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Ambidexterity and height. (English) Zbl 07358501
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Stable homotopy theory had developed considerably for the two decades since the late 1960s. Periodic phenomena based on the connections to formal group theory through complex cobordism theory MU and the Adams-Novikov spectral sequence were discovered on the computational side, while the localization techniques of Bousfield provided a means to classify generalized cohomology theories according to their relative strength on the structural side. These two strands of ideas were combined into a global version of stable homotopy theory as stratified by different lebels of periodicity in [*D. C. Ravenel*, Am. J. Math. 106, 351–414 (1984; Zbl 0586.55003)]. *E. S. Devinatz* et al. [Ann. Math. (2) 128, No. 2, 207–241 (1988; Zbl 0673.55008)] together with *M. J. Hopkins* [Lond. Math. Soc. Lect. Note Ser. 117, 73–96 (1987; Zbl 0657.55008)] established the nilpotence theorem as a settlement and an improvement of the most central of Ravenel's seven conjectures. *Chromatic homotopy thoery* is a subfield of stable homotopy theory that has sprung from the deep and astonishing connection between the ∞-category of spectra and the stack of formal groups, the height fibration on the latter being mirrored by the *chromatic height fibration* on the former.

The localizations $\mathrm{Sp}_{T(n)}$ and $\mathrm{Sp}_{K(n)}$ are known to possess several remarkable properties, among which are the vanishing of the Tate construction for finite group actions [*D. Clausen* and *A. Mathew*, Proc. Am. Math. Soc. 145, No. 12, 5413–5417 (2017; Zbl 1378.55005); *J. P. C. Greenlees* and *H. Sadofsky*, Math. Z. 222, No. 3 (1996; Zbl 0849.55005); *M. Hovey* and *H. Sadofsky*, Proc. Am. Math. Soc. 124, No. 11, 3579–3585 (1996; Zbl 0866.55011); *N. J. Kuhn*, Invent. Math. 157, No. 2, 345–370 (2004; Zbl 1069.55007)]. Hopkins and Lurie [http://people.math.harvard.edu/~ lurie/papers/Ambidexterity.pdf] reinterpreted this Tate vanishing property as 1-*semiadditivity*, showing that the ∞-categories $\mathrm{Sp}_{K(n)}$ are ∞-*semiadditive*, which was in turn exploited to get new structural results. The authors ["Ambidexterity in chromatic homotopy theory", Preprint, arXiv:1811.02057] have classified all the higher semiadditive localizations of Sp with respect to homotopy rings, showing that, among localizations of spectra with respect to homotopy rings, the higher semiadditive property singles out precisely the monochromatic localizations, which are parameterized by the chromatic height.

This paper introduces a natural notion of *semiadditive height* for higher semiadditive ∞-categories, reproducing the usual chromatic height n, in the exmaples $\mathrm{Sp}_{T(n)}$ and $\mathrm{Sp}_{K(n)}$, without appealing to the theory of formal groups. It is shown that the semiadditive height is a fundamental invariant of a higher semiadditive ∞-category, controlling many facets of its higher semiadditive structure and the behavior of of local systems on the π-finite spaces valud in it. It is also shown that the semiadditive height exhibits a compelling form of the "redshift principle", categorification increasing the height exactly by one. Restricting to *stable* ∞-categories, the authors demonstrate that higher semiadditive ∞-categories decompose completely according to the semiadditive height, which accounts for the monochromatic nature of the higher semiadditiv localizations of Sp. Building on the work of *Y. Harpaz* [Proc. Lond. Math. Soc. (3) 121, No. 5, 1121–1170 (2020; Zbl 07242523)], the authors introduce and investigate universal constructions of stable ∞-semiadditive ∞-categories of height n, comparing them with the chromatic examples.

This paper should be viewed as part of a more extensive program aiming to place chromatic phenomena within the categorical context of interaction between higher semiadditivity and stability.

A synopsis of the paper, consisting of five sections, goes as follows.

- §2 collects general facts regarding the notion of *ambidexterity* and its implications.
- §3 addresses the main notion of the paper, that of height in a higher semiadditive ∞-category, defined in terms of the cardinalities of Eilenberg-MacLane spaces. It is shown (Theorem 3.2.7) that the higher semiadditive structure trivializes above the height. It is also shown (Theorem 3.3.2) that the higher semiadditive structure exhibits a redshift principle of increasing by one under categorification.
- §4 studies semiadditivity and height for *stable* ∞-categories. It is shown (Theorem 4.2.7) that a stable higher semiadditive ∞-category splits as a product according to height. Interested in local

systems valued in a stable higher semiadditive ∞-category of height n, it is shown (Theorem 4.3.2) how the notion of height is related to the phenomenon of semismplicity of local systems. It is finally shown, by using nil-conservative functors, that semiadditive and chromatic heights coincide for monochromatic localizations of spectra.

- §5 is concerned with the theory of *modes*. The theory of modes allows not only of analyzing the implication of certain preperties of presentable ∞-categories, but also of enforcing them in a universal way so that for every mode $\mathcal{M}$ and a presentable ∞-category $\mathcal{C}$, one can view $\mathcal{M}\otimes\mathcal{C}$ as the universal approximation of $\mathcal{C}$ by a presentable ∞-category obeying the properties classified by $\mathcal{M}$. It is shown how algebraic operations on modes, such as tensor product and localization, translate into operations on the properties of presentable ∞-categories classified by them. It is also shown (e.g. Theorem 5.3.6) that semiadditivity and height, together with the more classical notion of chromatic height, are all encoded in modes. Using this theory, the authors study the interaction between the chromatic and the semiadditive height through the interaction between the correponding modes, to get Theorem 5.4.10 concerning with 1-semiadditivity decomposition, from which Theorem 5.5.17 as a partial result in the direction of [*G. Angelini-Knoll* and *J. D. Quigley*, "Chromatic complexity of the algebraic K-theory of $y(n)$, Preprint, arXiv:1908.09164, Conjecture 1.1.5] is deduced.

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MSC:

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