Roy, Michael
**The topos of ball complexes.** (English) Zbl 1464.18007
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This is a reproduction of the doctoral dissertation written in 1997 by Michael Roy under the direction of F. W. Lawvere, accompanied by a foreword by Lawvere himself.

This paper investigates the topos of ball complexes $\mathcal{E}$, which is a presheaf topos with site $\boldsymbol{B}$ the category of

- $n$-balls $B_n$ as objects,
- three morphisms for successive balls

$$B_n \; \begin{array}{c} \xrightarrow{\delta_0} \\ \xleftarrow{p} \\ \xrightarrow{\delta_1} \end{array} \; B_{n+1}$$

  where $\delta_0$ and $\delta_1$ are the inclusions of the upper and lower hemisphers while $p$ is a common retraction squashing the ball onto its solid equator, and
- the relation requiring the two compositions of

$$B_n \xrightarrow{\delta_i} B_{n+1} \; \begin{array}{c} \xrightarrow{\delta_j} \\ \xrightarrow{\delta_k} \end{array} \; B_{n+2}$$

  to be equal for any three successive balls.

Two other topoi are also considered for comparison with $\mathcal{E}$, namely, simplicial sets and the Boolean algebra classifier. All of the three allow one to truncate the site obtaining a full subcategory and, on taking Kan extensions, an essential subtopos or level in the ambient topos. Skeletal and coskeletal functors are obtained by composing these functors. The double negation sheaves are just sets, constituting level zero. In $\mathcal{E}$, level 1 is the truncation at $B_1$, the topos of reflexive graphs, which is also true of simplicial sets. In the Boolean algebra classifier, level 1 is the truncation at the second object in the site, being the topos of two-way reflexive graphs.

This paper gives the following considerations.

- It is shown that no level can generate $\mathcal{E}$, while Lawvere has shown that reflexive graphs generate simplicial sets and that the Boolean algebra classifier is codiscretely generated. Each subcategory $B_n \hookrightarrow B$ is of a left adjoint retraction, something not true of the other sites.
- For a given topos $\mathcal{X}$ and level $\mathcal{X}_n$ in $\mathcal{X}$, one can ask whether there is aa smallest level $\mathcal{X}_m$ for which the $n$-skeletal inclusion of $\mathcal{X}_n$ in $\mathcal{X}$ factors through the $m$-coskeletal inclusion of $\mathcal{X}_m$ in $\mathcal{X}$. Lawvere has given the name "Aufhebung of level $n$" to the level $m$ if it exists. In $\mathcal{E}$, the "Aufhebung function is simply the successor function, and the co"Aufhebung function, as defined in the obvious way, is also the successor function. *F. W. Lawvere* [Rev. Colomb. Mat. 20, No. 3–4, 179–186 (1986; Zbl 0648.18004)] has shown that, if $X$ is a reflexive graph, the subcategory $S(X)$ of $\mathcal{S}^{B_1^{\mathrm{op}}}/X$ consisting of $Y \to X$ discretely fibered over $X$ is a topos, $S(L)$ being the topos of irreflexive graphs provided that $X$ is replaced by the loop $L = \underset{\circlearrowleft}{\cdot}$. A similar result is true in $\mathcal{E}$. Using an appropriate $L \in \mathcal{E}$, one obtains

$$S(L) = \mathcal{S}^{B_{\mathrm{mono}}^{\mathrm{op}}}$$

  where $\mathcal{S}^{B_{\mathrm{mono}}^{\mathrm{op}}}$ is the subcategory of $\boldsymbol{B}$ consisting only of monomorphisms.
- The Dold-Kan-Moore Theorem for simplicial sets claiming that the category of abelian groups in simplicial sets is equivalent to the category of chain complexes is shared by $\mathcal{E}$, though, for an abelian object in simplicial sets, the homology groups of the associated chain complex and Moore complex are isomorphic, which does not hold for $\mathcal{E}$.

- Since $\mathcal{E}$ comes equipped with $\pi_n$ functors preserving finite products, homotopy categories $\mathcal{H}_n(\mathcal{E})$ are to be defined for each $n \in \mathbb{N}$. We have

  $$\mathcal{H}_n(\mathcal{E})(X, Y) = \pi_n\left(Y^X\right)(B_n)$$

  for $X, Y \in \mathcal{E}$. For $n > 0$, the only terminal objects in $\mathcal{H}_n(\mathcal{E})$ are already terminal in $\mathcal{E}$, while the representable $B_n$ are contractible in $\mathcal{E}$ in the sense that they are terminal objects of $\mathcal{H}_0(\mathcal{E})$. It is shown surprisingly that $S^n$ is contractible.
- The last section considers the points of $\mathcal{E}$.

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**MSC:**

18B25 Topoi
18F10 Grothendieck topologies and Grothendieck topoi
55U10 Simplicial sets and complexes in algebraic topology

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