Espíndola, Christian
Infinitary generalizations of Deligne's completeness theorem. (English) Zbl 1464.18005
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This paper is a continuation of the author's [Ann. Pure Appl. Logic 170, No. 2, 137–162 (2019; Zbl 1445.03077)], focusing on infinitary generalizations of *Deligne's completeness theorem* (in an appendix of [*M. Artin* (ed.) et al., Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki , P. Deligne, B. Saint-Donat. Tome 2. Exposes V à VIII. Springer, Cham (1972; Zbl 0237.00012)]), which claims that a coherent topos has enough points, being essentially Gödel completeness theorem for first-order classical logic. *M. Makkai* and *G. E. Reyes* [First order categorical logic. Model-theoretical methods in the theory of topoi and related categories. Berlin-Heidelberg-New York: Springer-Verlag (1977; Zbl 0357.18002)] established that the same is true for the so called *separable* toposes, those toposes of sheaves on a site that is of countably many objects and morphisms and whose topology is generated by countably many covering families. This result is intimately related to the completeness theorem of countably axiomatized theories in $\mathcal{L}_{\omega_1,\omega}$. This paper shows that this result can be generalized in a way that ω is replaced by any regular cardinal κ with $\kappa^{<\kappa}=\kappa$ and that the result is essentially a completeness theorem for what is called *κ-geometirc logic*, which is an extension of geometric logic where arities of function and relation symbols are cardinals less than κ, and one can take conjunctions of less than κ many formulas and existential quantification of less than κ many variables. The theory of well-orderings can not be expressed in finite-quantifier languages, but it is indeed κ-geometric.

Validity in this more expressive extension is grasped syntactically by a refined version of the rule of *transfinite transitivity* introduced in [*C. Espíndola*, Ann. Pure Appl. Logic 170, No. 2, 137–162 (2019; Zbl 1445.03077)], corresponding to an *exactness property* T satisfied by κ-geometric categories,by besides the usual axioms of geometric logic with an extension of the usual rules and axioms for conjunction and existential quantification. It is shown that κ-geometric theories are of what are called *κ-classifying toposes*, which is a Grothendieck topos abiding by the exactness condition T, where there is a generic model of the theory whose image along the inverse image of κ-geometric morphisms corresponds precisely to the model of the theory in any other topos satisfying the same exactness condition T.

Just as countably axiomatized geometric theories are complete with respect to $\mathcal{S}et$-valued models, it is shown that κ-geometric theories with at most κ many axioms are also complete. The restriction on the cardinality of the axiomatization implies that the corresponding κ-classifying topos is *κ-separable*, so that one can prove, by completeness, that it has enough *κ-points*. The existence of enough κ-points for a given topos necessarily implies that the topos has property T. In the particular case when κ is a weakly compact cardinal, the topos is equivalent to sheaves on a site where Grothendieck topology is generated by families of less than κ many morphisms, and its κ-separability is the precise generalization of Deligne's completeness theorem from coherent toposes to κ-coherent toposes.

It turns out that the corresponding completeness theorem for κ-coherent theories adopts, via Morleyization, the form of *C. R. Karp*'s [Languages with expressions of infinite length. Amsterdam: North-Holland Publishing Company (1964; Zbl 0127.00901)] completenss theorem for $\mathcal{L}_{\kappa,\kappa}$, the exactness property T corresponding to a combination of her distributivity and dependent choice axioms.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18B25 Topoi
18C50 Categorical semantics of formal languages
03G30 Categorical logic, topoi
03C75 Other infinitary logic

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