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From categories to homotopy theory. (English) Zbl 1465.18001
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The book is divided into two parts. Part I, consisting of 9 chapters (from Chapter 1 through Chapter 9), is concerned with category theory.

- Chapter 1 has to do with basic notions of category theory such as epimorphisms and monomorphisms. Chapter 2 is concerned with the Yoneda lemma, adjoint pairs, equivalence of categories, and so on, while Chapter 3 deals with limits and colimits.
- Chapter 4 is engaged in Kan extensions. The first two sections deal with left and right Kan extensions. §4.3 is concerned with functors preserving Kan extensions, while §4.4 deals with ends and coends, which are expressed as limits and as colimits in §4.5. §4.6 establishes the Fubini theorem for ends and coends. The last section is devoted to the famous slogan "all concepts are Kan extensions".
- Chapter 5 is concerned with comma categories and the Grothendieck construction. §5.1 gives the definition of comma categories and their basic properties. §5.2 establishes that a cofinal functor preserves colimits, and its dual. §5.3 deals with sifted colimits. §5.4 has to do with density results, establishing the so-called co-Yoneda lemma. The last section is devoted to Grothendieck constructions.
- Chapter 6 is concerned with monads and comonads, while Chapter 7 is an introduction to abelian categories.
- Chapter 8 is concerned with symmetric monoidal categories. §8.1 deals with monoidal categories, while §8.2 has to do with symmetric monoidal categories. §8.3 addresses monoidal functors, while §8.4 is concerned with closed symmetric monoidal categories. The category of compactly generated spaces and that of k-spaces are of closed symmetric monoidal structures, which §8.5 is engaged in. The last section is a terse introduction to braided monoidal categories.
- Chapter 9 is concerned with enriched categories. §9.1 deals with basic notions. §9.2 deals with the underlying category of an enriched category, while §9.3 addresses enriched Yoneda and co-Yoneda lemmas. §9.4 is concerned with cotensored and tensored categories. §9.5 has to do with strict 2-categories, while §9.6 is concerned with bicategories [*J. Bénabou*, Lect. Notes Math. 47, 1–77 (1967; Zbl 1375.18001)]. After a short exposition of enrichedness of functor categories in §9.7, §9.8 addresses Day convolution products [http://web.science.mq.edu.au/~street/DayPhD.pdf], which is a simple procedure to transfer symmetric monoidal structures to functor categories.

Part II, consisitng of 7 chapters (from Chapter 10 through Chapter 16), is concerned with applications to homotopy theory, though it is not intended at all for a comprehensive treatment of model categories.

- The author put an emphasis on simplicial methods, on functor categories, on some concepts that are crucial to algebraic K-theory, on models for iterated based loop spaces, and on applications to homological algebra. Therefore Chapter 10 is concerned with simplicial objects. §10.3 deals with Joyal's [`https://ncatlab.org/nlab/files/JoyalThetaCategories.pdf`] category of intervals. §10.4 addresses bar and cobar constructions. §10.5 is concerned with simplicial homotopies. §10.6 deals with geometric realization of a simplicial set [*J. W. Milnor*, Ann. Math. (2) 65, 357–362 (1957; Zbl 0078.36602)], while §10.8 has to do with geometric realization of bisimplicial sets. §10.7 is concerned with skeleta of simplicial sets. §10.9 has to do with the fat realization of a (semi)simplicial set or space [*G. Segal*, Topology 13, 293–312 (1974; Zbl 0284.55016), Appendix A]. §10.10 addresses the totalization of a cosimplicial space [*A. K. Bousfield* and *D. M. Kan*, Homotopy limits, completions and localizations. Berlin-Heidelberg-New York: Springer-Verlag (1972; Zbl 0259.55004)]. §10.11 is concerned with the so-called Dold-Kan correspondence [*A. Dold*, Ann. Math. (2) 68, 54–80 (1958; Zbl 0082.37701)]. §10.12 discusses Kan condition, and §10.13 addresses quasi-categories [*J. M. Boardman* and *R. M. Vogt*, Homotopy invariant algebraic structures on topological spaces. Berlin-Heidelberg-New York: Springer-Verlag (1973; Zbl 0285.55012), 4.8] and joins of simplicial sets. §10.14 is concerned with Segal sets. §10.15 presents a symmetric monoidal model for the stable homotopy category that is built out as simplicial sets.

- Chapter 11 is concerned with the nerve and the classifying space of a small category. §11.1 defines the nerve $N\mathcal{C}$ of a small category $\mathcal{C}$, which is shown to be a strongly symmetric monoidal and to respect the join construction. §11.2 introduces the classifying space $B\mathcal{C}$ of a small category $\mathcal{C}$ as the geometric realization of $N\mathcal{C}$, whose properties [*G. Segal*, Topology 13, 293–312 (1974; Zbl 0284.55016)] are established in Theorem 11.2.4. §11.3 discusses π_0 and π_1 of small categories. §11.4 is concerned with the Bousfield-Kan homotopy colimits [*A. K. Bousfield* and *D. M. Kan*, Homotopy limits, completions and localizations. Berlin-Heidelberg-New York: Springer-Verlag (1972; Zbl 0259.55004)]. §11.5 addresses an explicit description of coverings of classifying spaces [*D. Quillen*, Lect. Notes Math. None, 85–147 (1973; Zbl 0292.18004)]. §11.6 discusses fibers and homotopy fibers. §11.7 establishes Quillen's Theorem A by using a variant of the twisted arrow category, presenting Theorem B with referring to [*D. Quillen*, Lect. Notes Math. None, 85–147 (1973; Zbl 0292.18004); *V. Srinivas*, Algebraic K-theory. Boston, MA etc.: Birkhäuser (1991; Zbl 0722.19001); *P. G. Goerss* and *J. F. Jardine*, Simplicial homotopy theory. Basel: Birkhäuser (1999; Zbl 0949.55001)] for a proof. §11.8 describes how to package the data of a monoidal category into the requirement that a certain functor is a Grothendieck opfibration, discussing a brief overview how to generalize (symmetric) monoidal structures to the context of ∞-categories.
- Operads were introduced in the middle of the previous century in topology when Michael Boardman, Rainer Vogt and Peter May struggled to get a systematic understanding of iterated loop spaces [*J. M. Boardman* and *R. M. Vogt*, Bull. Am. Math. Soc. 74, 1117–1122 (1968; Zbl 0165.26204); *J. M. Boardman* and *R. M. Vogt*, Homotopy invariant algebraic structures on topological spaces. Berlin-Heidelberg-New York: Springer-Verlag (1973; Zbl 0285.55012); *J. P. May*, The geometry of iterated loop spaces. Berlin-Heidelberg-New York: Springer-Verlag (1972; Zbl 0244.55009)]. Chapter 12 is a brief introduction to operads.
- Chapter 13 is concerned with classifying spaces of symmetric monoidal categories, relating monoidal structures on categories to multiplicative properties of the corresponding classifying spaces. §13.1 claims that symmetric monoidal structures on a small category directly translates into H-space structures on the corresponding classifying space, which are upgraded to an action of the Barratt-Eccles operad [*M. G. Barratt* and *P. J. Eccles*, Topology 13, 25–45 (1974; Zbl 0292.55010)]. §13.2 is concerned with group comletion of discrete monoids. §13.3 has to do with Grayson-Quillen construction [*D. Grayson*, Lect. Notes Math. 551, 217–240 (1976; Zbl 0362.18015)]. §13.4 is concerned with group completion of H-spaces [*J. F. Davis* and *P. Kirk*, Lecture notes in algebraic topology. Providence, RI: AMS, American Mathematical Society (2001; Zbl 1018.55001), Theorem 6.71; *J. P. May*, in: New developments in topology. The edited and revised proceedings of the symposium on algebraic topology, Oxford, June 1972. Cambridge: University Press (1974; Zbl 0281.55003), Lemma 2.1; *F. R. Cohen* et al., The homology of iterated loop spaces. Berlin-Heidelberg-New York: Springer-Verlag (1976; Zbl 0334.55009), III.3.3; [*D. Grayson*, Lect. Notes Math. 551, 217–240 (1976; Zbl 0362.18015)].
- Since the 1970s, diagram categories have been used to model loop spaces Ω^n, mostly for $n = 1, 2, \infty$ [*G. Segal*, Topology 13, 293–312 (1974; Zbl 0284.55016)]. During the last decades, new approaches for modelling iterated loop spaces Ω^n for all n, have been developed [*C. Schlichtkrull*, Doc. Math. 14, 699–748 (2009; Zbl 1241.55005); *C. Schlichtkrull* and *M. Solberg*, Trans. Am. Math. Soc. 368, No. 10, 7305–7338 (2016; Zbl 1345.18006); *C. Balteanu* et al., Adv. Math. 176, No. 2, 277–349 (2003; Zbl 1030.18006); *J. E. Bergner*, Contemp. Math. 431, 59–83 (2007; Zbl 1134.18006); *M. A. Batanin*, Adv. Math. 217, No. 1, 334–385 (2008; Zbl 1138.18003)]. Chapter 14 is intended as an overview, mostly referring to the literature for proofs and technical details.
- Chapter 15 is concerned with functor homology. §15.1 deals with tensor products as coends, and §15.2 addresses axiomatic description of Tot and Ext ([*H. Cartan* and *S. Eilenberg*, Homological algebra. Princeton, New Jersey: Princeton University Press (1956; Zbl 0075.24305), III.5], [*V. Franjou* and *T. Pirashvili*, Panor. Synth. 16, 107–126 (2003; Zbl 1063.19002), Proposition 2.1]). There are several functor homology interpretations of homology theories and their applications in the literature [*A. Djament*, "Hodge decomposition for stable homology of automorphism groups of free groups", Preprint, `arXiv:1510.03546`; *V. Franjou* and *T. Pirashvili*, Topology 37, No. 1, 109–114 (1998; Zbl 0889.16002); *V. Franjou* et al., Rational representations, the Steenrod algebra and functor homology. Paris: Société Mathématique de France (SMF) (2003; Zbl 1061.18011); *M. Livernet*

and *B. Richter*, Math. Z. 269, No. 1–2, 193–219 (2011; Zbl 1235.13009); *T. Pirashvili* and *B. Richter*, K-Theory 25, No. 1, 39–49 (2002; Zbl 1013.16004); *T. Pirashvili* and *B. Richter*, Topology 39, No. 3, 525–530 (2000; Zbl 0957.18005)]. Hochschild homology is of a description as functor homology [*J.-L. Loday*, in: Algebraic K-theory and its applications. Proceedings of the workshop and symposium, ICTP, Trieste, Italy, September 1–19, 1997. Singapore: World Scientific. 234–254 (1999; Zbl 0964.19003)], which §15.3 addresses after [*T. Pirashvili* and *B. Richter*, K-Theory 25, No. 1, 39–49 (2002; Zbl 1013.16004)], while §15.4 deals with cyclic homology after [*T. Pirashvili* and *B. Richter*, K-Theory 25, No. 1, 39–49 (2002; Zbl 1013.16004)]. Gamma homology of a commutative k-algebra A with coefficients in an A-module was defined in [*A. Robinson* and *S. Whitehouse*, Math. Proc. Camb. Philos. Soc. 132, No. 2, 197–234 (2002; Zbl 0997.18004)] in terms of an explicit chain complex. §15.5 has to do with gamma homology as functor homology after [*T. Pirashvili* and *B. Richter*, Topology 39, No. 3, 525–530 (2000; Zbl 0957.18005)]. §15.6 is concerned with adjoint base-change, giving a result of [*V. Franjou* et al., Rational representations, the Steenrod algebra and functor homology. Paris: Société Mathématique de France (SMF) (2003; Zbl 1061.18011), Lemma 2.7].

- Chapter 16 is concerned with homology and cohomology of small categories. There are many variants of (co)homology of a small category with coefficients in a suitable system of coefficients. §16.1 and §16.2 discuss a description of the one that is the most general one, exposing the Thomason (co)homology of categories and its properties after [*I. Gálvez-Carrillo* et al., J. Pure Appl. Algebra 217, No. 11, 2163–2179 (2013; Zbl 1285.18020)]. §16.3 is concerned with a spectral sequence for homotopy colimits in chain complexes. §16.4 addresses Baues-Wirsching (co)homology [*H.-J. Baues* and *G. Wirsching*, J. Pure Appl. Algebra 38, 187–211 (1985; Zbl 0587.18006)]. *M. Dzhibladze* and *T. Pirashvili* [J. Algebra 137, No. 2, 253–296 (1991; Zbl 0724.18005)] showed that under a mild projectivity asssumption, functor homology is to be expressed as Baues-Wirsching homology of categories with coefficients in a suitable tensor functor, which §16.5 addresses.

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MSC:

18-02 Research exposition (monographs, survey articles) pertaining to category theory

55-02 Research exposition (monographs, survey articles) pertaining to algebraic topology

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