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Monoidal characterisation of groupoids and connectors. (English) Zbl 1461.18008

Topology Appl. 273, Article ID 106966, 25 p. (2020).

Internal structures to categories have usually been studied under the assumption of suitable exactness properties, such as regular, Mal'tsev or semi-abelian, on the category they live in. One can also assume, in place of exactness properties, some appropriate monoidal structure on an ambient category, speaking of internal monoids or groups, in which graphical manipulations replace algebraic calculations [*P. Selinger*, Lect. Notes Phys. 813, 289–355 (2011; Zbl 1217.18002)]. Gtoupoifd in a regular category $\boldsymbol{C}$ are to be equivalently described as special dagger Frobenius monoids in the monoidal category Rel $(\boldsymbol{C})$ of relations over $\boldsymbol{C}$ [*C. Heunen* et al., J. Pure Appl. Algebra 217, No. 1, 114–124 (2013; Zbl 1271.18004); Proceedings of the 12th international workshop on quantum physics and logic, QPL'15, Oxford, UK, July 15–17, 2015. Waterloo: Open Publishing Association (OPA). 247–261 (2015; Zbl 1434.03015)], the correspondence turning out functorial for regular Mal'tsev categories $\boldsymbol{C}$.

This paper extends this correspondence from regular Mal'tsev categories to regular Goursat categories [*A. Carboni* et al., Appl. Categ. Struct. 1, No. 4, 385–421 (1993; Zbl 0799.18002)]. Mal'tsev categories obey 2-permutability

$$R \circ S = S \circ R$$

for any pair of equivalence relations R and S on the same object, while Goursat categories abide only by 3-permutability

$$R \circ S \circ R = S \circ R \circ S$$

They form a large class of categories $\boldsymbol{C}$ whose category of internal groupoids Grp $(\boldsymbol{C})$ is regular, which ensures that Rel (Grp $(\boldsymbol{C})$) is well-defined. It is established that Rel (Grp $(\boldsymbol{C})$) is equivalent to a category of special dagger Frobenius structures in Rel $(\boldsymbol{C})$. This result is extended to connectors [*D. Bourn* and *M. Gran*, Algebra Univers. 48, No. 3, 309–331 (2002; Zbl 1061.18006)]. It is shown that connectors in $\boldsymbol{C}$ are to be described as normal dagger Frobenius 3-structure in Rel $(\boldsymbol{C})$, where Frobenus 3-structures are defined by a ternary multiplication while Frobenius 2-structures have a binary multiplication.

Frobenius 2-structures in the category of Hilbert spaces are finite-dimensional C^*-algebras [*J. Vicary*, Commun. Math. Phys. 304, No. 3, 765–796 (2011; Zbl 1221.81146)], while Frobenus 3-structures include Hilbert triple systems and ternary rings of operators in finite-dimensional Hilbert spaces. The authors develop some of the theory of abstract Frobenus 3-structures, including a normal form theorem. The relationship between Frobenius 2-structures and Frobenus 3-structures in arbitrary monoidal categories is investigated, generalizing that between groupoids and connectors. It is left open whether there exists a useful notion of Frobenius n-structures for general n.

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MSC:

18E99 Categorical algebra
18B10 Categories of spans/cospans, relations, or partial maps
18B40 Groupoids, semigroupoids, semigroups, groups (viewed as categories)

Keywords:

Frobenius structure; groupoid; connector; regular category; category of relations; monoidal category; Goursat category

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