PER-extension pairs over $\boldsymbol{Asm}\left(\mathcal{P}^{\Theta}\left(\mathcal{A}\right)\right)$, which does have a generic family. It is shown in this chapter that the fibration of PER-extension pairs over $\boldsymbol{Asm}\left(\mathcal{P}^{\Theta}(\mathcal{A})\right)$ does not have Lawvere comprehension unless $\mathcal{A} = \Theta$.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

03-02	Research exposition (monographs, survey articles) pertaining to mathematical logic and foundations	Cited in **10** Documents
03F65	Other constructive mathematics	
03F50	Metamathematics of constructive systems	
03F60	Constructive and recursive analysis	

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