A study on maximal antipodal sets of compact symmetric spaces

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1 Introduction

Let M be a compact Riemannian symmetric space and denote the geodesic symmetry at $x \in M$ by s_x. If $s_x(y) = y$ for two points $x, y \in M$, we say that x, y are antipodal. A subset S of M is an antipodal set, if any two points of S are antipodal. The 2-number $\#_2 M$ of M is the maximum of cardinalities of antipodal sets of M. We call an antipodal set S of M great if $\#S = \#_2 M$. An antipodal set S is called maximal if there are no anitipodal sets including S properly. These notions were introduced by Chen-Nagano [2]. In general, any antipodal set of any Riemannian symmetric space of noncompact type is a one-point set, so we consider only compact symmetric spaces in this paper.

By the definition of antipodal sets, great antipodal sets are maximal. However, maximal antipodal sets are not necessarily great. It is known that any antipodal set is a finite set and $\#_2 M$ is finite for any compact symmetric space M. Moreover, it is known that the 2-number is an inavriant for compact symmetric spaces. For any two compact symmetric spaces M, N, if $\#_2 M \neq \#_2 N$, then M and N are not isomorphic. Also, it is known that the 2-number is related to the topology. Let M be a symmetric R space. Then, the followings are true [15]:

$$\#_2 M = \sum_i \dim_{\mathbb{Z}_2} H_i(M; \mathbb{Z}_2).$$

This equation is true in $M = SU(n), G_2, G_2/SO(4)...$, where these are not symmetric R spaces. By these properties, it is considered that to study antipodal sets is important to study compact symmetric spaces.

For example, Griess studied congruent classes of maximal antipodal sets in many simple compact Lie groups [5]. Wood classified congruent classes of maximal antipodal sets in low rank spinor groups by using the coding theory [27]. Tanaka-Tasaki classified congruent classes of maximal antipodal sets in many symmetric spaces explicitly [16],[17],[19]. Moreover, Tasaki classified congruent classes of maximal antipodal sets in oriented real Grassmannians with rank 3 and 4 explicitly [21]. Tanaka-Tasaki-Yasukura classified congruent classes of maximal antipodal sets in G_2 and $G_2/SO(4)$ explicitly [20].

Moreover, applications of antipodal sets are studied. Tanaka-Tasaki proved that if any two real forms in a Hermitian symmetric spaces of compact type intersect transversily, then the intersection is an antipodal set [18]. Iriye-Tasaki-Sakai applied this property to the study of Floer homology [7]. Kurihara-Okuda applied antipodal sets to the study of the design of complex Grassmannians [10].

However, we don't know properties of antipodal sets completely. In fact, the classification of congruent classes of maximal antipodal sets are not complete in some exceptional symmetric spaces of compact type, high rank spinor groups, high rank oriented real Grassmannians... .

In this paper, we study the following two topics with respect to antipodal sets.

- The homogeneity of maximal antipodal sets.
- Some relations of the 2-number and the topology in $G_2/SO(4)$.

Firstly, we discuss the homogeneity of maximal antipodal set in section 2. Section 2 is based on [12]. In this section, we assume that any compact symmetric space M is connected. We say that an antipodal set $A \subset M$ is homogeneous if there is a subgroup of the isometry group of M acting on A transitively. For example, it is known that any compact Lie group G is a Riemannian symmetric space with respect to a biinvariant metric and any maximal antipodal set including the unit element of G becomes a subgroup of

G. Therefore, we see that any maximal antipodal set of G is homogeneous. Moreover, Tanaka and Tasaki proved that any great antipodal set of any symmetric R-space is homogeneous [16]. Thus, we consider the following problem:

Problem 1.0.1. Is any maximal antipodal set of any compact Riemannian symmetric space homogeneous ?

We consider this problem in this section by introducing a concept of connectedness of antipodal sets. Moreover, we construct a method to make a bigger antipodal set from a given antipodal set using this connectedness.

Section 2 is organized as follows. In subsection 2.1, we consider shortest closed geodesics on a compact Riemannian symmetric space and prove that any two shortest closed geodesics through two antipodal points p, q are congruent under the action of some subgroup of the isometry group. In subsection 2.2, we construct a totally geodesic sphere from two antipodal points through which there is a shortest closed geodesic. Subsection 2.3 is the main content in this section. We introduce a concept of connectedness of antipodal sets. Using this connectedness, we construct a subgroup G_W of the isometry group from a given antipodal set A satisfying some condition and prove that $G_W(A) = \cup_{g \in G_W} g(A)$ is an antipodal set. This is the method to make a bigger antipodal set. We study this expanded antipodal sets in this subsection and we give a sufficient condition that maximal antipodal sets become homogeneous. In subsection 2.4, we observe examples of connected maximal antipodal sets in some compact symmetric spaces.

In section 3 we discuss Morse functions of $G_2/SO(4)$. Section 3 is based on [13]. In a symmetric R space M, we already know $\#_2 M = \dim H_*(M; \mathbb{Z}_2)$. As the background of this equation, there are Morse functions of symmetric R spaces. A symmetric R space M is realized as an orbit of the linear isotropy action of some Riemmanian symmetric pair. In particular, M is embedded into some vector space. This embedding is called the standard embedding of M. Then, it is known that there are height functions of M with respect to the standard embedding which are $\mathbb{Z}_2$-perfect Morse functions and whose set of all critical points is a great antipodal set of M.

In non symmetric R spaces, it is known that there are some symmetric spaces which have $\mathbb{Z}_2$-perfect Morse functions whose set of all critical points is a great antipodal set. The special unitary group $SU(n)$ is one of the examples [9]. We denote the set of all real $n \times n$ matrices by $M(n, \mathbb{R})$. In the embedding $SU(n) \subset M(n; \mathbb{R})$, it is known that there are some height functions satisfying such properties. In particular, it is true that $\#_2 SU(n) = \dim_{\mathbb{Z}_2} H_*(SU(n); \mathbb{Z}_2)$. Moreover, the exceptional compact Lie group G_2 is one of the examples [9]. It is known that G_2 is embedded into the special orthogonal group $SO(7) \subset M(7, \mathbb{R})$. Then, some height finctions with respect to this embedding satisfy such properties. Hence, it is true that $\#_2 G_2 = \dim_{\mathbb{Z}_2} H_*(G_2; \mathbb{Z}_2)$.

In $G_2/SO(4)$, the following proposition about $\#_2 G_2/SO(4)$ is proved by Chen-Nagano [2].

Proposition 1.0.2. ([2], Example 3.13) $\#_2 G_2/SO(4) = 7$

We denote the polynomial ring generated by $x_1, \cdots, x_n$ on $\mathbb{Z}_2$ by $\mathbb{Z}_2[x_1, \cdots, x_n]$. The cohomology group of $G_2/SO(4)$ is caluculated by Borel [1].

Proposition 1.0.3. ([1], Section 17.3) $H^*(G_2/SO(4); \mathbb{Z}_2) = \mathbb{Z}_2[y_2, y_3]/(y_2^3 + y_3^2, y_2^2 y_3)$, where the degree of

y_2 is 2 and of y_3 is 3. In particular,

$$\dim_{\mathbb{Z}_2} H_i(G_2/SO(4);\mathbb{Z}_2) = \begin{cases} 1 & (i = 0, 2, 3, 4, 5, 6, 8) \\ 0 & \text{(otherwise)} \end{cases}, \ \dim_{\mathbb{Z}_2} H_*(G_2/SO(4);\mathbb{Z}_2) = 7.$$

Therefore, $\#_2(G_2/SO(4)) = \dim_{\mathbb{Z}_2} H_*(G_2/SO(4);\mathbb{Z}_2) = 7$. Hence, it is expected that there is a $\mathbb{Z}_2$-perfect Morse function of $G_2/SO(4)$ whose set of all critical points is a great antipodal set. In this section we construct Morse functions of $G_2/SO(4)$ satisfying such properties.

Section 3 is organized as follows. In subsection 3.1, we define $G_2/SO(4)$ as a submanifold of G_2. For sake of this, we recall the definition of G_2 from [26]. Moreover, we recall some results of great antipodal sets of $G_2/SO(4)$ from [20]. In subsection 3.2, we consider some height functions of $G_2/SO(4)$ with respect to the embedding $G_2/SO(4) \subset G_2$. In particular, we prove that the set of all critical points of these functions is a great antipodal set. In subsection 3.3, we prove that functions considered in subsection 3.2 are $\mathbb{Z}_2$-perfect Morse functions by calculating the Hessian matrix of these functions at each critical point.

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2 Homogeneity of maximal antipodal sets

In this section, we introduce a concept of connectedness of antipodal sets and discuss the homogeneity of maximal antipodal set. This section is based on [12]. In this section, we assume that any compact Riemannian symmetric space is connected.

2.1 Shortest closed geodsics and meridians

We introduce some notations used in this section. Let (M, g) be a connected compact Riemannain symmetric space and G be the subgroup of the isometry group of M generated by all geodesic symmetries. We set $K_p := \{h \in G; h(p) = p\}$ $(p \in M)$. Then, (G, K_p) is a compact Riemannian symmetric pair. Let $\mathfrak{g}$ be the Lie algebra of G and σ_x be the involutive inner automorphism of G with respect to $s_x \in G$ $(x \in M)$. The involutive automorphism of $\mathfrak{g}$ indued by σ_x is denoted by the same notation σ_x. Fix $o \in M$. Let $\mathfrak{k}$ be the Lie algebra of K_o. Then, we obtain the eigenspace decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ of $\mathfrak{g}$ with respect to σ_o and $\mathfrak{k}, \mathfrak{m}$ are eigenspaces corresponding to the eigenvalues $+1, -1$ respectively. Then, $T_oM \cong \mathfrak{m}$. Let $\langle , \rangle$ be the K_o-invariant inner product on $\mathfrak{m}$ induced by g.

Let A be a maximal flat torus of M through $o \in M$ and $\mathfrak{a} = T_oA$. Then, $\mathfrak{a}$ becomes a maximal abelian subspace of $\mathfrak{m}$ under the identification of T_oM and $\mathfrak{m}$. We set the unit lattice $\Gamma = \{H \in \mathfrak{a}; \exp H \cdot o = o\} = \{H \in \mathfrak{a}; \exp H \in K_o\}$. In the following, for any geodesic $\gamma(t)$ in M we set $\gamma = \{\gamma(t) \in M; t \in \mathbb{R}\}$. Moreover, for any closed geodesic $\gamma(t)$ $(0 \leq t \leq c, \gamma(0) = \gamma(c))$ considered in the following, we assume that $\gamma(t) \neq \gamma(0)$ for any $0 < t < c$.

Proposition 2.1.1. Let A be a maximal flat torus through $o \in M$ and $\gamma(t)$ be a shortest closed geodesic in A such that $\gamma(0) = \gamma(1) = o$. Set $p = \gamma(1/2) \in A$. Then, there are no shortest closed geodesics of A containing o, p except for $\gamma(t)$ and $\gamma(-t)$.

Proof. We remark that $A = \mathfrak{a}/\Gamma$. For any closed geodesic $\delta(t)$ $(0 \leq t \leq 1)$ of A such that $\delta(0) = \delta(1) = o$ there is $H \in \Gamma$ such that $\delta(t) = \exp tH \cdot o$ and the length of δ is $||H|| = \langle H, H\rangle^{\frac{1}{2}}$. Let $c = \min_{H\in\Gamma} ||H||$ and $\Gamma_0 = \{H \in \Gamma; ||H|| = c\}$. The set of all shortest closed geodesics of A thorough $o \in M$ is $\{\exp tH \cdot o(t \in \mathbb{R}); H \in \Gamma_0\}$. Let $H_p \in \Gamma_0$ satisfy $\gamma(t) = \exp tH_p \cdot o$. We see $\exp H_p \cdot o = o$ and $\exp\frac{1}{2}H_p \cdot o = p$. It is sufficient to prove $\exp\frac{1}{2}H \cdot o \neq \exp\frac{1}{2}H_p \cdot o$ for any $H \in \Gamma_0$ and $H \neq \pm H_p$. It follows that

$$\exp\frac{1}{2}H \cdot o \neq \exp\frac{1}{2}H_p \cdot o \Leftrightarrow \frac{1}{2}(H + H_p) \notin \Gamma.$$

Hence, we show $\frac{1}{2}(H + H_p) \notin \Gamma$ for any $H \in \Gamma_0, H \neq \pm H_p$.

$||H|| = ||H_p|| = c$ from $H, H_p \in \Gamma_0$, so

$$\begin{aligned} ||\frac{1}{2}(H + H_p)||^2 =& \frac{1}{4}(c^2 + c^2 + 2||H||||H_p||\cos\theta) \\ \leq& \frac{1}{4}(4c^2) = c^2, \end{aligned}$$

where $0 \leq \theta \leq \pi$ is the angle made by H, H_p and the equality is valid if and only if $H = H_p$. Hence, for any $H \in \Gamma_0$ and $H \neq \pm H_p$ we see $||\frac{1}{2}(H + H_p)|| < c$. By the definition of c, we obtain $\frac{1}{2}(H + H_p) \notin \Gamma$.

□

We recall some fundamental results of polars and meridians introduced by Chen-Nagano[3].

Definition 2.1.2. For an isometry h of M, we set $F(h, M) = \{x \in M; h(x) = x\}$.

(1) A connected component of $F(s_o, M)$ is called a *polar* of o. The polar containing p ($p \in F(s_o, M)$) is denoted by $M_o^+(p)$. If a polar is a one-point set, then we call this polar a *pole*. We call $\{o\}$ the trivial pole.

(2) For every $p \in F(s_o, M)$ we denote the connected componet of $F(s_o s_p, M)$ containing p by $M_o^-(p)$. We call $M_o^-(p)$ a *meridian* of o through p.

Each polar and meridian is a totally geodesic submanifold of M, if their dimensions are larger than 1. In T_pM ($p \in F(s_o, M)$), $T_pM = T_pM_o^+(p) + T_pM_o^-(p)$ is the eigenspace decomposition with respect to s_o. In the following, we recall some properties of polars and meridians from [3].

Lemma 2.1.3. [3] The following three conditions are equivalent for $o, p \in M$.

(1) p is a pole of o.

(2) $K_o = K_p$.

(3) $s_o = s_p$.

Lemma 2.1.4. [3] Let p be an antipodal point of o. The followings are true.

(1) If A is a maximal flat torus containing o, p, then $A \subset M_o^-(p)$.

(2) p is a pole of o in $M_o^-(p)$.

(3) Any closed geodesic of M through o and p is included in $M_o^-(p)$.

Let $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$ be the eigenspace decomposition with respect to the involutive automorphism σ_p of $\mathfrak{g}$ and $\mathfrak{g}^+, \mathfrak{g}^-$ be the eigenspaces corresponding to the eigenvalues $+1, -1$ respectively. Because o and p are antipodal, s_o and s_p are commutative. Hence, σ_o and σ_p are commutative and $\sigma_p(\mathfrak{k}) \subset \mathfrak{k}, \sigma_p(\mathfrak{m}) \subset \mathfrak{m}$. We set $\mathfrak{k} = \mathfrak{k}^+ + \mathfrak{k}^-$ and $\mathfrak{m} = \mathfrak{m}^+ + \mathfrak{m}^-$ as eigenspace decompositions of $\mathfrak{k}, \mathfrak{m}$ with respect to σ_p. We see $\mathfrak{g}^+ = \mathfrak{k}^+ + \mathfrak{m}^+$ and $\mathfrak{g}^- = \mathfrak{k}^- + \mathfrak{m}^-$.

Lemma 2.1.5. [3] $M_o^-(p) = \exp\mathfrak{m}^- \cdot o$.

Let G^- be the identity component of $F(\sigma_o\sigma_p, G)$. The Lie algebra of G^- is $\mathfrak{k}^+ + \mathfrak{m}^-$ and the Lie algebra of $G^- \cap K_o$ is $\mathfrak{k}^+$.

Lemma 2.1.6. [3] $M_o^-(p) = G^- \cdot o \cong G^-/G^- \cap K_o$.

The pair $(G^-, G^- \cap K_o)$ becomes a compact Riemannian symmetric pair by the involutive automorphism σ_o of G^- and $M_o^-(p) = G^- \cdot o \cong G^-/G^- \cap K_o$. We define $K(o, p)$ as the identity component of $G^- \cap K_o$. The Lie algebra of $K(o, p)$ is $\mathfrak{k}^+$. The Lie algebra of K_o is $\mathfrak{k} = \mathfrak{k}^+ + \mathfrak{k}^-$ and that of K_p is $\mathfrak{k}_\mathfrak{p} = \mathfrak{k}^+ + \mathfrak{m}^+$, so that of $K_o \cap K_p$ is $\mathfrak{k}^+$. Hence, $K(o, p)$ is the identity component of $K_o \cap K_p$. We remark that for any two maximal flat tori A_1, A_2 of $M_o^-(p)$ through o there is $k \in K(o, p)$ such that $A_1 = k(A_2)$.

Proposition 2.1.7. Let $\gamma(t)$ be a shortest closed geodesic of M and $\gamma(0) = \gamma(1) = o$. Set $p = \gamma(\frac{1}{2})$. If $\delta(t)$ ($\delta \neq \gamma$) is a shortest closed geodesic such that $\delta(0) = \delta(1) = o$ and $\delta(\frac{1}{2}) = p$, then there is $k \in K(o, p)$ such that $k\delta = \gamma$.

Proof. Let A and B be maximal flat tori such that $\gamma \subset A$ and $\delta \subset B$. We see $A, B \subset M_o^-(p)$ from Lemma 2.1.4. There is $k \in K(o,p)$ such that $kB = A$. $k\delta(t)$ is a shortest closed geodesic on A and satisfies $k\delta(0) = k\delta(1) = o$ and $k\delta(\frac{1}{2}) = p$ since $K(o,p) \subset K_o \cap K_p$. Thus, we obtain $k\delta = \gamma$ from Proposition 2.1.1. □

Let $M_{o.p}$ be as follows:

$$M_{o,p} := \left\{ \delta(t)\,; t \in \mathbb{R}, \begin{array}{c} \delta(s)\ (s \in \mathbb{R}) \text{ is a shortest closed geodesic of } M \\ \text{such that } \delta(0) = \delta(1) = o, \delta(\frac{1}{2}) = p. \end{array} \right\}.$$

Then, from Propsotion 2.1.7, we obtain for any shortest closed geodesic $\gamma(t)$ such that $\gamma(0) = \gamma(1) = o$ and $\gamma(\frac{1}{2}) = p$,

$$M_{o.p} = K(o,p)\gamma = (K_o \cap K_p)\gamma.$$

In the next subsection we study $M_{o,p}$.

2.2 Totally geodesic spheres and shortest closed geodesics

Firstly, we prepare the restricted root system. In the following, we denote σ_o by σ simply. By the definition, $\mathfrak{g}$ is a compact Lie algebra, so the direct sum decomposition $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}(\mathfrak{g})$ is true, where $[\mathfrak{g}, \mathfrak{g}]$ becomes a compact semisimple subalgebra of $\mathfrak{g}$ and $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. We denote $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}(\mathfrak{g})$ by $\mathfrak{g}_s$ and $\mathfrak{g}_c$. Since $\sigma : \mathfrak{g} \to \mathfrak{g}$ is an involutive automorphism, $\sigma(\mathfrak{g}_s) \subset \mathfrak{g}_s$ and $\sigma(\mathfrak{g}_c) \subset \mathfrak{g}_c$. We obtain eigenspace decompositions

$$\mathfrak{g}_s = \mathfrak{k}_s + \mathfrak{m}_s, \quad \mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{m}_c,$$

with respect to σ, where $\mathfrak{k}_s, \mathfrak{k}_c$ are corresponding to the eigenvalue $+1$ and $\mathfrak{m}_s, \mathfrak{m}_c$ are corresponding to the eigenvalue -1. Moreover, it is true that $\mathfrak{k}, \mathfrak{m}$ have following direct sum decompositions:

$$\mathfrak{k} = \mathfrak{k}_s + \mathfrak{k}_c, \quad \mathfrak{m} = \mathfrak{m}_s + \mathfrak{m}_c.$$

We denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}^{\mathbb{C}}$. Then, we obtain a direct sum decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_s^{\mathbb{C}} + \mathfrak{g}_c^{\mathbb{C}}$. Reamrk that $\mathfrak{g}_s^{\mathbb{C}}$ is complex semisimple and $\mathfrak{g}_s$ is the compact real form of $\mathfrak{g}_s^{\mathbb{C}}$. Let $\mathfrak{n} = \mathfrak{k}_s + (\mathfrak{m}_s)_*$, where $(\mathfrak{m}_s)_* = \sqrt{-1}\mathfrak{m}_s$. Then, $\mathfrak{n}$ is a non-compact real form of $\mathfrak{g}_s^{\mathbb{C}}$. Let $(\mathfrak{a}_s)_*$ be a maximal abelian subspace of $(\mathfrak{m}_s)_*$, put $\mathfrak{a}_s = \sqrt{-1}(\mathfrak{a}_s)_*$ and extend $\mathfrak{a}_s$ to a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}_s$. Then, $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_s^{\mathbb{C}}$. Let Δ be the corresponding root system and Σ be the corresponding restricted root system. We denote the multiplicity of each restricted root $\lambda \in \Sigma$ by $n(\lambda)$. Since it is known that each restricted root takes real values on $(\mathfrak{a}_s)_*$, we select some linear order of $(\mathfrak{a}_s)_*$ and denote the set of all positive restricted roots by Σ^+.

For each linear form λ on $(\mathfrak{a}_s)^{\mathbb{C}}$ set

$$\mathfrak{k}_\lambda = \{T \in \mathfrak{k}_s; (\mathrm{ad}H)^2 T = \lambda(H)^2 T \text{ for } H \in \mathfrak{a}_s\},$$
$$\mathfrak{m}_\lambda = \{X \in \mathfrak{m}_s; (\mathrm{ad}H)^2 X = \lambda(H)^2 X \text{ for } H \in \mathfrak{a}_s\}.$$

Then it is true that $\mathfrak{k}_\lambda = \mathfrak{k}_{-\lambda}$ and $\mathfrak{m}_\lambda = \mathfrak{m}_{-\lambda}$. $\mathfrak{k}_0$ is the centrailzer of $\mathfrak{a}_s$ in $\mathfrak{k}_s$. In this setting, it is known that the following direct sum decompositions are true.

$$\mathfrak{k}_s = \mathfrak{k}_0 + \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda, \quad \mathfrak{m}_s = \mathfrak{a}_s + \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda.$$

We set $\mathfrak{a} = \mathfrak{a}_s + \mathfrak{m}_c$ and $\mathfrak{s} = \mathfrak{k}_0 + \mathfrak{k}_c$. Then, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{m}$. We extend every root $\lambda \in \Sigma$ to $\mathfrak{a}^{\mathbb{C}}$ to be 0 on $\mathfrak{m}_c^{\mathbb{C}}$ and denote the extended root and the set of all extended roots by the same symbol λ and Σ. In these setting, we obtain direct sum decompositions of $\mathfrak{k}$ and $\mathfrak{m}$ as follows:

$$\mathfrak{k} = \mathfrak{s} + \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda, \quad \mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda.$$

Let $\langle\, , \,\rangle$ be the K-invariant inner product on $\mathfrak{m}$ induced by the G-inavriant metric g on M. The restriction of $\langle\, , \,\rangle$ to $\mathfrak{m}_s$ is the restriction of a negative constant multiple of the Killing form on the semisimple algebra $\mathfrak{g}_s$. It is known that $\mathfrak{m} = \mathfrak{m}_s + \mathfrak{m}_c$ is an orthogonal direct sum decomposition of $\mathfrak{m}$ with respect to $\langle\, , \,\rangle$. We see that $\mathfrak{a} = \mathfrak{a}_s + \mathfrak{m}_c$ is an orthogonal direct sum decomposition. Then we obtain the inner product on $\sqrt{-1}\mathfrak{a}$ by $\langle\, , \,\rangle$ and denote it by the same letter, that is $\langle \sqrt{-1}H_1, \sqrt{-1}H_2 \rangle = \langle H_1, H_2 \rangle$ for $H_1, H_2 \in \mathfrak{a}$. Every restricted root takes real values on $\sqrt{-1}\mathfrak{a}$, so there is some $(A_\lambda)_* \in \sqrt{-1}\mathfrak{a}$ such that

$$\langle (A_\lambda)_*, H \rangle = \lambda(H) \text{ for } H \in \sqrt{-1}\mathfrak{a}.$$

Set $A_\lambda = \sqrt{-1}(A_\lambda)_*$. We see $A_\lambda \in \mathfrak{a}_s$ easily. Denote $\mathbb{R}A_\lambda$ by $\mathfrak{a}_\lambda$.

Lemma 2.2.1. [6, Ch.VII, Lemma 11.4, Lemma 11.5] Let $\lambda, \mu \in \Sigma^+ \cup \{0\}$ $(\lambda \neq \mu)$ and $H \in \mathfrak{a}$. Then it follows that

$$\begin{aligned}
&[\mathfrak{k}_\lambda, \mathfrak{k}_\mu] \subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}, \quad [\mathfrak{m}_\lambda, \mathfrak{m}_\mu] \subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}, \\
&[\mathfrak{k}_\lambda, \mathfrak{m}_\mu] \subset \mathfrak{m}_{\lambda+\mu} + \mathfrak{m}_{\lambda-\mu}, \quad [\mathfrak{k}_\lambda, \mathfrak{m}_\lambda] \subset \mathfrak{m}_{2\lambda} + \mathfrak{a}_\lambda, \\
&\mathrm{ad}(H)\mathfrak{k}_\lambda \subset \mathfrak{m}_\lambda, \quad \mathrm{ad}(H)\mathfrak{m}_\lambda \subset \mathfrak{k}_\lambda.
\end{aligned}$$

Set $\langle \lambda, \mu \rangle = \langle A_\lambda, A_\mu \rangle$ for $\lambda, \mu \in \Sigma$ and $\hat{A}_\lambda = \frac{2\pi}{\langle \lambda, \lambda \rangle} A_\lambda$ for any $\lambda \in \Sigma$. We recall the unit lattice Γ of $\mathfrak{a}$.

Lemma 2.2.2. [6, Ch.VII, Proposition 11.9 Proof] $\hat{A}_\lambda \in \Gamma$ for any $\lambda \in \Sigma$.

Lemma 2.2.3. [6, Ch.VII, Section 8] $\lambda(H) \in \pi\sqrt{-1}\mathbb{Z}$ for any $H \in \Gamma, \lambda \in \Sigma$.

Suppose that A is the maximal flat torus corresponding to $\mathfrak{a}$. Let $\gamma(t) = \mathrm{exp} tH_p \cdot o$ $(H_p \in \mathfrak{a})$ be a shortest closed geodesic of A such that $\gamma(0) = \gamma(2) = o$ and put $p = \gamma(1)$. In this setting we see $2H_p \in \Gamma$. Let $\Gamma_p = \{H \in \mathfrak{a}\,; \mathrm{exp}H \cdot o = p\} = \{H_p + J\,; J \in \Gamma\}$. We define a subset Σ_p of Σ as follows:

$$\Sigma_p = \{\lambda \in \Sigma\,; \lambda(X) \in \pi\sqrt{-1}\mathbb{Z} \text{ for any } X \in \Gamma_p\} = \{\lambda \in \Sigma\,; \lambda(H_p) \in \pi\sqrt{-1}\mathbb{Z}\}.$$

We introduce an order of Σ satisfying $\lambda \in \Sigma^+ \Rightarrow \lambda(-\sqrt{-1}H_p) \geq 0$. Set $\Sigma_p^+ = \Sigma_p \cap \Sigma^+$. We recall the identity component of $K_o \cap K_p$ is $K(o,p)$ and the Lie algebra of $K(o,p)$ is $\mathfrak{k}^+$.

Lemma 2.2.4. $\mathfrak{k}^+ = \mathfrak{s} + \sum_{\lambda \in \Sigma_p^+} \mathfrak{k}_\lambda$.

Proof. Let $X \in \mathfrak{k}$. Then,

$$\begin{aligned}
X \in \mathfrak{k}_p &\Leftrightarrow \mathrm{exp} tX \cdot p = p \ (t \in \mathbb{R}) \\
&\Leftrightarrow \mathrm{exp} t\mathrm{Ad}(\mathrm{exp}(-H_p))X \cdot o = o \ (t \in \mathbb{R}) \\
&\Leftrightarrow \mathrm{Ad}(\mathrm{exp}(-H_p))X \in \mathfrak{k}.
\end{aligned}$$

Suppose that $X = X_0 + \sum_{\lambda\in\Sigma^+} X_\lambda$ is the decompostion of X corresponding to the direct sum decomposition $\mathfrak{k} = \mathfrak{s} + \sum_{\lambda\in\Sigma^+} \mathfrak{k}_\lambda$. Then,

$$\begin{aligned} e^{\mathrm{ad}(-H_p)}\big(X_0 + \sum_{\lambda\in\Sigma^+} X_\lambda\big) &= \sum_{n=0}^{\infty} \frac{1}{n!}\mathrm{ad}(-H_p)^n\big(X_0 + \sum_{\lambda\in\Sigma^+} X_\lambda\big) \\ &= X_0 + \sum_{\lambda\in\Sigma^+,\lambda(H_p)\neq 0} \Big(\cos\big(-\sqrt{-1}\lambda(H_p)\big)X_\lambda + \frac{\sin\big(-\sqrt{-1}\lambda(H_p)\big)}{\sqrt{-1}\lambda(H_p)}[H_p, X_\lambda]\Big) \\ &\quad + \sum_{\lambda\in\Sigma^+,\lambda(H_p)=0} X_\lambda. \end{aligned}$$

We remark $[H_p, X_\lambda] \in \mathfrak{m}$. Hence, $\mathrm{Ad}(\exp(-H_p))X \in \mathfrak{k} \Leftrightarrow X_\lambda = 0$ for $\lambda(H_p) \notin \pi\sqrt{-1}\mathbb{Z}$. Thus, we showed that $X \in \mathfrak{k}^+$ holds if and only if $X \in \mathfrak{s} + \sum_{\lambda\in\Sigma_p^+} \mathfrak{k}_\lambda$.

□

Since $\hat{A}_\lambda \in \Gamma$ for any $\lambda \in \Sigma$ by Lemma 2.2.2, we see that $\exp t\hat{A}_\lambda \cdot o$ $(0 \le t \le 1)$ is a closed geodesic of M. Therefore, $||2H_p|| \le ||\hat{A}_\lambda||$ because of the minimality of the length of $\gamma(t)$. We consider the following three cases (A-1),(A-2) and (B).

(A-1) $||2H_p|| < ||\hat{A}_\lambda||$ for any $\lambda \in \Sigma$.

(A-2) $||2H_p|| = ||\hat{A}_\lambda||$ for some $\lambda \in \Sigma$ and $2H_p \neq \hat{A}_\mu$ for any $\mu \in \Sigma$.

(B) $2H_p = \hat{A}_\lambda$ for some $\lambda \in \Sigma$.

Lemma 2.2.5. For three cases (A-1),(A-2) and (B), followings are true:
(1) (A-1),(A-2) $\Rightarrow$ $\mu(2H_p) = 0$ or $\mu(2H_p) = \pi\sqrt{-1}$ for any $\mu \in \Sigma^+$.
(2) (B) $\Rightarrow$ $\lambda(2H_p) = 2\pi\sqrt{-1}$ and $\mu(2H_p) = 0$ or $\mu(2H_p) = \pi\sqrt{-1}$ for any $\mu \in \Sigma^+, \mu \neq \lambda$.

Proof. Let $m \in \mathbb{Z}$ and set $L_\mu(m\pi) = \{H \in \mathfrak{a}\,; \langle H, A_\mu\rangle = m\pi\} = \{H \in \mathfrak{a}\ ;\ \mu(H) = m\sqrt{-1}\mathbb{Z}\}$ for any $\mu \in \Sigma^+$ which is a hyper plane of $\mathfrak{a}$. The point of $L_\mu(m\pi)$ which has the shortest length from 0 is $\frac{m\pi}{\langle\mu,\mu\rangle}A_\mu$, so $||\frac{m\pi}{\langle\mu,\mu\rangle}A_\mu|| \le ||H||$ for any $H \in L_\mu(m\pi)$. For any $\mu \in \Sigma^+$, it follows that $\mu(2H_p) \in \pi\sqrt{-1}\mathbb{Z}$ by Lemma 2.2.3. We see $\mu(2H_p) = 0, \pi\sqrt{-1}, 2\pi\sqrt{-1}$. In fact, if $\mu(2H_p) = m\pi\sqrt{-1}(m \ge 3)$, then

$$||\hat{A}_\mu|| = ||\frac{2\pi}{\langle\mu,\mu\rangle}\hat{A}_\mu|| < ||\frac{m\pi}{\langle\mu,\mu\rangle}\hat{A}_\mu|| \le ||2H_p||$$

from the above remark. However, this contradicts to the minimality of $||2H_p||$ by Lemma 2.2.2.

- the case (A-1),(A-2)
  We assume $\lambda(2H_p) = 2\pi\sqrt{-1}$ for some $\lambda \in \Sigma^+$. Then it follows that $||\hat{A}_\lambda|| \le ||2H_p||$ from $H \in L_\lambda(2\pi)$. From the minimality of $||2H_p||$, $\hat{A}_\lambda = 2H_p$. However this contradicts to the assumption of (A-1),(A-2). Thus $\lambda(2H_p) = 0, \pi\sqrt{-1}$.
- the case (B)
  Suppose $2H_p = \hat{A}_\lambda$. It is obvious that $\lambda(2H_p) = 2\pi\sqrt{-1}$. We assume $\mu(2H_p) = 2\pi\sqrt{-1}$ for some $\mu \in \Sigma^+, \mu \neq \lambda$. Then $||A(\mu)|| \le ||2H_p||$ from $H \in L_\lambda(2\pi)$. Moreover, it follows that $\hat{A}_\lambda = 2H_p$ from the minimality of $||2H_p||$. This implies $\lambda = \mu$. However, this is a contradiction. Thus $\mu(2H_p) = 0, \pi\sqrt{-1}$.

□

We consider three subsets $\Sigma^+(0), \Sigma^+(\frac{\pi}{2}), \Sigma^+(\pi)$ of Σ^+:

$$\Sigma^+(0) = \{\lambda \in \Sigma^+ \,;\, \lambda(2H_p) = 0\} = \{\lambda \in \Sigma^+ \,;\, \lambda(H_p) = 0\},$$
$$\Sigma^+(\frac{\pi}{2}) = \{\lambda \in \Sigma^+ \,;\, \lambda(2H_p) = \pi\sqrt{-1}\} = \{\lambda \in \Sigma^+ \,;\, \lambda(H_p) = \frac{\pi}{2}\sqrt{-1}\},$$
$$\Sigma^+(\pi) = \{\lambda \in \Sigma^+ \,;\, \lambda(2H_p) = 2\pi\sqrt{-1}\} = \{\lambda \in \Sigma^+ \,;\, \lambda(H_p) = \pi\sqrt{-1}\}.$$

By the proof of Lemma 2.2.5, it is true that $\Sigma^+ = \Sigma^+(0) \sqcup \Sigma^+(\frac{\pi}{2}) \sqcup \Sigma^+(\pi)$. Moreover, we see $\Sigma^+_p = \Sigma^+(0) \sqcup \Sigma^+(\pi)$. The following lemma is obvious.

Lemma 2.2.6. (A-1),(A-2)$\Rightarrow \Sigma^+(\pi) = \phi$, (B)$\Rightarrow \Sigma^+(\pi) = \{\lambda\}$.

Set $\mathfrak{a}_p = \mathbb{R}H_p$. We define a subspace $\mathfrak{m}_p$ of $\mathfrak{m}$ as follows:

$$\mathfrak{m}_p = \mathfrak{a}_p + \sum_{\lambda \in \Sigma^+(\pi)} \mathfrak{m}_\lambda.$$

Proposition 2.2.7. $\mathfrak{m}_p$ is a Lie triple system of $\mathfrak{m}$.

Proof. In (A-1),(A-2), we see $\Sigma^+(\pi) = \phi$ from Lemma 2.2.6, so $\mathfrak{m}_p = \mathfrak{a}_p$. Hence the statement is obvious.

In (B), suppose $2H_p = \hat{A}_\lambda$. Then $\mathfrak{a}_p = \mathfrak{a}_\lambda$ and $\mathfrak{m}_p = \mathfrak{a}_\lambda + \mathfrak{m}_\lambda$. In this case, we see $2\lambda \notin \Sigma^+$. In fact, if $2\lambda \in \Sigma^+$, then $\exp t\hat{A}_{2\lambda} \cdot o (t \in \mathbb{R})$ is a closed geodesic of M and its length is $||\hat{A}_{2\lambda}||$. Then,

$$||\hat{A}_{2\lambda}|| = ||\frac{2\pi}{\langle 2\lambda, 2\lambda\rangle}\hat{A}_{2\lambda}|| = \frac{2\pi}{\langle 2\lambda, 2\lambda\rangle}||2\hat{A}_\lambda|| = \frac{1}{2}||\hat{A}_\lambda|| < ||\hat{A}_\lambda|| = ||2H_p||.$$

This contradicts to the minimality of $||2H_p||$. Hence, $2\lambda \notin \Sigma^+$. By Lemma 2.2.1,

$$\left[\mathfrak{a}_\lambda + \mathfrak{m}_\lambda, [\mathfrak{a}_\lambda + \mathfrak{m}_\lambda, \mathfrak{a}_\lambda + \mathfrak{m}_\lambda]\right] \subset [\mathfrak{a}_\lambda + \mathfrak{m}_\lambda, \mathfrak{k}_\lambda + \mathfrak{s}] \subset \mathfrak{a}_\lambda + \mathfrak{m}_\lambda.$$

Therefore, we showed that $\mathfrak{m}_p$ is a Lie triple system of $\mathfrak{m}$.

□

From Proposition 2.2.7, we see that $\exp\mathfrak{m}_p \cdot o$ is a totally geodesic submanifold of M. In the following we denote $\exp\mathfrak{m}_p \cdot o$ as M_p. In particular, M_p is a compact Riemannian symmetric space of rank one since $\mathfrak{a}_p$ is a maximal abelian subspace of $\mathfrak{m}_p$ and $\dim \mathfrak{a}_p = 1$.

Lemma 2.2.8. [6, Ch.VII, Theorem 10.3] Let N be a compact Riemannian symmetric space of rank one and $q \in N$. Let $2L$ denote the common length of the geodesics in N. Then the exponential map $\mathrm{Exp} : T_qN \longrightarrow N$ is a diffeomorphism of the open ball $B(0, L) = \{X \in T_qN \,;\, ||X|| < L\}$ in T_qN onto $N - F(s_q, N)$.

Theorem 2.2.9. $K(o, p)\gamma = M_p$. Moreover, M_p is a totally geodesic sphere of M. Moreover, (A-1),(A-2)$\Rightarrow \dim M_p = 1$ and (B)$\Rightarrow \dim M_p = \dim \mathfrak{m}_p = n(\lambda) + 1$.

Proof. Since $K(o, p)\gamma(t) = \exp t\mathrm{Ad}(K(o, p))H_p \cdot o$, we consider $\mathrm{Ad}(K(o, p))H_p$ in every cases (A-1),(A-2),(B).

- the case (A-1),(A-2)

  In this case, we see $M_p = \exp\mathfrak{m}_p \cdot o = \exp\mathfrak{a}_p \cdot o = \gamma$. For the Lie algebra $\mathfrak{k}^+$ of $K(o,p)$, it follows that $\mathfrak{k}^+ = \mathfrak{s} + \sum_{\lambda\in\Sigma^+(0)} \mathfrak{k}_\lambda$ from Lemma 2.2.6. Hence, $\mathrm{Ad}(K(o,p))H_p = H_p$ because $[\mathfrak{k}^+, H_p] = \{0\}$. Thus,

$$K(o,p)\gamma = \{\exp tkH_p \cdot o\ ;\ k \in \mathrm{Ad}(K(o,p)), 0 \le t \le 2\} = \exp\mathfrak{a}_p \cdot o = \gamma = M_p.$$

  In paticular, $K(o,p)\gamma$ is a totally geodesic sphere of M since γ is a closed geodesic.

- the case (B)

  Let $2H_p = \hat{A}_\lambda$. Then, $\mathfrak{m}_p = \mathfrak{a}_\lambda + \mathfrak{m}_\lambda$ and $\mathfrak{k}^+ = \mathfrak{s} + \mathfrak{k}_\lambda + \sum_{\mu\in\Sigma^+(0)} \mathfrak{k}_\mu$ from Lemma 2.2.6. It follows that

$$[\mathfrak{k}^+, \mathfrak{a}_\lambda] = [\mathfrak{k}_\lambda, \mathfrak{a}_\lambda] = \mathfrak{m}_\lambda,$$
$$[\mathfrak{k}^+, \mathfrak{m}_\lambda] = [\mathfrak{s} + \mathfrak{k}_\lambda + \sum_{\mu\in\Sigma^+(0)} \mathfrak{k}_\mu, \mathfrak{m}_\lambda] \subset \mathfrak{m}_\lambda + \sum_{\mu\in\Sigma^+(0)} \Big(\mathfrak{m}_{\lambda+\mu} + \mathfrak{m}_{\lambda-\mu}\Big) + \mathfrak{a}_\lambda,$$

  from Lemma 2.2.1 and the proof of Proposition 2.2.7. We see $\mathfrak{m}_{\lambda\pm\mu} = 0$. In fact if $\lambda \pm \mu$ ($\mu \in \Sigma^+(0), \mu \neq 0$) is a root, then $(\lambda \pm \mu)(H_p) = \pi\sqrt{-1}$ from $\mu \in \Sigma^+(0)$. This means $\lambda \pm \mu \in \Sigma^+(\pi)$. However, this contradicts to $\Sigma^+(\pi) = \{\lambda\}$. Thus, $\lambda \pm \mu$ is not a root and $\mathfrak{m}_{\lambda\pm\mu} = \{0\}$. Therefore, we obtain $[\mathfrak{k}^+, \mathfrak{m}_\lambda] \subset \mathfrak{a}_\lambda + \mathfrak{m}_\lambda$. Since it follows that $[\mathfrak{k}^+, \mathfrak{m}_p] \subset \mathfrak{m}_p$, we see $\mathrm{Ad}(K(o,p))H_p \subset \mathfrak{m}_p$. Then, $\mathrm{Ad}(K(o,p))H_p$ is a compact submanifold of the round sphere $S(0, ||H_p||)$ centered at 0 in $\mathfrak{m}_p$ with radius $||H_p||$. Moreover, the tangent space $T_{H_p}(\mathrm{Ad}(K(o,p))H_p)$ of $\mathrm{Ad}(K(o,p))H_p$ at H_p is $[\mathfrak{k}^+, H_p] = \mathfrak{m}_\lambda$. Thus, we obtain $\mathrm{Ad}(K(o,p))H_p = S(0, ||H_p||)$. We see $\exp\mathfrak{m}_p \cdot o = \{\exp tX \cdot o\,; t \in \mathbb{R}, X \in S(0, ||H_p||)\}$. Therefore,

$$K(o,p)\gamma = \{\exp t\mathrm{Ad}(K(o,p))H_p \cdot o; t \in \mathbb{R}\} = \{\exp tX \cdot o\,; t \in \mathbb{R}, X \in S(0, ||H_p||)\} = M_p.$$

  In this case, $F(s_o, M) = \{p\}$, so the open ball $B(0, ||H_p||)$ in $\mathfrak{m}_p$ centerd at 0 with radius $||H_p||$ is diffeomorphic to $M_p - \{p\}$. Hence, M_p is a sphere. Thus, M_p is a totally geodesic sphere of M.

□

Summarizing Section 2 and 3, we obtain the following theorem.

Theorem 2.2.10. Let $o, p \in M$ be anitpodal two points. Suppose that there is a shortest closed geodesic of M through o and p. Then, there is a totally geodesic sphere N satisfying the following properties:

(1) Any shortest closed geodesic of M through o and p is included in N.

(2) If $K(o,p)$ is the identity component of $K_o \cap K_p$ and γ is a shortest closed geodesic of M through o and p, then $N = (K_o \cap K_p)\gamma = K(o,p)\gamma$. In particular, $N = M_{o,p}$.

2.3 Expansion of antipodal sets and homogeneous antipodal sets

This subsection is the main contains of this section. In this subsection, we introduce a concept of a connectedness of antipodal sets. Using the connectedness, we will define a subgroup G_W of the isometry group of M and construct the method to make a bigger antipodal set from a given antipodal set by using G_W. Moreover, we will prove that the connectedness is a sufficient condition that a maximal antipodal set is homogeneous.

2.3.1 Preliminary

Firstly, we introduce a concept of a connectedness of antipodal sets and the subgroup G_W.

Definition 2.3.1. Let p and q $(p \neq q)$ be two antipodal points of M. If there is a shortest closed geodesic on M through p, q, then we say that p is *connected* to q or p, q are *connected*.

Let S be an antipodal set of M and $o \in S$. We set $S_o = \{x \in S\,; x \text{ is connected to } o.\}$. In the following, we suppose $S_o \neq \phi$.

Proposition 2.3.2. Let $p \in S_o$. Then, there is a shortest closed geodesic of M through o and p which is invariant under every s_q $(q \in S)$.

Proof. It is sufficient to show that there is a closed geodesic of $M_{o,p}$ through o and p which is invariant under every s_q $(q \in S)$. In this proof, $M_{o,p}$ is denoted by N simply. Since N is invariant under the action of $K_o \cap K_p$ by Theorem 2.2.10, $s_q(N) \subset N$ for any $q \in S$. We can regard every $s_q|_N$ $(q \in S)$ as an isometry of N. We consider the subgroup Z of the isometry group of N which is generated by $\{s_q|_N; q \in S\}$.

If N is a closed geodesic, the statement follows from $s_q(N) \subset N$ $(q \in S)$. Suppose $N \cong S^{n-1}$ $(n \geq 3)$. We denote the differential of $g \in Z$ at T_oN by g_*. For any g_*, h_* $(g, h \in Z)$, $g_* : T_oN \to T_oN$ is an isometric automorphism and $g_*h_* = h_*g_*$. Moreover, since $g_*^2 = \mathrm{id}_{T_oN}$, eigenspaces of g_* are 1 or -1. Thus, there is a basis $X_1, \cdots, X_{n-1}$ such that $g_*(X_i) = \pm X_i$ for any $g \in Z$ and $1 \leq i \leq n-1$. Then, each $\exp(tX_i) \cdot o$ $(t \in \mathbb{R})$ is a shortest closed geodesic which is invariant under every s_q $(q \in S)$.

□

We introduce a concept of connectedness of antipodal sets.

Definition 2.3.3. Let S be an antipodal set.

(1) If a point series $\{p_i\}_{i=1}^{l}$ of S satisfies that p_i is connected to p_{i+1}, then we say this point series is a *connected point series*.

(2) If for any $p, q \in S$ there is a connected point series $\{p_i\}_{i=1}^{l}$ of S containing p and q, then we say that S is *connected*.

(3) Let S be connected. If there are no connected anitpodal sets containing S properly, then we say that S is a *maximally connected* antipodal set.

(4) Let S be not necessarily connected and T be a connectd subset of S. If there are no connected antipodal subsets of S containing T properly, we say that T is a *connected component* of S.

Remark 2.3.4. It is true that any connected maximal antipodal set is maximally connected. However, any maximally connected antipodal set is not necessarily maximal.

We introduce some notations to use later.

Notation 2.3.5. Let S be an antipodal set and $o \in S$.

- $\bar{S_o} := S_o \cup \{o\}$.
- $L(o, p, S)$ $(p \in S_o)$: the set of all shortest closed geodesics through o, p invariant under all s_q $(q \in S)$.
- $L(o, S) := \bigcup_{p \in S_o} L(o, p, S)$.
- $L(S) := \bigcup_{p,q \in S} L(p, q, S)$, where p, q are connected

- $CL(o,p,S)$ $(p \in S_o)$: all middle points between o and p on every closed geodesic in $L(o,p,S)$.
- $CL(o,S) := \bigcup_{p \in S_o} CL(o,p,S)$.
- $CL(S) := \bigcup_{p,q \in S} CL(p,q,S)$, where p, q are connected.

For any subset W of $CL(S)$, let G_W be the group generated by $\{s_q; q \in \mathrm{W}\}$. We denote $G_{CL(s)}$ and $G_{CL(o,S)}$ by G_S and $G_{o,S}$.

2.3.2 Expansions of antipodal sets

We will construct a big antipodal set from a given antipodal set using G_W. For any antipodal set S, remark that $CL(S) \neq \phi$ is equivalent to that S contains connected two points by Proposition 2.3.2. In the following, we often use the notation

$$x = \begin{cases} a, \\ b. \end{cases}$$

This means $x = a$ or $x = b$. In this section, we recall that we assume that our compact symmetric spaces M is connected. Then, for any isometry g and $x \in M$, it is true that $g s_x g^{-1} = s_{g(x)}$.

Lemma 2.3.6. Let S be an antipodal set. Suppose that S has connected two points. Then, the followings are true.

(1) $s_q(x) = x$ or $s_q(x) = s_p(x)$ for any $q \in S$ and $x \in CL(p,S)$ $(p \in S)$. Hence,

$$s_q s_x = \begin{cases} s_x s_q, \\ s_p s_x s_p s_q. \end{cases}$$

(2) Let $m \in M$ be antipodal to all points of S. Then, for any $x \in CL(p,S)$ $(p \in S)$

$$s_p s_x s_p(m) = s_x(m).$$

Proof. Let $x \in CL(p,r,S)$ $(r \in S_p)$ and $\gamma(t) \in L(p,r,S)$ such that $\gamma(0) = \gamma(2) = p, \gamma(1) = r$ and $\gamma(\frac{1}{2}) = x$. Firstly, we will show (1). We see $s_q(\gamma(t)) = \gamma(t)$ or $s_q(\gamma(t)) = \gamma(-t)$ since $s_q(\gamma) \subset \gamma$, $s_q^2 = \mathrm{id}_M$ and s_q fixes $p = \gamma(0)$ and $r = \gamma(1)$. In the former case, $s_q(x) = s_q(\gamma(\frac{1}{2})) = \gamma(\frac{1}{2}) = x$, so we obtain $s_q s_x = s_{s_q(x)} s_q = s_x s_q$. In the latter case, $s_q(x) = s_q(\gamma(\frac{1}{2})) = \gamma(-\frac{1}{2}) = s_p(x)$, so we obtain $s_q s_x = s_{s_q(x)} s_q = s_{s_p(x)} s_q = s_p s_x s_p s_q$. We consider (2). We see $r = s_x(p) \in S$ is antipodal to m by the definitions of x and m. Therefore, $s_x s_p s_x s_p(m) = s_{s_x(p)} s_p = s_r s_p(m) = m$. Thus, we obtain $s_p s_x s_p(m) = m$.

□

Using G_w we will construct a method to make a big antipodal set containing S.

Proposition 2.3.7. Let S be an antipodal set containing connected two points . Let W be any subset of $CL(S)$ and $g \in G_W$. Then, $S \cup gS$ is an antipodal set.

Proof. Since each of S and gS is an antipodal set, it is sufficient to show that any $r \in S$ is antipodal to any $g(q) \in gS$ $(q \in S)$. From the definition of G_W, we may write $g \in G_W$ as $g = s_{x_l} \cdots s_{x_2} s_{x_1}$ $(x_1, x_2, \cdots, x_l \in W)$. Let $x_i \in CL(p_i, S)$ $(p_i \in S, 1 \leq i \leq l)$. We will prove the statement by induction for l.

By Lemma 2.3.6, for $x_1 \in CL(p_i, S)$

$$s_r(s_{x_1}(q)) = \begin{cases} s_{x_1}s_r(q) = s_{x_1}(q), \\ s_{p_1}s_{x_1}s_{p_1}s_r(q) = s_{p_1}s_{x_1}s_{p_1}(q) = s_{x_1}(q). \end{cases}$$

Hence, r is antipodal to $s_{x_1}(q)$. We assume that the statement is true until $l-1$. Then, by using Lemma 2.3.6 again we obtain

$$\begin{aligned} s_r\Big(s_{x_l}\cdots s_{x_1}(q)\Big) &= \big(\epsilon_l s_{x_l}\epsilon_l\big)s_r\Big(s_{x_{l-1}}\cdots s_{x_1}(q)\Big) \\ &= \big(\epsilon_l s_{x_l}\epsilon_l\big)\Big(s_{x_{l-1}}\cdots s_{x_1}(q)\Big) \\ &= s_{x_l}s_{x_{l-1}}\cdots s_{x_1}(q), \end{aligned}$$

where ϵ_l is s_{p_l} or id_M. Therefore, r is antipodal to $s_{x_l}\cdots s_{x_1}(q)$.

□

We obtain the following theorem from this proposition immediately.

Theorem 2.3.8. Let S be an antipodal set containing connected two points. Let W be any subset of $CL(S)$. Then, $G_W(S) = \bigcup_{g\in G_W} gS$ is an antipodal set.

Proof. It is sufficient to prove that $g_1(S) \cup g_2(S)$ is an antipodal set for any $g_1, g_2 \in G_W$. However, since we see that $S \cup g_1^{-1}g_2(S)$ is an antipodal set, $g_1(S) \cup g_2(S)$ is an antipodal set. Thus, $G_W(S)$ is an antipodal set.

□

Definition 2.3.9. Let S be an antipodal set containing connected two points. Let W be any subset of $CL(S)$. Then, we call the antipodal set $G_W(S)$ the G_W-*expanded* set of S. It is obvious that $S \subset G_W(S)$.

The next proposition is obvious from the definition of maximal antipodal sets.

Corollary 2.3.10. Let S be a maximal antipodal set containing connected two points. Let W be any subset of $CL(S)$. Then, $G_W(S) \subset S$.

We use the following lemma later.

Lemma 2.3.11. Let S be an antipodal set containing connected two points. Let W be any subset of $CL(S)$ and set $T = G_W(S)$. Then, it follows that $L(p, q, T) = L(p, q, S)$ for any connected two points $p, q \in T$.

Proof. Since $S \subset T$, it is obvious that $L(p, q, T) \subset L(p, q, S)$. We will show $L(p, q, S) \subset L(p, q, T)$. Let $\gamma \in L(p, q, S)$. It is sufficient to show $s_{g(r)}(\gamma) \subset \gamma$ for any $g \in G_W$ and $r \in S$. By the definition of G_W, there are $x_1, x_2, \cdots, x_l \in W$ such that $g = s_{x_l}\cdots s_{x_2}s_{x_1}$. Let $x_i \in CL(p_i, q_i, S)$ ($p_i, q_i \in S, p_i, q_i$ are connected. $1 \leq i \leq l$). We prove $s_{g(r)}(\gamma) \subset \gamma$ by induction for l.

For $x_1 \in CL(p_1, q_1, S)$,

$$s_{s_{x_1}(r)}(\gamma) = s_{x_1}s_r s_{x_1}(\gamma) = \begin{cases} s_{x_1}s_{x_1}s_r(\gamma) = s_r(\gamma) \subset \gamma, \\ s_{x_1}s_{p_1}s_{x_1}s_{p_1}s_r(\gamma) = s_{q_1}s_{p_1}s_r(\gamma) \subset \gamma, \end{cases}$$

by Lemma 2.3.6 and $p_1, q_1, r \in S$. Hence $s_{s_{x_1}(r)}(\gamma) \subset \gamma$. We assume that it is true until $l-1$. We see

$$s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l} = \begin{cases} s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}, \\ s_{p_l} s_{x_l} s_{p_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}, \end{cases}$$

for $x_1, \cdots, x_l \in W$ as follows. Let $\delta(t) \in L(p_l, q_l, S)$ $(0 \leq t \leq 2)$ satisfy $\delta(0) = \delta(2) = p_l, \delta(1) = q_l$ and $\delta(\frac{1}{2}) = x_l$. By the assumption of induction, it follows that $s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\delta) \subset \delta$. Moreover, $s_{s_{x_{l-1}} \cdots s_{x_1}(r)}$ fixes $p_l = \delta(0)$ and $q_l = \delta(1)$. Hence, it follows that $s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\delta(t)) = \delta(t)$ or $s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\delta(t)) = \delta(-t)$ since $s_{s_{x_{l-1}} \cdots s_{x_1}(r)}$ is involutive. In the former case, because $s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(x_l) = x_l$ it is true that

$$s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l} = s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}.$$

In the latter case, because $s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(x_l) = s_{p_l}(x_l)$ it is true that

$$s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l} = s_{p_l} s_{x_l} s_{p_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}.$$

From above arguments, we obtain

$$s_{s_{x_l} \cdots s_{x_1}(r)}(\gamma) = s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l}(\gamma) = \begin{cases} s_{x_l} s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\gamma) \subset \gamma, \\ s_{x_l} s_{p_l} s_{x_l} s_{p_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}(\gamma) \subset s_{q_l} s_{p_l}(\gamma) \subset \gamma. \end{cases}$$

Hence, it follows that $s_{s_{x_l} \cdots s_{x_1}(r)}(\gamma) \subset \gamma$. By induction, we proved $s_{g(r)}(\gamma) \subset \gamma$ that is $\gamma \in L(p, q, T)$.

□

From the proof of Lemma 2.3.11, we obtain the following lemma which will be used later.

Lemma 2.3.12. Let S be an antipodal set containing connected two points. Let $x_1, \cdots, x_l \in CL(S)$ and $x_l \in CL(p_l, q_l, S)$ $(p_l, q_l \in S, p_l, q_l$ are connected). Then, for any $r \in S$

$$s_{s_{x_{l-1}} \cdots s_{x_1}(r)} s_{x_l} = \begin{cases} s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}, \\ s_{p_l} s_{x_l} s_{p_l} s_{s_{x_{l-1}} \cdots s_{x_1}(r)}. \end{cases}$$

Let $S = S_1$ be an antipodal set containing connected two points. We consider an expanded-series of S

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset S_{k+1} \subset \cdots,$$

where $S_{k+1} = G_{W_k}(S_k)$ for some subset W_k of $CL(S_k)$. Then we see that the following proposition follows immediately from Lemma 2.3.11.

Proposition 2.3.13. Let $k \leq 1$ and p, q be connected points of S_k. Then, $L(p, q, S_k) = L(p, q, S_1)$.

2.3.3 Orbits of G_W

Let S be an antipodal set containing connected two points. Let W be any subset of $CL(S)$. Then, we see that $G_W(S)$ is an antipodal set. We say that the group G_w acts on the antipodal set $G_W(S)$. In this subsection, we study G_W-orbit through each point in $G_W(S)$.

Proposition 2.3.14. Let S be an antipodal set containing connected two points. Let $x \in CL(r_1, r_2, S)$ ($r_1, r_2 \in S, r_1, r_2$ are connected). Suppose that $\gamma \in L(r_1, r_2, S)$ satisfies $\gamma(0) = \gamma(2) = r_1, \gamma(1) = r_2$ and $\gamma(\frac{1}{2}) = x$. Let $m \in M$ be antipodal to every point of S and $s_m(\gamma) \subset \gamma$. If $m \neq s_x(m)$, then m is antipodal and connected to $s_x(m)$.

Proof. Firstly we will show that m is antipodal to $s_x(m)$. We see $s_m(\gamma(0)) = \gamma(0)$ and $s_m(\gamma(1)) = \gamma(1)$ since m is antipodal to every point of S. Hence $s_m(\gamma(t)) = \gamma(t)$ or $s_m(\gamma(t)) = \gamma(-t)$ since $s_m(\gamma) \subset \gamma$, so we obtain $s_m(x) = x$ in the former case and $s_m(x) = s_{r_1}(x)$ in the latter case. Hence, it is true that

$$s_m s_x = \begin{cases} s_x s_m \\ s_{r_1} s_x s_{r_1} s_m \end{cases}$$

Therefore, it is true that

$$s_{s_x(m)}(m) = s_x s_m s_x(m) = \begin{cases} s_x s_x s_m(m) = m, \\ s_x s_{r_1} s_x s_{r_1} s_m(m) = s_{r_2} s_{r_1}(m) = m. \end{cases}$$

We showed that m is antipodal to $s_x(m)$.

Secondly we will show that m is connected to $s_x(m)$. From the homogeneity of M, we may let $o = r_1$ and denote r_2 by r simply. There is some $X \in \mathfrak{m}$ such that $\gamma(t) = \exp tX \cdot o$. We consider the map $\iota : M \to G; p \mapsto s_p s_o$. Since it is known that ι maps geodesics of M to geodesics of G, $s_{\gamma(t)} s_o$ ($t \in \mathbb{R}$) is a geodesic of G through unit element of G. In particular, $s_{\gamma(t)} s_o = \exp 2tX$. We will show that $s_{\gamma(t)} s_o(m) = \exp t2X \cdot m$ is a geodesic of M. It is sufficient to prove $X \in \mathrm{Ad}(g)\mathfrak{m}$, where $m = g \cdot o$ ($g \in G$). We obtain

$$s_m(\exp 2tX)s_m = s_m s_{\gamma(t)} s_o s_m = s_m s_{\gamma(t)} s_m s_o = s_{s_m \gamma(t)} s_o = \begin{cases} s_{\gamma(t)} s_o = \exp 2tX, & \text{(A)} \\ s_{\gamma(-t)} s_o = \exp 2(-t)X, & \text{(B)} \end{cases}$$

from the first part of this proof. Since $\sigma_m = \mathrm{Ad}(g)\sigma_o \mathrm{Ad}(g^{-1})$ in $\mathfrak{g}$, we see (A) $\Rightarrow X \in \mathrm{Ad}(g)\mathfrak{k}$ and (B) $\Rightarrow X \in \mathrm{Ad}(g)\mathfrak{m}$. If $X \in \mathrm{Ad}(g)\mathfrak{k}$, then

$$s_{\gamma(t)}(m) = s_{\gamma(t)} s_o(g \cdot o) \subset (gKg^{-1})(g \cdot o) = g \cdot o = m.$$

This contradicts to $s_x(m) \neq m$. Therefore $X \in \mathrm{Ad}(g)\mathfrak{m}$, so we showed that $s_{\gamma(t)}(m) = s_{\gamma(t)} s_o(m)$ ($t \in \mathbb{R}$) is a geodesic of M. In particular, $s_{\gamma(t)}(m)$ ($0 \leq t \leq 1$) is a closed geodesic since $s_{\gamma(0)}(m) = s_{r_1} m = m = s_{r_2}(m) = s_{\gamma(1)}(m)$. Moreover, since the length of $\gamma(t)$ ($0 \leq t \leq 2$) is $|2X|$ and the length of $s_{\gamma(t)}(m)$ ($0 \leq t \leq 1$) is $|\mathrm{Ad}(g^{-1})(2X)|$, we see that these two closed geodesics have the same length. In particular, $s_{\gamma(t)}(m)$ ($0 \leq t \leq 1$) is a shortest closed geodesic. Hence, we showed that m and $s_x(m) = s_{\gamma(\frac{1}{2})}(m)$ are connected.

□

By Lemma 2.3.11 and Proposition 2.3.14, we obtain the following theorem immediately.

Theorem 2.3.15. Let S be an antipodal set containing connected two points. Let W be any subset of $CL(S)$. Then, $G_W(p)$ is a connected antipodal set for any $p \in S$.

We obtain the following corollary from the definition of connected antipodal sets and Theorem 2.3.15.

Corollary 2.3.16. Let S be a connected antipodal set and W be any subset of $CL(S)$. Then, $G_W(S)$ is a connected antipodal set.

By the definition of maximally connectedness, we obtain the following corollary immediately.

Corollary 2.3.17. Let S be a maximally connected antipodal set and W be any subset of $CL(S)$. Then, $G_W(S) \subset S$.

2.3.4 $G_{o,S}$-homogeneous antipodal sets

We already see that if S is a maximal antipodal set or a maximally connected antipodal set, the group G_W acts on S. In particular, $G_{o,S}$ acts on S. In this subsection, we study such antipodal sets.

Definition 2.3.18. Let S be an antipodal set and $o \in S$. If $G_{o,S}(o) = S$, we say that S is *$G_{o,S}$-homogeneous.*

We see that $G_{o,S}$-homogeneous antipodal set is connected from Theorem 2.3.15.

Theorem 2.3.19. Let S be a connected antipodal set, $o \in S$ and $G_{o,S}(S) \subset S$. Then, S is $G_{o,S}$-homogenoeus.

Proof. It is sufficient to show $S - G_{o,S}(o) = \phi$. We see $S_o \subset G_{o,S}(o)$. In fact, for any $p \in S_o$ there is some $x \in CL(o,p,S)$ and $p = s_x(o) \in G_{o,S}(o)$. We assume that $S - G_{o,S}(o) \neq \phi$. Let $p \in G_{o,S}(o)$ and $q \in S - G_{o,S}(o)$. From the connectedness of S, there is a connected point series $\{p_i\}_{i=0}^{l}$ such that $p_0 = p$ and $p_l = q$. We see that there is some $0 \leq i \leq l-1$ such that $p_i \in G_{o,S}(o)$ and $p_{i+1} \in S - G_{o,S}(o)$ since S is connected. Then, there is some $g \in G_{o,S}$ such that $p_i = g(o)$. In particular g is an isometry of M, so $g^{-1}(p_{i+1}) \in S_o$. From the above remark, we obtain $p_{i+1} \in G_{o,S}(o)$. However, this is a contradiction. Therefore, $S - G_{o,S}(o) \neq \phi$ is wrong, so $S - G_{o,S}(o) = \phi$.

□

From Corollary 2.3.10 and Theorem 2.3.19, we obtain the following theorem immediately.

Theorem 2.3.20. Let S be a maximal antipodal set and $o \in S$. If S is connected, then S is $G_{o,S}$-homogeneous.

From Corollary 2.3.17 and Theorem 2.3.19, we obtain the following theorem similarly.

Theorem 2.3.21. If S is a maximally connected antipodal set and $o \in S$, then S is $G_{o,S}$-homogeneous.

In the followings, we study $G_{o,S}$-homogeneous sets. Firstly we study G_S of $G_{o,S}$-homogeneous sets.

Lemma 2.3.22. Let S be $G_{o,S}$-homogeneous. Then, for any $p = g(o) \in S$ ($g \in G_{o,S}$) it follows that $L(p,S) = g(L(o,S))$.

Proof. We remark $S_p = g(S_o)$ since g is an isometry of M. Let $r \in S_o$ and $\gamma \in L(o,r,S)$. Then, $g(\gamma)$ is a shortest closed geodesic through $p = g(o)$ and $g(r)$. Let q be any point of S. Because there is $u \in S$ such that $q = g(u)$, we obtain

$$s_q(g(\gamma)) = s_{g(u)}(g(\gamma)) = g s_u g^{-1} g(\gamma) = g s_u(\gamma) \subset g(\gamma).$$

Therefore, $g(\gamma) \in L(p, g(r), S)$, so $g(L(o, r, S)) \subset L(p, g(q), S)$. Thus, $g(L(o, S)) \subset L(p, S)$. Repeating the above argument replacing o by p we obtain $g^{-1}(L(p, S)) \subset L(o, S)$. Hence, $L(p, S) \subset g(L(o, S))$. Thus, we conclude $L(p, S) = g(L(o, S))$.

□

Proposition 2.3.23. If S is $G_{o,S}$-homogeneous, $G_S = G_{o,S}$.

Proof. Let p_1, p_2 be connected points in S and $y \in CL(p_1, p_2, S)$. Then, there are $x \in CL(o, S)$ and $g \in G_{o,S}$ such that $y = g(x)$, so $s_y = s_{g(x)} = g s_x g^{-1}$. Hence, $G_S \subset G_{o.S}$, so $G_S = G_{o,S}$.

□

Next, we study that $G_{o,S}$ is decided by only $\bar{S}_o = S_o \cup \{o\} \subset S$.

Proposition 2.3.24. Let S be an antipodal set and $o \in S$. Suppose that S is a $G_{o,S}$-homogeneous set. Then, $G_{o,S} = G_{o,\bar{S}_o}$. Hence $S = G_{o,\bar{S}_o}(o)$.

Proof. It is sufficient to prove $L(o, S) = L(o, \bar{S}_o)$. Since $\bar{S}_o \subset S$, we see $L(o, S) \subset L(o, \bar{S}_o)$ immediately. We show $L(o, \bar{S}_o) \subset L(o, S)$. Because $S = G_{o,S}(o)$, for every $p \in S$ there are $x_1, \cdots, x_l \in CL(o, S)$ such that $s_{x_l} \cdots s_{x_1}(o) = p$. Let $x_i \in CL(o, p_i, S)$ $(p_i \in S_o, 1 \leq i \leq l)$. We prove $s_p(\gamma) \subset \gamma$ by induction for l.

In $l = 1$, $s_{s_{x_1}(o)}(\gamma) \subset \gamma$ since $s_{x_l}(o) \subset \bar{S}_o$. We assume that it is true until $l - 1$. From Lemma 2.3.12, we see

$$s_{s_{x_l} \cdots s_{x_1}(o)}(\gamma) = s_{x_l} s_{s_{x_{l-1}} \cdots s_{x_1}(o)} s_{x_l}(\gamma) = \begin{cases} s_{x_l} s_{x_1} s_{s_{x_{l-1}} \cdots s_{x_1}(o)}(\gamma) \subset \gamma, \\ s_{x_l} s_o s_{x_l} s_o s_{s_{x_{l-1}} \cdots s_{x_1}(o)}(\gamma) \subset s_{p_l} s_o(\gamma) \subset \gamma. \end{cases}$$

Hence, it follows that $s_{s_{x_l} \cdots s_{x_1}(o)}(\gamma) \subset \gamma$, so we showed $s_{g(o)}(\gamma) \subset \gamma$ for any $g \in G_{o,S}$ by induction. Therefore, $L(o, \bar{S}_o) \subset L(o, S)$ and we conclude $L(o, \bar{S}_o) = L(o, S)$.

□

Using Proposition 2.3.24, we may construct a maximally connected antipodal set easily.

Proposition 2.3.25. Let $M_1^+, \cdots, M_k^+$ be all polars of o in M of which for every point there is a shortest closed geodesic of M through o and it. Let S be an antipodal set of M and $S \subset M_1^+ \cup \cdots \cup M_k^+$. Then, $G_{o,\bar{S}}(o)$ is a connected antipodal set of M. Moreover, set $\mathcal{A} = \{A\,;\, A$ is an antipiodal set of M and $A \subset M_1^+ \cup \cdots \cup M_k^+\}$ and let $T \in \mathcal{A}$ be maximal with respect to the inclusion relation in $\mathcal{A}$. Then, $G_{o,\bar{T}}(o)$ is a maximally connected antipodal set.

Proof. It is obvious that $\bar{S}$ is a connected antipodal set of M. Hence, $G_{o,\bar{S}}(o)$ is a connected antipodal set by Theorem 2.3.16.

We prove the later half of the statement. Let U be a maximally connected antipodal subset of M containing $G_{o,\bar{T}}(o)$. Then, $T \subset U_o$. However, $T = U_o$ by the definition of T, because $U_o \subset M_1^+ \cup \cdots \cup M_k^+$. Since U is a maximally conneted antipodal set, U is $G_{o,U}$-homogeneous and $U = G_{o,U}(o) = G_{o,\bar{U}_o}(o)$ by Proposition 2.3.24. Hence, we obtain $G_{o,\bar{T}}(o) = G_{o,\bar{U}_o}(o) = U$.

□

Remark 2.3.26. In Proposition 2.3.25, if $k = 1$, then considering T is equivalent to considering a maximal antipodal set in M_1^+.

Next, we consider a connected component of a maximal antipodal set.

Theorem 2.3.27. Let T be a maximal antipodal set and not connected. Let S be a connected component of T and $o \in S$. Then, S is $G_{o,S}$-homogeneous.

Proof. Firstly, we show that every point of S is antipodal to evey point of $g(T)$ for any $g \in G_{o,S}$. It is sufficient to prove $s_q(g(r)) = g(r)$ for any $q \in S, r \in T$. By the definition, we may write $g = s_{x_l} \cdots s_{x_1}$, where $x_1, \cdots, x_l \in CL(o, S)$. Let $x_k \in CL(o, p_k, S)$ $(p_k \in S_o, 1 \leq k \leq l)$. We prove it by induction for l.

In $l = 1$, we obtain from Lemma 2.3.6

$$s_q(s_{x_1}(r)) = \begin{cases} s_{x_1} s_q(r) = s_{x_1}(r), \\ s_o s_{x_1} s_o s_q(r) = s_o s_{x_1} s_o(r) = s_{x_1}(r). \end{cases}$$

We assume that it is true until $l - 1$. Then, from Proposition 2.3.6 again we obtain

$$\begin{aligned} s_q(s_{x_l} s_{x_l} \cdots s_{x_1}(r)) &= (\epsilon s_{x_l} \epsilon) s_q(s_{x_{l-1}} \cdots s_{x_1})(r) \\ &= (\epsilon s_{x_l} \epsilon) s_{x_{l-1}} \cdots s_{x_1}(r) \\ &= s_{x_l} s_{x_{l-1}} \cdots s_{x_1}(r), \end{aligned}$$

where ϵ is s_o or id_M. Therefore, we proved $s_q(g(r)) = g(r)$ by induction. Thus, we showed that every point of S is antipodal to evey point of $g(T)$ for any $g \in G_{o,S}$.. Then, every point of $g(S)$ is antipodal to every point of T. Hence, $g(S) \subset T$ because of the maximality of T. Thus, $G_{o,S}(S) \subset T$. $G_{o,S}(S)$ is connected and $S \subset G_{o,S}(S)$. Since S is a connected component of T, $G_{o,S}(S) \subset S$. Thus we conclude that S is $G_{o,S}$-homogeneous by Theorem 2.3.19.

□

2.3.5 Properties of G_W-expansions

In this subsection, let S be an antipodal set containing connected two points and W be any subset of $CL(S)$. Let $S_1 = S$ and we consider a series of antipodal sets

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset S_{k+1} \subset \cdots,$$

where $S_{k+1} = G_{S_k}(S_k)$ $(k \in \mathbb{N})$. Then, there is a natural number $m \in \mathbb{N}$ such that $S_m = S_{m+1} = \cdots$ since $\#_2 M$ is finite. If $S_i = S_{i+1}$ for some natural number $i < m$, then $S_i = S_{i+1} = S_{i+2} = \cdots$ because $S_{i+2} = G_{S_{i+1}}(S_{i+1}) = G_{S_i}(S_i) = S_{i+1}$. Hence, we can rewrite the above sequence as follows;

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subsetneq S_{k+1} \subsetneq \cdots \subsetneq S_m = S_{m+1} = \cdots.$$

Let $X = S_m$. Then $G_{o,X}(X) \subset X$. If S is connected, then every S_k is connected by Corollary 2.3.16. Hence, $X = G_{o,X}(o)$ by Theorem 2.3.19.

On the other hand, let $T_1 = S$ and we consider a series of antipodal sets

$$T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_k \subsetneq T_{k+1} \subsetneq \cdots,$$

where $T_{k+1} = G_{W_k}(T_k)$ for some subset W_k of $CL(T_k)$ such that $T_k \subsetneq T_{k+1}$. By the finiteness of $\#_2 M$, there is a natural number $n \in \mathbb{N}$ such that $G_W(T_n) \subset T_n$ for any subset $W \subset CL(T_n)$. Thus, we rewrite the sequence as follows:

$$T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_k \subsetneq T_{k+1} \subsetneq \cdots \subsetneq T_n.$$

Let $Y = T_n$. Then, $G_{o,Y}(Y) \subset Y$. If S is connected, then every T_k $(1 \leq k \leq n)$ is connected by Corollary 2.3.16, so $Y = G_{o,Y}(o)$ by Theorem 2.3.19.

Theorem 2.3.28. In the above setting, $X = Y$.

Proof. Firstly we will prove $T_k \subset S_k$ for any $k \leq \min(m, n)$. In $k = 1$, it is obvious by $T_1 = S = S_1$. We assume that it is true until $k - 1$. We see that W_{k-1} is a subset of $CL(S_{k-1})$ by Lemma 2.3.11, so $G_{W_{k-1}}$ is a subgroup of $G_{S_{k-1}}$. Hence, $T_k = G_{W_{k-1}}(T_{k-1}) \subset G_{S_{k-1}}(T_{k-1}) \subset G_{S_{k-1}}(S_{k-1}) = S_k$. Therefore, it is true that $T_k \subset S_k$ for any $1 \leq k \leq n$.

Divide the problem into two cases to show $Y \subset X$: (i) $n < m$, (ii) $m < n$. In (i),

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subsetneq S_{k+1} \subsetneq \cdots \subsetneq S_n \subsetneq \cdots \subsetneq S_m = X,$$
$$T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_k \subsetneq T_{k+1} \subsetneq \cdots \subsetneq T_n = Y.$$

It is obvious that $Y \subset X$ since $Y = T_n \subset S_n \subset S_m = X$. In (ii),

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset S_{k+1} \subset \cdots \subset S_m = X,$$
$$T_1 \subset T_2 \subset \cdots \subset T_k \subset T_{k+1} \subset \cdots \subset T_m \subset \cdots T_n = Y.$$

We prove $T_{m+a} \subset S_m$ for any a $(0 \leq a \leq n - m)$ by induction for a. It is obvious that $T_m \subset S_m$ from above arguments. We assume that it is true until a. Then, $G_{W_{m+a}}$ is a subgroup of $G_{T_{m+a}}$ since W_{m+a} is a subset of $CL(T_{m+a})$. By Proposition 2.3.13, $G_{T_{m+a}}$ is a subgroup of G_{S_m} since $T_{m+a} \subset S_m$. Thus, $G_{W_{m+a}}$ is a subgroup of G_{S_m}. Hence $T_{m+a+1} = G_{W_{m+a}}(T_{m+a}) \subset G_{W_{m+a}}(S_m) \subset G_{S_m}(S_m) = S_m$. Therefore, we proved $T_{m+a} \subset S_m = X$ for any $0 \leq a \leq n - m$ by induction and $Y = T_n \subset S_m = X$.

Next we will show $X \subset Y$. For the sake of this, we prove $S_k \subset Y$ $(1 \leq k \leq m)$ by induction for k. In $k = 1$, this is obvious. We assume that it is true until $k-1$. By Proposition 2.3.13, $G_{S_{k-1}}$ is a subgroup of G_Y since $S_{k-1} \subset Y$ and for any connected $p, q \in S_{k-1}$ it is true that $L(p, q, S_{k-1}) = L(p, q, S) = L(p, q, T_1) = L(p, q, Y)$. Hence, $S_k = G_{S_{k-1}}(S_{k-1}) \subset G_Y(Y) = Y$, so $S_k \subset Y$ for any $1 \leq k \leq m$. Therefore, $X = S_m \subset Y$. Thus, we conclude $X = Y$.

□

Let S be a connected antipodal set and $o \in S$, then we see that $X (= Y)$ is obtained by $G_{o,S}$. Let $U_1 = S$ and consider a series of antipodal sets

$$U_1 \subset U_2 \subset \cdots \subset U_k \subset U_{k+1} \subset \cdots \subset U_l \subset U_{l+1} \subset \cdots,$$

where $U_k = G_{o,U_k}(U_k)$. Then, there is a natural number l such that $U_{l+1} = U_l$. If there is some i $(i < l)$ such that $U_i = U_{i+1}$, then $U_i = U_{i+1} = U_{i+2} = \cdots$ since $U_{i+2} = G_{o,U_{i+1}}(U_{i+1}) = G_{o,U_i}(U_i) = U_{i+1}$. Hence, we may rewrite the above sequence as follows:

$$U_1 \subset U_2 \subset \cdots \subset U_k \subset U_{k+1} \subset \cdots \subset U_l = U_{l+1} = \cdots.$$

Let $Z = U_l$.

Corollary 2.3.29. In the above setting, $Z = X$.

Proof. It is sufficient to show $G_Z(Z) \subset Z$. By Corollary 2.3.16, Z is connected. Moreover, $G_{o,Z}(Z) \subset Z$ by the definition of Z. Hence, Z is $G_{o,Z}$-homogeneous. Therefore, $G_{o,Z} = G_Z$ by Proposition 2.3.23. Thus, $G_Z(Z) \subset Z$, so $Z = Y = X$.

□

2.4 Examples of connected maximal antipodal sets

From Theorem 2.3.20 we see that if a maximal antipodal set S is connected, then S is homogeneous. In this section, we decide whether a given maximal antipodal set is connected in some compact symmetric spaces.

2.4.1 Oriented real Grassmannians

Let $\tilde{G}_k(\mathbb{R}^n)$ be the set of all $k-$dim oriented subspaces of $\mathbb{R}^n$. In oriented real Grassmannians $\tilde{G}_k(\mathbb{R}^n)$, any maximal antipodal set is not necessarily great. Moreover, any two maximal antipodal sets are not necessarily congruent to each other. When $k = 3, 4$, maximal antipodal sets are classified completely[21]. However, when $k \geq 5$, the classification is incomplete. We will decide whether each maximal antipodal set is connected in $k = 3, 4$.

Let $v_1, \cdots, v_k \in \mathbb{R}^n$ be linearly independent. We denote the subspace spanned by $v_1, \cdots, v_k$ with the positive orientation or the negative orientation by

$$\pm V = \pm[v_1 \wedge v_2 \wedge \cdots \wedge v_k].$$

We often denote $+V$ by V simply. Let us denote

$$v_i = \begin{pmatrix} v_{i1} \\ \vdots \\ v_{in} \end{pmatrix} \in \mathbb{R}^n.$$

Moreover, we write $\pm V$ as follows:

$$\pm V = \pm \begin{bmatrix} v_{11} & \cdots & v_{kn} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ v_{1n} & \cdots & v_{kn} \end{bmatrix}.$$

Let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{R}^n$. We recall some results of anitpodal sets of oriented real Grassmannians $\tilde{G}_k(\mathbb{R}^n)$ from the work of Tasaki [21].

Proposition 2.4.1. [21] Let S be any antipodal set of $\tilde{G}_k(\mathbb{R}^n)$. Then, there is an orthonormal basis $v_1, \cdots, v_n$ of $\mathbb{R}^n$ satisfying the following condition:

$$S \subset \{\pm[v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)}];\ \alpha \in \mathrm{Inc}_k(n)\},$$

where $\mathrm{Inc}_k(n) = \{\alpha : \{1, \ldots, k\} \to \{1, \ldots, n\}\ ;\ 1 \leq i < j \leq k \Rightarrow \alpha(i) < \alpha(j)\}$.

For $\alpha, \beta \in \mathrm{Inc}_k(n)$, we denote $\beta - \alpha = \{b \in \mathrm{Im}\beta;\ b \notin \mathrm{Im}\alpha\}$. We fix an orthonormal basis $v_1, \cdots, v_n$ of $\mathbb{R}^n$.

Proposition 2.4.2. [21] Let $V_\alpha = [v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)}]$ and $V_\beta = [v_{\beta(1)} \wedge \cdots \wedge v_{\beta(k)}] \in \tilde{G}_k(\mathbb{R}^n)$ $(\alpha, \beta \in \mathrm{Inc}_k(n))$. Then, following two conditions are equivalent. Moreover, this is true for any pair of $(\pm V_\alpha, \pm V_\beta)$.

(1) V_α is antipodal to V_β.

(2) The cardinality of $\beta - \alpha$ is even.

We consider the condition that V_α and V_β are connected.

Proposition 2.4.3. Let $V_\alpha = [v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)}]$ and $V_\beta = [v_{\beta(1)} \wedge \cdots \wedge v_{\beta(k)}] \in \tilde{G}_k(\mathbb{R}^n)$ $(\alpha, \beta \in \mathrm{Inc}_k(n))$. Suppose that V_α is antipodal to V_β. Then, following two conditions are equivalent. Moreover, this is true for any pair of $(\pm V_\alpha, \pm V_\beta)$.

(1) V_α is connected to V_β.

(2) The cardinality of $(\beta - \alpha)$ is 2.

Proof. By the homogeneity of $\tilde{G}_k(\mathbb{R}^n)$, we may assume $V_\alpha, V_\beta \in \{\pm[e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}];\ \sigma \in \mathrm{Inc}_k(n)\}$ and $V_\alpha = o = +[e_1 \wedge \cdots \wedge e_k]$.

In a general compact Riemannian symmetric space M, the following two conditions are equivalent: (i) $p_1, p_2 \in M$ are antipodal and connected. (ii) p_2 is included in a polar of p_1 whose each point is contained in some shortest closed geodesic on M through p_1. In $\tilde{G}_k(\mathbb{R}^n)$, the one of such polars of o is given as follows and the following polar is the only such polar :

$$N = \left\{ \pm \left[\left(\begin{array}{c|c} \overbrace{\begin{matrix} * & \cdots & * \\ \vdots & \ddots & * \end{matrix}}^{k-2} & \overbrace{\phantom{\begin{matrix} * & * \end{matrix}}}^{2} \\ \hline & \begin{matrix} * & * \\ \vdots & \vdots \\ * & * \end{matrix} \end{array} \right) \right] \begin{matrix} \}k \\ \\ \}n-k \end{matrix} \in \tilde{G}_k(\mathbb{R}^n) \right\},$$

where the component of blank parts is 0. Denote $V_\beta = [e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}]$. Then, $V_\beta \in N$ if and only if $\#(\mathrm{Im}(\sigma) \cap \{1, \cdots, k\}) = k - 2$. Therefore, the statement follows.

□

In the following, denote $\pm[e_{i_1} \wedge \cdots \wedge e_{i_k}]$ $(1 \leq i_1, \cdots, i_k \leq n)$ by $\pm[i_1 \wedge \cdots \wedge i_k]$. Congruent classes of maximal antipodal sets are competely classified in $\tilde{G}_3(\mathbb{R}^n)$ and $\tilde{G}_4(\mathbb{R}^n)$ by Tasaki [21], although the classification is incomplete in $\tilde{G}_k(\mathbb{R}^n)$ $(k \geq 5)$. We set $B(3,6), B(3,7), B(4,8), A(2k, 2l)$ and $A(2k+1, 2l+1)$ $(k, l \in \mathbb{N},\ k < l)$ as follows.

$$B(3,6) = \{\ \pm[1 \wedge 2 \wedge 3], \pm[1 \wedge 4 \wedge 5], \pm[2 \wedge 4 \wedge 6], \pm[3 \wedge 5 \wedge 6]\ \},$$

$$B(3,7) = \{\ \pm[1 \wedge 2 \wedge 3], \pm[1 \wedge 4 \wedge 5], \pm[2 \wedge 4 \wedge 6], \pm[3 \wedge 5 \wedge 6], \pm[1 \wedge 6 \wedge 7], \pm[2 \wedge 5 \wedge 7], \pm[3 \wedge 4 \wedge 7]\ \},$$

$$B(4,7) = \left\{ \begin{matrix} \pm[4 \wedge 5 \wedge 6 \wedge 7], & \pm[2 \wedge 3 \wedge 6 \wedge 7], & \pm[1 \wedge 3 \wedge 5 \wedge 7], & \pm[1 \wedge 2 \wedge 4 \wedge 7], & \pm[2 \wedge 3 \wedge 4 \wedge 5], \\ \pm[1 \wedge 3 \wedge 4 \wedge 6], & \pm[1 \wedge 2 \wedge 5 \wedge 6] & & & \end{matrix} \right\},$$

$$B(4,8) = \left\{\begin{array}{l} \pm[4\wedge 5\wedge 6\wedge 7],\quad \pm[2\wedge 3\wedge 6\wedge 7],\quad \pm[1\wedge 3\wedge 5\wedge 7],\quad \pm[1\wedge 2\wedge 4\wedge 7],\quad \pm[2\wedge 3\wedge 4\wedge 5],\\ \pm[1\wedge 3\wedge 4\wedge 6],\quad \pm[1\wedge 2\wedge 5\wedge 6],\\ \pm[1\wedge 2\wedge 3\wedge 8],\quad \pm[1\wedge 4\wedge 5\wedge 8],\quad \pm[2\wedge 4\wedge 6\wedge 8],\quad \pm[3\wedge 5\wedge 6\wedge 8],\quad \pm[1\wedge 6\wedge 7\wedge 8],\\ \pm[2\wedge 5\wedge 7\wedge 8],\quad \pm[3\wedge 4\wedge 7\wedge 8] \end{array}\right\},$$

$$A(2k,2l) = \left\{ \pm\Big[\Big(\alpha(1)\wedge(\alpha(1)+1)\Big)\wedge\cdots\wedge\Big(\alpha(k)\wedge(\alpha(k)+1)\Big)\Big] \;;\; \begin{array}{c} 1\le\alpha(1)<\cdots<\alpha(k)\le 2l-1,\\ \alpha(i)\ (1\le i\le k) \text{ is odd.}\end{array} \right\},$$

$$A(2k+1,2l+1) = \Big\{ \pm\Big[V\wedge(2l+1)\Big] \;; V\in A(2k,2l)\Big\},$$

The following proposition tells us the classification of congruent classes of maximal antipodal sets in $\tilde{G}_3(\mathbb{R}^n)$ and $\tilde{G}_4(\mathbb{R}^n)$. For $\pm v = \pm[e_{v_1}\wedge\cdots\wedge e_{v_k}] \in \tilde{G}_k(\mathbb{R}^n)$ $(1\le v_1<\cdots<v_k\le n)$ and $1\le m\le n-v_k$, we set $\pm v+m = \pm[e_{v_1+m}\wedge\cdots\wedge e_{v_k+m}]$.

Proposition 2.4.4. [21] In $\tilde{G}_3(\mathbb{R}^n)$, any maximal antipodal set is congruent to one of the following maximal antipodal sets.

n	$3,4$	5	6	$7,8$	$9\le n$
	$A(3,3)$	$A(3,5)$	$B(3,6)$	$B(3,7)$	$A(3,2[\frac{n-1}{2}]+1), B(3,7)$

Proposition 2.4.5. [21] In $\tilde{G}_4(\mathbb{R}^n)$, any maximal antipodal set is congruent to one of the following maximal antipodal sets.

n	$4,5$	6	7	$8,9$	10
	$A(4,4)$	$A(4,6)$	$B(4,7)$	$B(4,8)$	$A(4,10), B(4,8)$

n	$11\le n$
	$A(4,2[\frac{n}{2}])$, $B(4,7)\cup\{X+7\ ;\ X$ is a maximal antipodal set of $\tilde{G}_4(\mathbb{R}^{n-7})$ in this list$\}$ $B(4,8)\cup\{Y+8\ ;\ Y$ is a maximal antipodal set of $\tilde{G}_4(\mathbb{R}^{n-8})$ in this list $\}$

Using Proposition 2.4.3 we check whetehr the above maximal antipodal sets are connected.

Proposition 2.4.6. $B(3,6), B(3,7), B(4,7)$ and $B(4,8)$ are connected.

Proof. By the definition of the connectedness, we fix o and it is sufficient to show that for any point p there is a connected point series $\{p_i\}_{i=0}^{l}$ containing o and p in each case.

In $B(3,6)$ and $B(3,7)$, let $o=[1\wedge 2\wedge 3]$. Then we see that any point except for $-o$ is connected to o. Let $p_1=[1\wedge 4\wedge 5]$. Then $\{p_0=o,p_1,p_2=-o\}$ is a connected point series containing o and $-o$. We see that $B(4,7)$ is connected by the similar way. In $B(4,8)$, let $o=[4\wedge 5\wedge 6\wedge 7]$. Then we see that any point except for $-o, \pm p=\pm[1\wedge 2\wedge 3\wedge 8]$ is connected to o. Let $p_1=[2\wedge 3\wedge 6\wedge 7]$ and $q_1=[3\wedge 4\wedge 7\wedge 8]$. Then $\{p_0=o,p_1,p_2=-o\}$ is a connected point series containing $o,-o$ and $\{q_0=o,q_1,q_2=\pm p\}$ is a connected point series containing o,p and $\{r_0=o,r_1=q_1,r_2=-p\}$ is a connected point series containing $o,-p$.

□

Proposition 2.4.7. $A(2k,2l)\subset\tilde{G}_{2k}(\mathbb{R}^{2l}),\tilde{G}_{2k}(\mathbb{R}^{2l+1})$ is connected.

Proof. Fix $o = [(1 \wedge 2) \wedge \cdots \wedge (2k-1 \wedge 2k)]$. It is sufficient to show that for any point $p \in A(2k, 2l)$ $(p \neq o)$ there is a connected point series containing o and p. Let

$$p = \pm[\Big(\alpha(1) \wedge (\alpha(1)+1)\Big) \wedge \cdots \wedge \Big(\alpha(k) \wedge (\alpha(k)+1)\Big)] \neq -o,$$

where $1 \leq \alpha(1) < \cdots < \alpha(k) \leq 2l-1$ and every $\alpha(i)$ is odd. We see $\alpha(i) \geq 2i-1$ for $1 \leq i \leq k$ obviously. Let $\{p_i\}_{i=0}^{k}$ be as follows:

$$\begin{aligned}
&p_0 = o \\
&,p_1 = [(1 \wedge 2) \wedge (3 \wedge 4) \wedge \cdots \wedge (2k-3 \wedge 2k-2) \wedge \Big(\alpha(k) \wedge (\alpha(k)+1)\Big)], \\
&p_2 = [(1 \wedge 2) \wedge (3 \wedge 4) \wedge \cdots \wedge \Big(\alpha(k-1) \wedge (\alpha(k-1)+1)\Big) \wedge \Big(\alpha(k) \wedge (\alpha(k)+1)\Big)], \\
&\vdots \\
&p_{k-1} = [(1 \wedge 2) \wedge \Big(\alpha(2) \wedge (\alpha(2)+1)\Big) \wedge \cdots \wedge \Big(\alpha(k-1) \wedge (\alpha(k-1)+1)\Big) \wedge \Big(\alpha(k) \wedge (\alpha(k)+1)\Big)], \\
&p_k = p.
\end{aligned}$$

We can take a connected subseries of $\{p_i\}_{i=0}^{k}$ containing o, p because $p_i = p_{i+1}$ or p_i is connectd to p_{i+1} for $1 \leq i \leq k-1$. Moreover, for $-o$ we consider a point series $\{q_1 = o, q_2 = p_1, q_3 = -o\}$. This point series is a connected point series of $A(2k, 2l)$ containing o and $-o$. Thus, we conclude that $A(2k.2l)$ is connected.

□

We can prove the following proposition by the similar way.

Proposition 2.4.8. $A(2k+1, 2l+1) \subset \tilde{G}_{2k+1}(\mathbb{R}^{2l+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2l+2})$ is connected.

Summarizing above results we obtain the following corollary by Theorem 2.3.20.

Corollary 2.4.9. In $\tilde{G}_3(\mathbb{R}^n)$, any maximal antipodal set is homogeneous.

In $\tilde{G}_4(\mathbb{R}^n)$ $(n \geq 11)$, any maximal antipodal set is congruent to some antipodal set contained in

$$\begin{Bmatrix} A(4, 2[\frac{n}{2}]), & I(4,7) = B(4,7) \cup \{X + 7\ ;\ X \text{ is a maximal antipodal set of } \tilde{G}_4(\mathbb{R}^{n-7}) \text{ in the list.}\}, \\ & I(4,8) = B(4,8) \cup \{Y + 7\ ;\ Y \text{ is a maximal antipodal set of } \tilde{G}_4(\mathbb{R}^{n-8}) \text{ in the list.}\} \end{Bmatrix}.$$

By Propositions 2.4.3 we see that any antipodal set in $I(4,7)$ and $I(4,8)$ is not connected. We see that any connected component of any maximal antipodal set in $I(4,7)$ and $I(4,8)$ is congruent to one of $B(4,7), B(4,8)$ and $A(4, 2l)$ $(l \in \mathbb{N})$.

Remark 2.4.10. It is true that any connected maximal antipodal set is maximally connected, but any maximally connected antipodal set is not necessarily maximal. $B(4,7)$ and $B(4,8)$ is maximally connected in $\tilde{G}_4(\mathbb{R}^n)$ $(n \geq 11)$, but they are not maximal.

Proposition 2.4.11. Let A be a maximal antipodal set in $I(4,7)$. If there is some connected component of A which is not congrurent to $B(4,7)$, then A is not homogeneous. Similarly, let B be a maximal antipodal set in $I(4,8)$. If there is some connected component of B which is not congrurent to $B(4,8)$, then B is not homogeneous.

Proof. We prove the former part of the statement. The latter part is proved by the similar way. Let A_0 be a connected component of A and $A_0 = B(4,7)$. It is sufficient to show that if there is a connected component A_1 of A which is not congruent to $B(4,7)$, then there are no isometries g such that $g(A_0) = A_1$. Then, A_1 is congruent to $B(4,8)$ or $A(4,2l)$ ($i \in \mathbb{N}$). We see $\#B(4,7) = 14$ and $\#B(4,8) = 28$. Moreover, we see $\#A(4,2l)$ increases as l increases and $12 = \#A(4,8) < \#B(4,7) < \#A(4,10) = 20 < \#B(4,8) < \#A(4,12) = 30$. Therefore, we conclude that there are no isometries g such that $g(A_0) = A_1$.

□

Proposition 2.4.12. Let A be a maximal antipodal set in $I(4,7)$. If every connected component of A is congruent to $B(4,7)$ that is $A = B(4,7) \sqcup (B(4,7)+7) \sqcup (B(4,7)+14) \sqcup \cdots$, then A is homogeneous. Similarly, let B be a maximal antipodal set in $I(4,8)$. If every connected component of B is congruent to $B(4,8)$ that is $B = B(4,8) \sqcup (B(4,8)+8) \sqcup (B(4,8)+16) \sqcup \cdots$, then B is homogeneous.

Proof. We consider the former part of the statement. The latter part is proved by the similar way. Then A is the following maximal antipodal set

$$\bigcup_{m=0}^{k-1} \Big\{ \pm [(n_1+7m) \wedge (n_2+7m) \wedge (n_3+7m) \wedge (n_4+7m)] \ ; \ \pm[n_1 \wedge n_2 \wedge n_3 \wedge n_4] \in B(4,7) \Big\},$$

where $7k \leq n \leq 7k+3$ and every connected component of A is

$$B(4,7)_m = \Big\{ \pm [(n_1+7m) \wedge (n_2+7m) \wedge (n_3+7m) \wedge (n_4+7)] \ ; \ \pm[n_1 \wedge n_2 \wedge n_3 \wedge n_4] \in B(4,7) \Big\},$$

where $0 \leq m \leq k-1$. Then, we see that any $g \in G_{B(4,7)_m}$ $(0 \leq m \leq k-1)$ fixes every point of $B(4,7)_l$ $(l \neq m)$ by caliculations. Moreover, we consider permutation matrices corresponding to following permutations : for $0 \leq m \leq k-1$,

$$\sigma_m : \{1,\cdots,n\} \to \{1,\cdots,n\}; \sigma_m(a) = \begin{cases} a+7m & (1 \leq a \leq 7), \\ a-7m & (1+7m \leq a \leq 7+7m), \\ a & (a \text{ is otherwise}). \end{cases}$$

We denote the permutation matrix corresponding to σ_m by the same letter. Then, we obtain

$$\sigma_m(B(4,7)_0) = B(4,7)_m, \ \sigma_m(B(4,7)_m) = B(4,7)_0, \ \sigma_m|_{B(4,7)_l} = \mathrm{Id}|_{B(4,7)_l} \ (l \neq 0, m).$$

We consider the subgroup of the isometry group generated by every element of $G_{B(4,7)_0}$ and σ_m $(0 \leq m \leq k-1)$. Then, we see that this group acts on A and this action is transitive. Thus, A is homogeneous.

□

By Theorem 2.3.20 we obtain the following corollary summarizing above propositions.

Corollary 2.4.13. In $\tilde{G}_4(\mathbb{R}^n)$, the followings are true.

(i) In $4 \leq n \leq 10$, any maximal antipodal set is homogeneous.

(ii) In $11 \leq n$, a maximal antipodal set A is homogeneous if and only if A satisfies one of the following three conditions:

  (1) A is congruent to $A(4, 2[\frac{n}{2}])$.
  (2) Each connected component of A is congruent to $B(4,7)$.
  (3) Each connected component of A is congruent to $B(4,8)$.

2.4.2 Compact symmetric spaces having one polar except for the trivial pole

If a compact Riemannian symmetric space M has one polar except for the trivial pole, then any two antipodal points are connecetd, so we easily obtain the folloiwng theorem.

Theorem 2.4.14. If M has only one polar except for the trivial pole, then any antipodal set is connected.

In particular, any maximal antipodal set in M is connected and homogeneous. By the classification of polars [2][3][11], we obtain the following example about exceptional compact symmetric spaces.

Example 2.4.15. Any maximal antipodal set of $E_6/F_4, (E_6/F_4)^*, F_4/\mathrm{Spin}(9)$ and $G_2/SO(4)$ is connected and homogeneous, where $(E_6/F_4)^*$ is the bottom space of E_6/F_4.

Remark 2.4.16. $F_4/\mathrm{Spin}(9)$ is a symmetric R-space, so it has been known that any maximal antipodal set is homogeneous [16].

2.4.3 Symmetric R-spaces

In symmetric R-spaces it is known that all maximal antipodal sets are congruent to each other and any maximal antipodal set is great and homogeneous [16]. We will study the connectedness of great antipodal sets in symmetric R-spaces.

Let M be an irreducible symmetric R-space. The followings are known. Let (G, K) be some compact simple Riemannian symmetric pair and $\mathfrak{g}$ and $\mathfrak{k}$ be Lie algebras of G and K. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the standard decomposition of $\mathfrak{g}$ with respect to (G, K). Then, there is $E \in \mathfrak{m}$ such that $N \cong \mathrm{Ad}(K)E$. The metric of N is induced by the K-inavariant inner product of $\mathfrak{m}$ which is the restriction of a negative constant multiple of the Killing form of $\mathfrak{g}$. Let $\mathfrak{h}$ be a maximal abelian subspace of $\mathfrak{m}$ containing E and W be the Weyl group of $\mathfrak{h}$. Then, it is known that $A = W(E)$ is a great antipodal set of M and any great antipodal set of M is congruent to A. In these setting, it is known that following lemmas are true.

Lemma 2.4.17. [14] Let T be a maximal flat torus of M through E and $T_E(T)$ be the tangent space at E of T. Then, there is a basis $X_1, \cdots, X_r$ ($r = \mathrm{rank}(M)$) of $T_E(T)$ satisfies the following properties.

(1) $\langle X_i, X_j \rangle = 0$ ($i \neq j$), where $\langle , \rangle$ is the inner product of $T_E(T)$ induced by the metric of N and $|\cdot|$ is the norm induced by $\langle , \rangle$.

(2) $\{X \in T_E(T)\ ;\ \mathrm{Ad}(\exp X)E = E\} = \{X_1, \cdots, X_r\}_{\mathbb{Z}}$.

(2) $\{\mathrm{Ad}(\exp tX_i)E;\ 0 \leq t \leq 1\}$ is a shortest closed geodesic in M.

Lemma 2.4.18. [14] For the great antipodal set A, there is a maximal flat torus T of M through E satisfying the following properties.

(1) There is a basis $X_1, \cdots, X_r$ of $T_E(T)$ satisfying properties of Lemma 2.4.17 and

$$A \cap T = \{\mathrm{Ad}(\exp(\epsilon_1 X_1 + \cdots + \epsilon_r X_r))E;\ \epsilon_i = 0 \text{ or } \frac{1}{2}\ (1 \leq i \leq r)\}.$$

(2) Let $W_0 = \{s \in W;\ s(E) = E\}$. Then any point of A is congruent to some point of $A \cap T$ by the action of W_0.

We obtain the following proposition by above two lemmas.

Proposition 2.4.19. A is connected.

Proof. It is sufficient to prove that for any $p \in A$ there is a connected point series containing E and p. Accroding to Lemma 2.4.18, there is some $w \in W_0$ such that

$$q = w(p) = \mathrm{Ad}(\exp(\frac{1}{2}X_{i_1} + \cdots + \frac{1}{2}X_{i_k}))E \in T,$$

where $1 \leq i_1 < \cdots i_k \leq r$. We define $\{q_j\}_{j=0}^k \subset A \cap T$ as follows:

$$\begin{aligned}
q_0 &= E, \\
q_1 &= \mathrm{Ad}(\exp(\frac{1}{2}X_{i_1}))E, \\
q_2 &= \mathrm{Ad}(\exp(\frac{1}{2}X_{i_1} + \frac{1}{2}X_{i_2}))E, \\
&\vdots \\
q_{k-1} &= \mathrm{Ad}(\exp(\frac{1}{2}X_{i_1} + \frac{1}{2}X_{i_2} + \cdots + \frac{1}{2}X_{i_{k-1}}))E, \\
q_{k-1} &= \mathrm{Ad}(\exp(\frac{1}{2}X_{i_1} + \frac{1}{2}X_{i_2} + \cdots + \frac{1}{2}X_{i_{k-1}} + \frac{1}{2}X_{i_k}))E.
\end{aligned}$$

Then, we see that q_i is connected to q_{i+1} for $0 \leq j \leq k-1$ by Lemma 2.4.17 and Lemma 2.4.18. Therefore, $\{q_j\}_{j=0}^k$ is a connected point series in $A \cap T$ containing E and q. Let $p_j = w^{-1}(q_j)$ $(0 \leq j \leq k)$. Then, $\{p_j\}_{j=0}^k$ is included in A and becomes a connected point series containing E and p. Hence, A is connected.

□

Summarizing this subsection and results of Tanaka and Tasaki [16] we obtain the following theorem.

Theorem 2.4.20. Let M be an irreducible symmetric R-space. Then, any great antipodal set of M is connected and homogeneous.

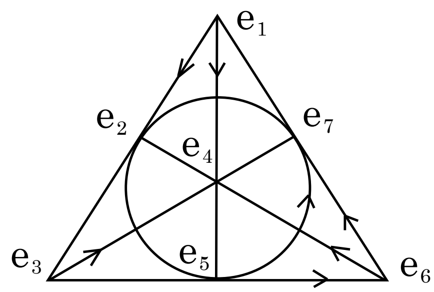
3 Morse functions and maximal antipodal sets of $G_2/SO(4)$

We will study Morse functions of $G_2/SO(4)$ in this section. This section is based on [13].

3.1 Preliminary

3.1.1 Octonion and G_2

Let $\mathbb{O} = \sum_{i=0}^{7} \mathbb{R}e_i$ be the octonion algebra where $\{e_0, e_1, \cdots, e_7\}$ is a basis of $\mathbb{O}$. We define the multiplication as follows. In the following fingure, we define multiplications between e_1, e_2, e_3 are $e_1e_2 = e_3,\ e_2e_3 = e_1,\ e_3e_1 = e_2$ and define similarly on the other lines. Moreover, we define that e_0 is the unit element of this multiplication and $e_i^2 = -1\ (1 \leq i \leq 7),\ e_ie_j = -e_je_i\ (1 \leq i \neq j \leq 7)$. We often denote e_0 by 1. Moreover, we assume the distribution law.



We call the map $f : \mathbb{O} \to \mathbb{O}$ an automorphism of $\mathbb{O}$ when f is a linear automorphism and satisfies $f(xy) = f(x)f(y)$ for any $x, y \in \mathbb{O}$. Then, the exceptional compact Lie group G_2 is realized as the automorphism group of $\mathbb{O}$, that is $G_2 = \{f : \mathbb{O} \to \mathbb{O}\ ;\ f \text{ is an automorphism of } \mathbb{O}\}$. We define the inner product (x, y) and the norm $|x|$ for $x = x_0 + \sum_{i=0}^{7} x_ie_i$ and $y = y_0 + \sum_{i=0}^{7} y_ie_i$ as follows:

$$(x, y) = \sum_{i=0}^{7} x_iy_i, \quad |x| = \sqrt{(x, y)}.$$

The following properties of G_2 are known. For example, there are proofs of these properties in [26].

Proposition 3.1.1. ([26], Section 1.2) The followings are true.
(1) $(gx, gy) = (x, y)$ for any $g \in G_2$ and $x, y \in \mathbb{O}$. In particular, $G_2 \subset O(8)$.
(2) $g(1) = 1$ for any $g \in G_2$.

It is known that G_2 is connected, so $G_2 \subset SO(7)$. We say $x = x_0 + \sum_{i=1}^{7} x_ie_i \in \mathbb{O}$ is a pure octonion if $x_0 = 0$. Denote the set of all pure octonions by $\mathbb{O}_0$. Then, it is true that $g(\mathbb{O}_0) \subset \mathbb{O}_0$ for any $g \in G_2$. We easily see that if $x, y \in \mathbb{O}_0$ are orthogonal with respect to $(,)$, then $xy \in \mathbb{O}_0$. We identify $\mathbb{O}_0$ with $\mathbb{R}^7$ and denote $x = \sum_{i=1}^{7} x_ie_i \in \mathbb{O}_0$ by

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_7 \end{pmatrix} \in \mathbb{R}^7.$$

We denote the standard inner product of $\mathbb{R}^7$ and the norm by same symbols $(,)$ and $|\cdot|$. We define the multiplication $xy \in \mathbb{R}^7$ between $x, y \in \mathbb{R}^7$ satisfying $(x,y) = 0$ as the expression of the multiplication between $x, y \in \mathbb{O}_0$. Let $x = {}^t(x_1, \cdots, x_7)$ and ${}^t y = (y_1, \cdots, y_7)$ satisfy $(x,y) = 0$. Then, $xy \in \mathbb{R}^7$ is given by

$$xy = \begin{pmatrix} (x_0y_1 + x_1y_0) + (x_2y_3 - x_3y_2) + (x_4y_5 - x_5y_4) + (x_6y_7 - x_7y_6) \\ (x_0y_2 + x_2y_0) + (x_3y_1 - x_1y_3) + (x_6y_4 - x_4y_6) + (x_5y_7 - x_7y_5) \\ (x_0y_3 + x_3y_0) + (x_1y_2 - x_2y_1) + (x_4y_7 - x_7y_4) + (x_5y_6 - x_6y_5) \\ (x_0y_4 + x_4y_0) + (x_5y_1 - x_1y_5) + (x_2y_6 - x_6y_2) + (x_7y_3 - x_3y_7) \\ (x_0y_5 + x_5y_0) + (x_1y_4 - x_4y_1) + (x_7y_2 - x_2y_7) + (x_6y_3 - x_3y_6) \\ (x_0y_6 + x_6y_0) + (x_7y_1 - x_1y_7) + (x_4y_2 - x_2y_4) + (x_3y_5 - x_5y_3) \\ (x_0y_7 + x_7y_0) + (x_1y_6 - x_6y_1) + (x_2y_5 - x_5y_2) + (x_3y_4 - x_4y_3) \end{pmatrix} \in \mathbb{R}^7.$$

In these notations, the embedding $G_2 \subset SO(7)$ is characterized as follows [25]. For $g \in M(7, \mathbb{R})$, we denote the i-th low by g_i.

Proposition 3.1.2. ([25], Lemma 120) The following is true:

$$\begin{aligned} G_2 &= \left\{ g = (g_1, g_2, \cdots, g_7) \in SO(7); \quad g_3 = g_1g_2, \ g_5 = g_1g_4, \ g_6 = g_4g_2, \ g_7 = g_3g_4 \right\} \\ &= \left\{ g = (g_1, g_2, \cdots, g_7) \in M(7, \mathbb{R}); \ \begin{array}{r} |g_1| = |g_2| = |g_4| = 1 \\ (g_1, g_2) = (g_2, g_4) = (g_4, g_1) = (g_1g_2, g_4) = 0 \\ g_3 = g_1g_2, \ g_5 = g_1g_4, \ g_6 = g_4g_2, \ g_7 = g_3g_4 \end{array} \right\}. \end{aligned}$$

Let $\mathfrak{g}_2$ be the Lie algebra of G_2 and $\mathfrak{o}(7)$ be the set of all 7×7 real skew symmetric matrices. Then, $\mathfrak{g}_2 \subset \mathfrak{o}(7)$ is given as follows [26]. Let E_{ij} be the matrix whose (i,j)-component is 1 and the other components are 0. We set $G_{ij} = E_{ij} - E_{ji}$ for $1 \le i < j \le 7$.

Proposition 3.1.3. ([26], Theorem 1.4.3) The following $\{V_1, \cdots, V_{14}\}$ is a basis of $\mathfrak{g}_2$:

$$\begin{array}{lllll} V_1 = G_{23} - G_{67}, & V_2 = G_{45} - G_{67}, & V_3 = G_{13} + G_{57}, & V_4 = G_{46} + G_{57}, & V_5 = G_{12} - G_{56}, \\ V_6 = G_{47} - G_{56}, & V_7 = G_{15} - G_{37}, & V_8 = G_{26} + G_{37}, & V_9 = G_{14} + G_{36}, & V_{10} = G_{27} - G_{36}, \\ V_{11} = G_{17} + G_{35}, & V_{12} = G_{24} + G_{35}, & V_{13} = G_{16} - G_{34}, & V_{14} = G_{25} - G_{34}. & \end{array}$$

Let $p_1 \in G_2$ be as follows:

$$p_1 = \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& -1 &&& \\ &&&& -1 && \\ &&&&& -1 & \\ &&&&&& -1 \end{pmatrix},$$

where empty componets are 0. We set $K_{p_1} = \{g \in G_2 \ ; \ gp_1 = p_1g\}$. Then, $K_{p_1} \cong SO(4)$ is known [26]. In the following, we denote K_{p_1} by $SO(4)$.

3.1.2 Polars and $G_2/SO(4)$

It is known that any compact Lie group G is a compact symmetric space with respect to a biinvariant metric and the geodesic symmetry s_g at $g \in G$ is given by $s_g : G \to G; h \mapsto gh^{-1}g$. Hence, we see that G_2 is a compact symmetric space.

We recall the fundamental property of polars from [3].

Lemma 3.1.4. ([3], Lemma 2.4) Let M be a symmetric space and M_o^+ be a polar of $o \in M$. Let K be the isotropy subgroup of the isometry group of M at o. If $p \in M_o^+$, then $M_o^+ = K(p)$. In particular, if M is a compact Lie group and o is the unit element, then a polar M_o^+ of o is given by $M_o^+ = \bigcup_{g \in M} gpg^{-1}$ for any $p \in M_o^+$. Let $M_p = \{g \in M \ ; \ gp = pg\}$. Then, $M_o^+ = M/M_p$.

We see polars on a compact Lie group G. Let e be the unit element of G. Then, $F(s_e, G)$ is given by $\{g \in G \ ; \ g^2 = e\}$. Moreover, each polar of e is given by a G-orbit in $F(s_e, G)$, where G-action on $F(s_e, G)$ is as follows : for any $g \in G$ and $x \in F(s_e, G)$, $g(x) = gxg^{-1}$. In other words, for each polar M_e^+ of e there is some $x \in F(s_e, G)$ such that $M_e^+ = \bigcup_{g \in G} gxg^{-1}$.

Let $o = I_7 \in G_2$ be the identity matrix of size 7. Polars of G_2 is studied by Nagano [11].

Proposition 3.1.5. ([11]) The number of polars of o in G_2 is 2. One is $\{o\}$ and the other is $M = \bigcup_{g \in G_2} gp_1g^{-1}$. In partcular, $M \cong G_2/SO(4)$ since $K_{p_1} \cong SO(4)$.

M is an realization of the compact symmetric space $G_2/SO(4)$. In this article, we study Morse functions of M. Next, we recall some results about maximal antipodal sets of $G_2/SO(4)$ from [20]. We set $p_2, \cdots, p_7 \in M$ as follows:

$$p_2 = \begin{pmatrix} 1 &&&&&& \\ & -1 &&&&& \\ && -1 &&&& \\ &&& 1 &&& \\ &&&& 1 && \\ &&&&& -1 & \\ &&&&&& -1 \end{pmatrix}, p_3 = \begin{pmatrix} -1 &&&&&& \\ & 1 &&&&& \\ && -1 &&&& \\ &&& 1 &&& \\ &&&& -1 && \\ &&&&& 1 & \\ &&&&&& -1 \end{pmatrix}, p_4 = \begin{pmatrix} -1 &&&&&& \\ & -1 &&&&& \\ && 1 &&&& \\ &&& -1 &&& \\ &&&& 1 && \\ &&&&& 1 & \\ &&&&&& -1 \end{pmatrix},$$

$$p_5 = \begin{pmatrix} 1 &&&&&& \\ & -1 &&&&& \\ && -1 &&&& \\ &&& -1 &&& \\ &&&& -1 && \\ &&&&& 1 & \\ &&&&&& 1 \end{pmatrix}, p_6 = \begin{pmatrix} -1 &&&&&& \\ & 1 &&&&& \\ && -1 &&&& \\ &&& -1 &&& \\ &&&& 1 && \\ &&&&& -1 & \\ &&&&&& 1 \end{pmatrix}, p_7 = \begin{pmatrix} -1 &&&&&& \\ & -1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& -1 && \\ &&&&& -1 & \\ &&&&&& 1 \end{pmatrix}.$$

Proposition 3.1.6. ([20]) $A = \{p_1, \cdots, p_7\}$ is a great antipodal set of M and for any maximal antipodal set B of M there is some $g \in G_2$ such that $B = gAg^{-1}$. In particular, $\#_2 M = 7$.

3.2 Height functions and the set of critical points

3.2.1 Height functions of M

In this section, we consider height functions of M with respect to the embedding $M \subset G_2 \subset M(7, \mathbb{R})$. We define the inner product $\langle , \rangle$ of $M(7, \mathbb{R})$ as follows:

$$\langle X, Y \rangle = \mathrm{tr}({}^t XY) \quad \text{for } X, Y \in M(7, \mathbb{R}).$$

Definition 3.2.1. We set the function $h_A : M(7, \mathbb{R}) \to \mathbb{R}; B \mapsto \langle A, B \rangle$ for $A \in M(7, \mathbb{R})$. We call h_A the height function with respect to A.

We consider the following lemma which is proved in [9].

Lemma 3.2.2. ([9], Lemma 5) There are positive number $0 < c_1 < c_2 < c_4$ satisfying the following conditions:
(1) $2c_1 < c_2, \ 2c_2 < c_4$

(2) For any $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$,

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 \equiv 0 \mod 2\pi \\ c_1 \sin\theta_1 = c_2 \sin\theta_2 = c_4 \sin\theta_3 \end{cases} \implies \theta_1 \equiv \theta_2 \equiv \theta_3 \equiv 0 \mod \pi$$

Let c_1, c_2, c_4 be such positive numbers and

$$X = \begin{pmatrix} c_1 & & & & & \\ & c_2 & & & & \\ & & 0 & & & \\ & & & c_4 & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix} \in M(7, \mathbb{R}).$$

We define $f : M \to \mathbb{R}$ as $f = h_X|_M$. In this article, we will prove that f is a $\mathbb{Z}_2$-perfect Morse function and the set of all critical points of f is a great antipodal set of M. Denote the set of all critical points of f by $C(f)$. We will decide $C(f)$ in this subsection.

Remark 3.2.3. It is known that the restriction of h_X to G_2 is a $\mathbb{Z}_2$-perfect Morse function and the set of all critical points is a great antipodal set of G_2 ([25] [20]).

Before we decide $C(f)$, we study properties of fixed points of the action of each $\exp(tV_i)(1 \leq i \leq 14, t \in \mathbb{R})$. For $p \in M$, we set $K_p = \{g \in G_2; gpg^{-1} = p\}$ and denote the Lie algebra of K_p by $\mathfrak{k}_p$.

Proposition 3.2.4. For $V_i \in \mathfrak{g}_2$ and $p = (p_{ij})_{1 \leq i,j \leq 7} \in M$, followings are true.

(1) $V_1 \in \mathfrak{k}_p \Longrightarrow p_{23} = 0$
(2) $V_2 \in \mathfrak{k}_p \Longrightarrow p_{45} = 0$
(3) $V_3 \in \mathfrak{k}_p \Longrightarrow p_{13} = 0$
(4) $V_4 \in \mathfrak{k}_p \Longrightarrow p_{46} = 0$
(5) $V_5 \in \mathfrak{k}_p \Longrightarrow p_{12} = 0$
(6) $V_6 \in \mathfrak{k}_p \Longrightarrow p_{47} = 0$
(7) $V_7 \in \mathfrak{k}_p \Longrightarrow p_{15} = 0$
(8) $V_8 \in \mathfrak{k}_p \Longrightarrow p_{26} = 0$
(9) $V_9 \in \mathfrak{k}_p \Longrightarrow p_{14} = 0$
(10) $V_{10} \in \mathfrak{k}_p \Longrightarrow p_{27} = 0$
(11) $V_{11} \in \mathfrak{k}_p \Longrightarrow p_{17} = 0$
(12) $V_{12} \in \mathfrak{k}_p \Longrightarrow p_{24} = 0$
(13) $V_{13} \in \mathfrak{k}_p \Longrightarrow p_{16} = p_{34} = 0$
(14) $V_{14} \in \mathfrak{k}_p \Longrightarrow p_{25} = p_{34} = 0$

Proof. We will prove (1). The others may be proved by the similar way. Since $V_1 \in \mathfrak{k}_p$, it follows that $\exp(tV_1)A = A\exp(tV_1)$. In particular, $\exp(\pi V_1)A = A\exp(\pi V_1)$ and $\exp(\frac{\pi}{2}V_1)A = A\exp(\frac{\pi}{2}V_1)$, so we easily see that

$$\begin{pmatrix} p_{32} & p_{33} \\ -p_{22} & -p_{23} \end{pmatrix} = \begin{pmatrix} -p_{23} & p_{22} \\ -p_{33} & p_{32} \end{pmatrix}.$$

Hence, $p_{23} = -p_{32}$. Since $p \in M$, p is a symmetric matrix and $p_{23} = p_{32}$. Thus, $p_{23} = 0$.

□

3.2.2 A necessary condition for $p \in C(f)$

Let Δ be as follows:

$$\Delta = \left\{ p \in M \ ; \ \frac{d}{dt}\bigg|_{t=0} f(\exp(tV_i)\ p\ \exp(-tV_i)),\ (1 \leq i \leq 14) \right\}.$$

We easily see $C(f) \subset \Delta$. It is hard to decide $C(f)$ directly, so we firstly consider Δ.

Proposition 3.2.5. The followings are true for any $p = (p_{ij})_{1 \leq i,j \leq 7} \in \Delta$.

(1) $p_{23}=0$ (4) $p_{46}=0$ (7) $p_{15}=0$ (10) $p_{27}=0$

(2) $p_{45}=0$ (5) $p_{12}=0$ (8) $p_{26}=0$ (11) $p_{17}=0$

(3) $p_{13}=0$ (6) $p_{47}=0$ (9) $p_{14}=0$ (12) $p_{24}=0$.

Proof. We will prove (1). We may prove $(2),\cdots,(12)$ by the similar way. If $V_1\in\mathfrak{k}_p$, then (1) is true by Proposition 3.2.4. Hence, we assume $V_1\notin\mathfrak{k}_p$. Since p is a critical point of f and $\exp(tV_1)\ p\ \exp(-tV_1)(t\in\mathbb{R})$ is a curve on M,

$$\frac{d}{dt}\bigg|_{t=0} f(\exp(tV_1)\ p\ \exp(-tV_1))=0.$$

Moreover, it follows that

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} f(\exp(tV_1)\ p\ \exp(-tV_1)) &= \frac{d}{dt}\bigg|_{t=0}\langle X,\exp(tV_1)\ p\ \exp(-tV_1)\rangle\\ &= f(\frac{d}{dt}\bigg|_{t=0}\exp(tV_1)\ p\ \exp(-tV_1)))\\ &= f(V_1p-pV_1)\\ &= c_2(p_{32}+p_{23})=2c_2p_{23}.\end{aligned}$$

By the definition, $c_2\neq 0$. Therefore, $p_{23}=0$. Thus, (1) is true.

□

Proposition 3.2.6. It is true that for any $p=(p_{ij})_{1\le i,j\le 7}\in\Delta$,

$$p_{16}=p_{25}=p_{34}=0 \text{ or } \begin{cases} c_1a_{16}+c_4p_{34}=0\\ c_2p_{25}+c_4p_{34}=0\end{cases}.$$

Proof. To proof this proposition we consider the following four cases: (1) $V_{13}\in\mathfrak{k}_p$ and $V_{14}\in\mathfrak{k}_p$, (2) $V_{13}\in\mathfrak{k}_p$ and $V_{14}\notin\mathfrak{k}_p$, (3) $V_{13}\notin\mathfrak{k}_p$ and $V_{14}\in\mathfrak{k}_p$, (4) $V_{13}\notin\mathfrak{k}_p$ and $V_{14}\notin\mathfrak{k}_p$.

(1) In this case, $p_{16}=p_{25}=p_{34}=0$ from Proposition 3.2.4

(2) From Proposition 3.2.4, $p_{16}=p_{34}=0$. Moreover, we see easily $\frac{d}{dt}\big|_{t=0} f(\exp tV_{14}(p))=2(c_2p_{25}+c_4p_{34})$. Hence, $p_{25}=0$ because $c_2\neq 0$.

(3) We see $p_{16}=p_{25}=p_{34}=0$ in the similar way to (2).

(4) Since $\frac{d}{dt}\big|_{t=0} f(\exp tV_{13}(p))=2(c_1p_{16}+c_4p_{34})$ and $\frac{d}{dt}\big|_{t=0} f(\exp tV_{14}(p))=2(c_2p_{25}+c_4p_{34})$, we see that the proposition is true.

□

From Proposition 3.2.5, Proposition 3.2.6 and the characterization of $G_2\subset SO(7)$, we obtain the following proposition.

Proposition 3.2.7. It is true that

$$p\in\Delta=\left\{\begin{pmatrix} p_{11} & & & & & p_{16} & \\ & p_{22} & & & p_{25} & & \\ & & p_{33} & p_{34} & & & \\ & & p_{34} & p_{44} & & & \\ & p_{25} & & & p_{55} & & \\ p_{16} & & & & & p_{66} & \\ & & & & & & p_{77}\end{pmatrix}\in M\ ;\ \begin{matrix}(p_{16},p_{25},p_{34})=(0,0,0)\\ \text{or}\\ \begin{cases} c_1p_{16}+c_4p_{34}=0\\ c_2p_{25}+c_4p_{34}=0\end{cases}\end{matrix}\right\},$$

where empty componets are 0.

We set Δ_1 and Δ_2 as follows:

$$\Delta_1 = \left\{ \begin{pmatrix} \cos\theta_1 & & & & & \sin\theta_1 & \\ & \cos\theta_2 & & & \sin\theta_2 & & \\ & & \cos\theta_3 & \sin\theta_3 & & & \\ & & \sin\theta_3 & -\cos\theta_3 & & & \\ & \sin\theta_2 & & & -\cos\theta_2 & & \\ \sin\theta_1 & & & & & -\cos\theta_1 & \\ & & & & & & -1 \end{pmatrix} \in \Delta \ ; \ \theta_i \in \mathbb{R} \right\}$$

$$\Delta_2 = \left\{ \begin{pmatrix} \epsilon_1 & & & & & & \\ & \epsilon_2 & & & & & \\ & & \epsilon_3 & & & & \\ & & & \epsilon_3 & & & \\ & & & & \epsilon_2 & & \\ & & & & & \epsilon_1 & \\ & & & & & & 1 \end{pmatrix} \in \Delta \ ; \ \epsilon_i = \pm 1 \right\} = \{p_5, p_6, p_7\}.$$

Then, we see $\Delta = \Delta_1 \sqcup \Delta_2$ easily. The following lemma about Δ_1 is true.

Lemma 3.2.8. $\Delta_1 = \{p_1, p_2, p_3, p_4\}$

Proof. By $\Delta_1 \subset G_2$ and the characterization of $G_2 \subset O(7)$, we see

$$\Delta_1 = \left\{ \begin{pmatrix} \cos\theta_1 & & & & & \sin\theta_1 & \\ & \cos\theta_2 & & & \sin\theta_2 & & \\ & & \cos\theta_3 & \sin\theta_3 & & & \\ & & \sin\theta_3 & -\cos\theta_3 & & & \\ & \sin\theta_2 & & & -\cos\theta_2 & & \\ \sin\theta_1 & & & & & -\cos\theta_1 & \\ & & & & & & -1 \end{pmatrix} \in O(7) \ ; \ \begin{matrix} \theta_1 + \theta_2 + \theta_3 \equiv 0 \mod 2\pi \\ (\sin\theta_1, \sin\theta_2, \sin\theta_3) = (0,0,0) \\ \text{or} \\ c_1 \sin\theta_1 = c_2 \sin\theta_2 = c_4 \sin\theta_3 \end{matrix} \right\}.$$

By the definition of c_1, c_2, c_4, we see $\sin\theta_1 = \sin\theta_2 = \sin\theta_3 = 0$. Thus, we see $\Delta_1 = \{p_1, \cdots, p_4\}$.

□

Summarizing results of this subsection, we obtain $\Delta = \{p_1, p_2, \cdots, p_7\}$. In particular, the following proposition is true.

Proposition 3.2.9. $C(f) \subset \{p_1, p_2, \cdots, p_7\}$.

3.2.3 critical points of f

In this subsecion, we will show $C(f) = \{p_1, \cdots, p_7\}$. We denote the gradient vector field of f on M by $\mathrm{grad}^M f$. We will prove $\mathrm{grad}^M f_{p_i} = 0$ for each p_i $(1 \leq i \leq 7)$. Let $p \in M$ and $\pi_p : T_pM(7, \mathbb{R}) \to T_pM$ be the orthogonal projection with respect to $\langle , \rangle$. Then, $\mathrm{grad}^M f_p$ is given by $\pi_p(\mathrm{grad}(h_X)_p)$, where $\mathrm{grad}(h_X)$ is the gradient vector field of h_X on $M(7, \mathbb{R})$. In particular, since h_X is a height function, $\mathrm{grad}(h_X)_A = X$ for any $A \in M(7, \mathbb{R})$. Hence, $\mathrm{grad}^M f_p = \pi_p(X)$.

To prove $\mathrm{grad}^M f_{p_i} = 0$ for each p_i $(1 \leq i \leq 7)$ we show that X is perpendicular to $T_{p_i}M$ with respect to the inner product $\langle , \rangle$ in $T_pM(7, \mathbb{R}) = M(7, \mathbb{R})$. For sake of this, it is sufficient to prove that X is perpendicular to $T_{p_i}G_2$ because $T_{p_i}M \subset T_{p_i}G_2$. We see

$$p_i X \in \left\{ \begin{pmatrix} \pm c_1 & & & & & \\ & \pm c_2 & & & & \\ & & 0 & & & \\ & & & \pm c_4 & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix} \right\}$$

by the definition of p_i $(1 \leq i \leq 7)$. Therefore,

$$\langle p_i X, E_{kl} \rangle = 0$$

for any $1 \leq k < l \leq 7$. The tangent space $T_{p_i}G_2$ is given by $\sum_{j=1}^{14} \mathbb{R}p_iV_j$. By the definition of V_j $(1 \leq j \leq 14)$, we see that X is perpendicular to $T_{p_i}G_2$ for any p_i $(1 \leq i \leq 7)$ since $\langle p_iA, p_iB \rangle = \langle A, B \rangle$ $(A, B \in M(7, \mathbb{R}))$ and $p_i^2 = I_7$. Thus, $\{p_1, \cdots, p_7\} \subset C(f)$. Summarizing this result and Prop 3.2.9, we obtain the following theorem.

Theorem 3.2.10. $C(f) = \{p_1, \cdots, p_7\}$

3.3 Morse functions of $G_2/SO(4)$

Let $n_f(p_i)$ be the number of negative eigenvalues of the Hessian matrix of f at p_i, that is the index of f at p_i. In this section, we calculate eigenvalues of the Hessian matrix of f at p_i and $n_f(p_i)$ for each p_i $(1 \leq i \leq 7)$ and prove that f is a Morse function. We put $U_j = V_{j+6}$ for $1 \leq j \leq 8$. We see $\mathfrak{k}_{p_1} = \sum_{j=1}^{6} \mathbb{R}V_i$ easily. Hence, $T_{p_1}M \cong \sum_{i=1}^{8} \mathbb{R}U_j$. In particular, the map

$$\phi_1 : \mathbb{R}^8 \longrightarrow M; (t_1, \cdots, t_8) \mapsto \exp t_1U_1 \cdots \exp t_8U_8 \; p_1 \; \exp(-t_8U_8) \cdots \exp(-t_1U_1)$$

is a local coodinate around p_1 when $|t_1|, \cdots, |t_8|$ are sufficiently small. Moreover, we set $g_1 = I_7$ and $g_2, \cdots, g_7 \in G_2$ as follows:

$$g_2 = \begin{pmatrix} 1 & & & & & & \\ & 0 & & 1 & & & \\ & & 0 & & 1 & & \\ & 1 & & 0 & & & \\ & & 1 & & 0 & & \\ & & & & & -1 & \\ & & & & & & -1 \end{pmatrix}, \; g_3 = \begin{pmatrix} 0 & & & 1 & & & \\ & 1 & & & & & \\ & & 0 & & & 1 & \\ 1 & & & 0 & & & \\ & & & & -1 & & \\ & & 1 & & & 0 & \\ & & & & & & -1 \end{pmatrix}, \; g_4 = \begin{pmatrix} 0 & & & & 1 & & \\ & 0 & & & & 1 & \\ & & 1 & & & & \\ & & & -1 & & & \\ 1 & & & & 0 & & \\ & 1 & & & & 0 & \\ & & & & & & -1 \end{pmatrix}$$

$$g_5 = \begin{pmatrix} 1 & & & & & & \\ & 0 & & & & 1 & \\ & & 0 & & & & 1 \\ & & & -1 & & & \\ & & & & -1 & & \\ & 1 & & & & 0 & \\ & & 1 & & & & 0 \end{pmatrix}, \; g_6 = \begin{pmatrix} 0 & & & & 1 & & \\ & -1 & & & & & \\ & & 0 & & & & 1 \\ & & & -1 & & & \\ 1 & & & & 0 & & \\ & & & & & 1 & \\ & & 1 & & & & 0 \end{pmatrix}, \; g_7 = \begin{pmatrix} 0 & & & 1 & & & \\ & 0 & & & & & 1 \\ & & 1 & & & & \\ 1 & & & 0 & & & \\ & & & & -1 & & \\ & & & & & -1 & \\ & 1 & & & & & 0 \end{pmatrix}.$$

Then, we see that the following lemma is true by calculations.

Lemma 3.3.1. $p_i = g_i p_1 g_i^{-1}$ for any $1 \leq i \leq 7$.

Since G_2 acts on M transitively, we see that the following map ϕ_i is a local coodinate around p_i when $|t_1|, \cdots, |t_8|$ are sufficiently small:

$$\phi_i : \mathbb{R}^8 \to M; (t_1, \cdots, t_8) \mapsto g_i \; (\phi_1(t_1, \cdots, t_8)) \; g_i^{-1}.$$

We set $f_{kl}(p_i) = \left.\frac{\partial^2}{\partial t_k \partial t_l}\right|_{t=0} f(\phi_i(t_1, \cdots, t_8))$ for $1 \leq k, l \leq 8$. Then, the matrix $(f_{kl}(p_i))_{1 \leq k,l \leq 8}$ is the Hessian matrix of f at p_i with respect to the local coodinate ϕ_i. Denote $(f_{kl}(p_i))_{1\leq k,l\leq 8}$ by Hf_{p_i}. Then, we see

$$\begin{aligned} f_{kl}(p_i) &= f\left(\left.\frac{\partial^2}{\partial t_k \partial t_l}\right|_{t=0} \phi_i(t_1,\cdots,t_8)\right) \\ &= f\left(\frac{\partial^2}{\partial t_k \partial t_l} g_i \ \mathrm{exp} t_1 U_1 \cdots \mathrm{exp} t_8 U_8 \ p_1 \ \mathrm{exp}(-t_8)U_8 \cdots \mathrm{exp}(-t_1)U_1 \ g_i^{-1}\Big|_{t=0}\right) \\ &= f\Big(g_i(U_kU_lp_1 - U_kp_1U_l - U_lp_1U_k + p_1U_lU_k)g_i^{-1}\Big). \end{aligned}$$

Followings are true for any $1 \leq k, l \leq 8$:

$$\begin{aligned} &{}^t(U_kU_lp_1) = {}^tp_1{}^tU_l{}^tU_k = p_1(-U_l)(-U_k) = p_1U_lU_k \\ &{}^t(U_lp_1U_k) = {}^tU_k{}^tp_1{}^tU_l = (-U_k)p_1(-U_l) = U_kp_1U_l \\ &p_1U_k = -U_kp_1. \end{aligned}$$

Thus, we obtain $f_{kl}(p_i) = f\Big(2g_i(U_kU_lp_1 + {}^t(U_kU_lp_1))g_i^{-1}\Big)$. Hence, we obtain Hf_{p_i} as follows:

$$\mathrm{H}f_{p_1} = \begin{pmatrix} -4c_1 &&&&&&& \\ & -4c_2 &&&&&& \\ && 4(c_4-c_1) &&&&& \\ &&& -4c_2 &&&& \\ &&&& -4c_1 &&& \\ &&&&& 4(c_4-c_2) && \\ &&&&&& 4(c_4-c_1) & 4c_4 \\ &&&&&& 4c_4 & 4(c_4-c_2) \end{pmatrix}$$

$$\mathrm{H}f_{p_2} = \begin{pmatrix} -4c_1 &&&&&&& \\ & -4c_4 &&&&&& \\ && 4(c_2-c_1) &&&&& \\ &&& -4c_4 &&&& \\ &&&& -4c_1 &&& \\ &&&&& 4(c_2-c_4) && \\ &&&&&& 4(-c_1+c_2) & 4c_2 \\ &&&&&& 4c_2 & 4(c_2-c_4) \end{pmatrix}$$

$$\mathrm{H}f_{p_3} = \begin{pmatrix} -4c_4 &&&&&&& \\ & -4c_2 &&&&&& \\ && 4(c_1-c_4) &&&&& \\ &&& -4c_2 &&&& \\ &&&& -4c_4 &&& \\ &&&&& 4(c_1-c_2) && \\ &&&&&& 4(c_1-c_4) & 4c_1 \\ &&&&&& 4c_1 & 4(c_1-c_2) \end{pmatrix}$$

$$\mathrm{H}f_{p_4} = \begin{pmatrix} 4c_1 &&&&&&& \\ & 4c_2 &&&&&& \\ && 4(c_2+c_4) & -4c_2 &&&& \\ && -4c_2 & 4c_2 &&&& \\ &&&& 4c_1 & 4c_1 && \\ &&&& 4c_1 & 4(c_1+c_2) && \\ &&&&&& 4(c_2+c_4) & 4c_4 \\ &&&&&& 4c_4 & 4(c_1+c_4) \end{pmatrix}$$

$$
\mathrm{H}f_{p_5} = \begin{pmatrix}
-4c_1 & & & & & & & \\
& 4c_2 & & & & & & \\
& & 4(-c_1+c_2+c_4) & -4c_2 & & & & \\
& & -4c_2 & 4c_2 & & & & \\
& & & & -4c_1 & & & \\
& & & & & 4c_4 & & \\
& & & & & & 4(-c_1+c_2+c_4) & 4c_4 \\
& & & & & & 4c_4 & 4c_4
\end{pmatrix}
$$

$$
\mathrm{H}f_{p_6} = \begin{pmatrix}
4c_1 & & & & & & & \\
& -4c_2 & & & & & & \\
& & 4c_4 & & & & & \\
& & & -4c_2 & & & & \\
& & & & 4c_1 & 4c_1 & & \\
& & & & 4c_1 & 4(c_1-c_2+c_4) & & \\
& & & & & & 4c_4 & 4c_4 \\
& & & & & & 4c_4 & 4(c_1-c_2+c_4)
\end{pmatrix}
$$

$$
\mathrm{H}f_{p_7} = \begin{pmatrix}
4(c_2-c_4) & -4c_2 & & & & & & \\
-4c_2 & 4c_2 & & & & & & \\
& & 4(c_1-c_4) & & & & & \\
& & & 4c_2 & & & & \\
& & & & 4(c_2-c_4) & & & \\
& & & & & 4c_1 & & \\
& & & & & & 4(c_1-c_4) & 4c_1 \\
& & & & & & 4c_1 & 4c_1
\end{pmatrix}
$$

Eigenvalues of Hf_{p_i} are as follows:

eigenvalues of $\mathrm{H}f_{p_1}$: $-4c_1, -4c_1, -4c_2, -4c_2, 4(c_4-c_1), 4(c_4-c_2), \lambda_1^{\pm} = 2\Big((c_4-c_1)+(c_4-c_2)\pm\sqrt{4c_4^2+(c_2-c_1)^2}\Big)$

eigenvalues of $\mathrm{H}f_{p_2}$: $-4c_1, -4c_1, 4(c_2-c_1), -4c_4, -4c_4, 4(c_2-c_4), \lambda_2^{\pm} = 2\Big((c_2-c_1)+(c_2-c_4)\pm\sqrt{4c_2^2+(c_4-c_1)^2}\Big)$

eigenvalues of $\mathrm{H}f_{p_3}$: $-4c_2, -4c_2, -4c_4, -4c_4, 4(c_1-c_2), 4(c_1-c_4), \lambda_3^{\pm} = 2\Big((c_1-c_2)+(c_1-c_4)\pm\sqrt{4c_1^2+(c_4-c_2)^2}\Big)$

eigenvalues of $\mathrm{H}f_{p_4}$: $4c_1, 4c_2, \lambda_4^{\pm} = 2\Big((2c_2+c_4)\pm\sqrt{4c_2^2+c_4^2}\Big)$,

$$\mu_4^{\pm} = 2\Big((2c_1+c_2)\pm\sqrt{4c_1^2+c_2^2}\Big), \nu_4^{\pm} = 2\Big((c_1+c_2+2c_4)\pm\sqrt{(c_2-c_1)^2+4c_4^2}\Big)$$

eigenvalues of $\mathrm{H}f_{p_5}$: $-4c_1, -4c_1, 4c_2, 4c_4$,

$$\lambda_5^{\pm} = 2\Big(2c_2+(c_4-c_1)\pm\sqrt{4c_2^2+(c_4-c_1)^2}\Big), \mu_5^{\pm} = 2\Big(2c_4+c_2-c_1\pm\sqrt{4c_4^2+(c_2-c_1)^2}\Big)$$

eigenvalues of $\mathrm{H}f_{p_6}$: $4c_1, -4c_2, -4c_2, 4c_4$,

$$\lambda_6^{\pm} = 2\Big(2c_1+c_4-c_2\pm\sqrt{4c_1^2+(c_4-c_2)^2}\Big), \mu_6^{\pm} = 2\Big(2c_4+c_1-c_2\pm\sqrt{4c_4^2+(c_2-c_1)^2}\Big)$$

eigenvalues of $\mathrm{H}f_{p_7}$: $4c_1, 4c_2, 4(c_1-c_4), 4(c_2-c_4)$,

$$\lambda_7^{\pm} = 2\Big(2c_2-c_4\pm\sqrt{4c_2^2+c_4^2}\Big), \mu_7^{\pm} = 2\Big(2c_1-c_4\pm\sqrt{4c_1^2+c_4^2}\Big)$$

Lemma 3.3.2. Followings are true:

$(1): \lambda_1^+ > 0, \lambda_1^- < 0$ $(2): \lambda_2^+ > 0, \lambda_2^- < 0$ $(3): \lambda_3^{\pm} < 0$ $(4): \lambda_4^{\pm}, \mu_4^{\pm}, \nu_4^{\pm} > 0$

$(5): \lambda_5^{\pm}, \mu_5^{\pm} > 0$ $(6): \lambda_6^{\pm}, \mu_6^{+} > 0, \mu_6^{-} < 0$ $(7): \lambda_7^{+}, \mu_7^{+} > 0, \lambda_7^{-}, \mu_7^{-} < 0$

Proof. (1) $\lambda_1^{\pm}$ are solutions of $x^2+4(2c_4-c_1-c_2)x+16(c_1c_2-c_1c_4-c_2c_4)=0$ with respect to x. Then, $c_1c_2-c_1c_4-c_2c_4 < c_1(c_2-c_4) < 0$ since $0 < c_1 < c_2 < c_4$. Thus, $\lambda_1^- < 0,\ \lambda_1^+ > 0$.

(2) $\lambda_2^\pm$ are solutions of $x^2+4(2c_2-c_1-c_4)x+16(c_1c_4-c_1c_2-c_2c_4)=0$. Then, $c_1c_4-c_1c_2-c_2c_4<(c_1-c_2)c_4<0$ since $0<c_1<c_2<c_4$. Thus, $\lambda_2^-<0,\ \lambda_2^+>0$.

(3) $\lambda_3^\pm$ are solutions of $x^2-4(2c_1-c_2-c_4)x+16(c_2c_4-c_1c_2-c_1c_4)=0$. Then, $c_2c_4-c_1c_2-c_1c_4>(c_2-2c_1)c_4>0$ since $2c_1<c_2$, and $2c_1-c_2-c_4=(c_1-c_2)+(c_1-c_4)<0$ since $0<c_1<c_2<c_4$. Thus, $\lambda_3^\pm<0$.

(4) $\lambda_4^\pm$, $\mu_4^\pm$ and $\nu_4^\pm$ are solutions of $x^2-4(2c_2+c_4)x+16c_2c_4=0$, $x^2-4(2c_1+c_2)x+16c_1c_2=0$ and $x^2-4(c_1+c_2+2c_4)x+16(c_2c_4+c_1c_2+c_1c_4)=0$. Then, $2c_2+c_4,\ 2c_1+c_2,\ c_1+c_2+2c_4,\ c_2c_4,\ c_1c_2,\ (c_1c_2+c_1c_4+c_2c_4)>0$ by the definition of c_1,c_2,c_4. Thus, $\lambda_4^\pm,\mu_4^\pm,\nu_4^\pm>0$.

(5) $\lambda_5^\pm$ and $\mu_5^\pm$ are solutions of $x^2-4(2c_2-c_1+c_4)x+16c_2(c_4-c_1)=0$ and $x^2-4(2c_4-c_1+c_2)x+16c_4(c_2-c_1)=0$. Then, $2c_2-c_1+c_4,\ 2c_4-c_1+c_2,\ c_2(c_4-c_1),\ c_4(c_2-c_1)>0$ by the definition of c_1,c_2,c_4. Thus, $\lambda_5^\pm,\mu_5^\pm>0$.

(6) $\lambda_6^\pm$ and $\mu_6^\pm$ are solutions of $x^2-4(2c_1-c_2+c_4)x+16c_1(c_4-c_2)=0$ and $x^2-4(2c_4+c_1-c_2)x+16c_4(c_1-c_2)=0$. Then, $2c_1-c_2+c_4,\ c_1(c_4-c_2)>0$ and $c_4(c_1-c_2)<0$ by the definition of c_1,c_2,c_4. Thus, $\lambda_6^\pm,\mu_6^+>0$ and $\mu_6^-<0$.

(7) $\lambda_7^\pm$ and $\mu_7^\pm$ are solutions of $x^2-4(2c_2-c_4)x+16(-c_1c_2)=0$ and $x^2-4(2c_1-c_4)x+16(-c_1c_4)=0$. Then, $c_2c_4,c_1c_4>0$ and $2c_2-c_4,2c_1-c_4<0$ by the definition of c_1,c_2,c_4. Thus, $\lambda_7^+,\mu_7^+>0$ and $\lambda_7^-,\mu_6^-<0$.

□

Summarizing this section we obtain the following theorem.

Theorem 3.3.3. The function f is a Morse function and the set of all critical points of f is the great antipodal set $\{p_1,\cdots,p_7\}$. The index $n_f(p_i)$ of f at each p_i is as follows:

	p_1	p_2	p_3	p_4	p_5	p_6	p_7
$n_f(p_i)$	5	6	8	0	2	3	4

Let $\#C_i(f)$ be the number of critical points of f whose index are i . Then,

$$\#C_i(f)=\begin{cases}1 & (i=0,2,3,4,5,6,8)\\ 0 & \text{(others)}\end{cases} \quad \text{and} \quad \#C(f)=7.$$

In particular, $\dim_{\mathbb{Z}_2} H_i(G_2/SO(4);\mathbb{Z}_2)=\#C_i(f)$ for any $i\in\mathbb{Z}$. Therefore, f is a $\mathbb{Z}_2$-perfect Morse function.

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