On metric geometry of convergences and topological distributions of metric structures

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February 2021

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Doctoral Program in Mathematics

Submitted to the Graduate School of
Pure and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in
Science

at the

University of Tsukuba

Abstract

In this doctoral thesis, we investigate metric geometry from the viewpoints of convergences and topological distributions of metric structure. This thesis mainly consists of two parts.

The fist part is a study on the Assouad dimension and limit spaces of subspaces of metric spaces. We introduce pseudo-cones of metric spaces as generalizations of tangent and asymptotic cones of metric spaces, and provide lower estimations of the Assouad dimensions of metric spaces by the dimensions of pseudo-cones. This is a generalization of the Mackay–Tyson estimation of the Assouad dimension by tangent cones. As another application of pseudo-cones, we study subsets of full Assouad dimension of metric spaces, and we introduce the notion of a tiling space. A tiling space is a pair of a metric space and a family of subsets called tiles of the metric spaces. The class of tiling spaces contains the Euclidean spaces, the p-adic numbers, the Sierpiński gasket, and various self-similar spaces appearing in fractal geometry. As our result, for a doubling tiling space, we characterize a subspace possessing the same Assouad dimension as that of the whole space in terms of pseudo-cones and tiles. Since the Euclidean spaces are tiling spaces, this result can be considered as a generalization of the Fraser–Yu characterization of a subspace of the Euclidean space of full Assouad dimension.

The second part is a study on topological distributions of sets of “singular” metrics in spaces of metrics. We first prove an interpolation theorem of metrics adapted for investigating topologies of spaces of metrics. We introduce the notion of the transmissible property, which unifies geometric properties determined by finite subsets of metric spaces. As an application of our interpolation theorem, we prove that the sets of all metrics not satisfying transmissible properties are dense and represented as an intersection of countable open subsets of spaces of metrics. We also prove analogues of these results for ultrametric spaces. It is often expected to prove ultrametric analogues of statements on ordinary metrics. As realizations of this expectation, we first prove an isometric embedding theorem stating that every ultrametric space can be isometrically embedded into an ultra-normed module over an integral domain, which is an analogue of the Arens–Eells isometric embedding theorem. Due to this embedding theorem, as an analogue of the Hausdorff metric extension theorem, we can prove a theorem on extending an ultrametric on a closed subset to an ultrametric on the whole space, while referring to the Toruńczyk's proof of the Hausdorff metric extension theorem by the Arens–Eells isometric embedding theorem. By our extension theorem on ultrametrics, we establish an interpolation theorem on ultrametrics, and theorems on topological distributions on spaces of ultrametrics.

This doctoral thesis is written as a comprehensive paper including the contents of the author's papers [60], [56], [59], and [58]. Most of the results stated in this doctoral thesis have appeared in [60], [56], [59], and [58]. The author has added some auxiliary explanations and statements for the sake of comprehension of readers.

Contents

Chapter 1

Introduction

In this doctoral thesis, we investigate metric geometry from the viewpoints of convergences and topological distributions of metric structure. We prove theorems concerning the Assouad dimension and convergences of metric spaces, and theorems on topological distributions of metrics satisfying geometric properties.

1.1 Background

We first review backgrounds of our studies.

1.1.1 The Assouad dimensions and cones

Assouad [3, 4, 5] introduced the notion which today we call Assouad dimension for metric spaces, and studied the relation between the Assouad dimension and bi-Lipschitz embeddability into a Euclidean space of metric spaces (see Theorem 2.3.4). For a metric space, its Assouad dimension is greater or equal to its Hausdorff dimension. As is the case with the Hausdorff dimension, in general, it seems to be difficult to estimate the Assouad dimension from below. Mackay–Tyson [76] proved that if (W, h) is a tangent space of a metric space (X, d), then $\dim_A(W, h) \leq \dim_A(X, d)$, where $\dim_A$ stands for the Assouad dimension.

The Assouad dimension and its variations are studied as geometric dimensions, and several estimations of these dimensions were proven in metric geometry and coarse geometry. Le Donne and Rajala [26] gave both-sides estimations of the Assouad dimension and the Nagata dimension by tangent spaces under a certain assumption (see [26, Theroms 1.2, 1.4]). Dydak and Higes [30] provided a lower estimation of the Assouad–Nagata dimension by asymptotic cones as ultralimits (see [30, Proposition 4.1]). Note that the Nagata dimension and the Assouad–Nagata dimension are identical notions.

1.1.2 A subset of the Euclidean space of full Assouad dimension

We now introduce another geometric dimension focusing on local behavior of the Hausdorff measures. Let $\delta \in (0, \infty)$. A metric space (X, d) is said to be *Ahlfors regular of dimension* δ if (X, d) is complete and has at least two elements, and if there exists $C \in (0, \infty)$ such that for all $x \in X$ and for all $r \in (0, \delta_d(X)]$, we have $C^{-1}r^{\delta} \leq \mu_{\delta}(B(x, r)) \leq Cr^{\delta}$, where μ_{δ} is the δ-dimensional Hausdorff measure, $B(x, r)$ is the ball of (X, d), and $\delta_d(X)$ is the diameter of (X, d) (see [25]). Dyatlov–Zahl [29] observed that if a subset S of $[0, 1]$ is Ahlfors regular of dimension $\delta \in (0, 1)$, then S does not contain long arithmetic progressions (see [29, Subsection 6.1.1]).

Focusing on the Assouad dimension, Fraser–Yu [35] developed Dyatlov–Zahl's observation mentioned above on the Ahlfors regularity and arithmetic progressions. In [35], they provided a characterization of subsets of the Euclidean space possessing the same Assouad dimension as that of the whole space. Before stating their precise statement, we introduce an arithmetic patch. For $k \in \mathbb{N}$, and for $\delta \in (0, \infty)$, a subset P of $\mathbb{R}^N$ is said to be an *arithmetic patch of size* k *and scale* δ if $P = \{t + \delta \sum_{i=1}^{N} x_i e_i \mid x_i \in \{0, \dots, k-1\}\}$, where $\{e_i\}_{i=1}^{N}$ is some linear basis of $\mathbb{R}^N$. The notion of an arithmetic patch is a higher dimensional generalization of arithmetic progressions. Fraser–Yu's characterization [35] states that for every subset F of the N-dimensional Euclidean space $\mathbb{R}^N$, the following are equivalent:

(1) F asymptotically contains arbitrary large arithmetic patches; namely, for every $k \in \mathbb{N}$ and for every $\epsilon \in (0, \infty)$, there exist a positive number $\delta \in (0, \infty)$, and an arithmetic patches of size k and scale δ, and a subset E of F such that $\mathcal{H}(E, P) \leq \delta\epsilon$, where $\mathcal{H}$ is the Hausdorff metric;

(2) F satisfies the asymptotic Steinhaus property; namely, for every finite subset P of $\mathbb{R}^N$, and for every $\epsilon \in (0, \infty)$, there exists a positive number $\delta \in (0, \infty)$, and $t \in \mathbb{R}^N$, and a subset E of F such that $\mathcal{H}(E, t + \delta P) \leq \delta\epsilon$;

(3) $\dim_A F = N$;

(4) $\mathrm{Cdim}_A F = N$, where Cdim_A stands for the conformal Assouad dimension;

(5) F has a weak tangent with non-empty interior;

(6) the closed unit ball $B(0, 1)$ in $\mathbb{R}^N$ is a weak tangent to F.

The notion of the weak tangent appearing in Fraser–Yu's characterization is a specialized concept of tangent cones for the Euclidean spaces. They used this characterization to solve a problem of whether specific subsets of the Euclidean spaces related to number theory, such as the Cartesian products of the set of all prime numbers, asymptotically contain arithmetic patches. This problem is an asymptotic version of the Erdős–Turán conjecture on arithmetic progressions in sets of natural numbers.

1.1.3 Extension of metrics

For a metrizable space X, we denote by $\mathrm{M}(X)$ the set of all metrics on X that generate the same topology as the original one on X.

In 1930, Hausdorff [45] proved the extension theorem stating that for every metrizable space X, for every closed subset A of X and for every $d \in \mathrm{M}(A)$, there exists $D \in \mathrm{M}(X)$ such that $D|_{A^2} = d$ (see Theorem 2.2.3). Hausdorff [45] made use of this extension theorem to give a simple proof of the Niemytzki–Tychonoff characterization theorem [86] which states that a metrizable space X is compact if and only if all metrics in $\mathrm{M}(X)$ are complete. Independently of Hausdorff's paper [45], Bing [9] proved the extension theorem for metrics in a context of metrization by coverings of topological spaces. Bacon [6] pointed out that the Hausdorff metric extension theorem for complete metrics holds true.

The proof of Bing in [9] is done by constructing concrete extended metrics by using coverings of spaces. In [45], Hausdorff regarded his extension theorem of metrics as a theorem on extension of homeomorphisms between metrizable spaces. In [46], Hausdorff proved a theorem on extending continuous maps, which contains the result in [45] as a special case (see also [70], [1], or [55]).

Some mathematicians gave proofs of the metric extension theorem by following Hausdorff's argument in [46] of extending homeomorphism between metric spaces in order to extend metrics. Arens [1] gave a proof using the Dugundji theorem on extension of maps from metric spaces into locally convex linear spaces (see Theorem 2.7.10). Toruńczyk [113] provided a simple proof of the Hausdorff extension theorem by using the Arens–Eells embedding theorem (see Theorem 2.2.2).

As a previous research on the space $\mathrm{M}(X)$, we explain a result of a simultaneous extension of metrics. For a metrizable space X, we define a function $\mathcal{D}_X : \mathrm{M}(X)^2 \to [0, \infty]$ by

$$\mathcal{D}_X(d, e) = \sup_{(x,y)\in X^2} |d(x, y) - e(x, y)|.$$

The function $\mathcal{D}_X$ is a metric on $\mathrm{M}(X)$ valued in $[0, \infty]$. For every metrizable space X, and for every closed subset A of X, Nguyen Van Khue and Nguyen To Nhu [68] constructed a Lipschitz metric extensor from $(\mathrm{M}(A), \mathcal{D}_A)$ into $(\mathrm{M}(X), \mathcal{D}_X)$, and a monotone continuous metric extensor from $\mathrm{M}(A)$ into $\mathrm{M}(X)$; moreover, if X is completely metrizable, then each of these metric extensors maps any complete metric in $\mathrm{M}(A)$ into a complete metric in $\mathrm{M}(X)$ (see Theorem 2.2.5). To obtain such metric extensors, they used the Dugundji extension theorem. Their construction can be considered to use Hausdorff's argument of extending homeomorphisms to extend metrics.

A central idea of our interpolation theorem of metrics (Theorems 1.2.9 and 1.2.15) is based on Hausdorff's argument of extending homeomorphisms to extend metrics.

1.1.4 Theorems on topological distributions

We now introduce some notions of the theory of Baire spaces. Let X be a topological space. A subset S of X is said to be *nowhere dense* if the complement of the closure of S is dense in X. A subset S of X is said to be *of first category* or *meager* if S is the union of countable nowhere dense subsets of X. A non-meager subset is called *a set of second category*. Baire spaces are characterized as spaces whose all non-empty open subsets are of second category. A subset S of X is said to be *comeager* or *residual* if the complement of S is a meager subset of X. Note that, for a Baire space X, a subset S of X is residual if and only if S contains a dense G_δ subset. Let P be a property on elements of X. Then P is said to be *typical* or *generic* if the set $\{ x \in X \mid x \text{ satisfies } P \}$ is residual.

The celebrated Baire category theorems state that complete metrizable spaces and locally compact Hausdorff spaces are Baire (these results can be unified by the concept of Čech complete spaces) (see [92], [32], [123], [105], or [66]). After the theory of Baire spaces arose, many mathematicians have proven theorems on typicality of interesting (or singular) mathematical objects in suitable spaces. The set of all Liouville numbers is dense G_δ in $\mathbb{R}$ (see [92, Chapter 2]). The set of all normal real numbers are dense and of first category in $\mathbb{R}$ (see [99]). Thom's transversality theorem concerns typicality of certain smooth mappings between smooth manifolds (see [39] and [50]). Banach [7] and Mazurkiewicz [77] independently proved that continuous nowhere differentiable functions are typical in $C([0, 1])$ (see also [61] and [92]). Bruckner–Garg [14] studied a residual set concerning level sets of continuous real-valued functions on $[0, 1]$. Kato [63] studied topological distributions of higher dimensional Bruckner–Garg type functions from compacta into manifolds. O'Neil [89] investigated spaces of measures on the Euclidean spaces, and proved the typicality of measures possessing large tangent measures. Chen–Rossi [18] proved the typicality of compact subsets of $[0, 1]^N$ whose all tangent cones coincide with all compact subsets of $[0, 1]^N$, which can be considered as an analogue of O'Neil's result for metric spaces. The results mentioned above are only a few examples of typicality theorems.

Theorems 1.2.10 and 1.2.16, and their local versions (Theorems 1.2.11 and 1.2.17) in this thesis are typicality theorems on spaces of metrics.

1.2 Main results

1.2.1 Pseudo-cones

To give a new lower estimation of the Assouad dimension, we introduce the notion of a pseudo-cone, which can be regarded as a generalization of tangent and asymptotic cones.

In this thesis, for a metric space (X,d), and for a subset A of X, we denote the restricted metric $d|_{A^2}$ by the same symbol d as the ambient metric d.

Definition 1.2.1 ([60]). Let (X,d) be a metric space. Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of subsets of X, and let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $(0,\infty)$. We say that a metric space (P,h) is a *pseudo-cone of X approximated by $(\{A_i\}_{i\in\mathbb{N}},\{u_i\}_{i\in\mathbb{N}})$* if $\mathcal{GH}((A_i,u_id),(P,h))\to 0$ as $i\to\infty$, where $\mathcal{GH}$ is the Gromov–Hausdorff distance.

Remark 1.2.1. We emphasize that, in this thesis, we define the Gromov–Hausdorff distance between not only compact metric spaces but also non-compact ones (see Section 2.4). Thus $\mathcal{GH}$ is not necessarily a metric in the usual sense.

For instance, every closed ball centered at a based point of a tangent cone or an asymptotic cone is a pseudo-cone. Indeed, if a pointed metric space (W,h,w) is a tangent (resp. asymptotic) cone of a metric space (X,d) at p, then for every $R\in(0,\infty)$ the closed ball $(B(w,R),h)$ centered at w with radius R is a limit space of the sequence $\{(B(p_i,R/r_i),r_id)\}_{i\in\mathbb{N}}$ in the Gromov–Hausdorff topology, where $\{p_i\}_{i\in\mathbb{N}}$ is a sequence in X with $p_i\to p$ and $r_i\to\infty$ (resp. 0) as $i\to\infty$; in particular, the space $(B(w,R),h)$ is a pseudo-cone of (X,d) approximated by $(\{B(p_i,R/r_i)\}_{i\in\mathbb{N}},\{r_i\}_{i\in\mathbb{N}})$.

For a metric space (X,d), we denote by $\mathcal{PC}(X,d)$ the class of all pseudo-cones of X. By using the notion of a pseudo-cone, we can formulate a generalization of the Mackay–Tyson estimation of the Assouad dimension.

Theorem 1.2.1 ([60]). *Let (X,d) be a metric space. Then for every $(P,h)\in\mathcal{PC}(X)$ we have*

$$\dim_A(P,h)\le\dim_A(X,d).$$

We also obtain a lower estimation by using ultralimits of metric spaces. The notion of an ultralimit of metric spaces is a method of constructing a space behaving as a limit space of a sequence of metric spaces (see Subsection 2.6). Let μ be a non-principal ultrafilter on $\mathbb{N}$. For a sequence $\{(X_i,d_i,p_i)\}_{i\in\mathbb{N}}$ of pointed metric spaces, we denote by $\lim_\mu(X_i,d_i,p_i)$ the ultralimit of $\{(X_i,d_i,p_i)\}_{i\in\mathbb{N}}$ with respect to μ. The existence of an ultralimit is always guaranteed. We obtain the following ultralimit analogue of Theorem 1.2.1:

Theorem 1.2.2 ([60]). *Let (X,d) be a metric space. Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of subsets of X, and let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $(0,\infty)$. Take $a_i\in A_i$ for each $i\in\mathbb{N}$. Let μ be a non-principal ultrafilter on $\mathbb{N}$. Then we have*

$$\dim_A\left(\lim_{\mu}(A_i,u_id,a_i),D\right)\le\dim_A(X,d),$$

where D is a canonical metric on $\lim_\mu(A_i,u_id,a_i)$.

The lower Assouad dimension is a variation of the Assouad dimension, and it is used for interpolation of the Assouad dimension. We also obtain similar estimations as Theorems 1.2.1 and 1.2.2 for the lower Assouad dimension (see Theorems 4.1.5 and 4.1.6).

The conformal Assouad dimension is studied as a variation of the ordinary Assouad dimension invariant under quasi-symmetric maps in the conformal dimension theory (see [76]). By comparing the conformal Assouad dimensions of metric spaces with each other, we can distinguish their quasi-symmetric equivalent classes. In general, it seems to be quite difficult to find the exact value of the conformal Assouad dimension.

For a metric space (X, d), we denote by $\mathcal{KPC}(X, d)$ the class of all pseudo-cones approximated by a pair of a sequence $\{A_i\}_{i\in\mathbb{N}}$ of compact sets of X and a sequence $\{u_i\}_{i\in\mathbb{N}}$ in $(0, \infty)$. Namely, members of $\mathcal{KPC}(X, d)$ are pseudo-cones approximated by compact subsets. We obtain the following lower estimation of the conformal Assouad dimensions:

Theorem 1.2.3 ([60])**.** *Let (X, d) be a metric space. Then for every $(P, h) \in \mathcal{KPC}(X, d)$ we have*

$$\mathrm{Cdim}_A(P, h) \leq \mathrm{Cdim}_A(X, d),$$

where Cdim_A *stands for the conformal Assouad dimension.*

As a consequence of Theorem 1.2.3, for every metric space (X, d), we also give a lower estimation of the conformal Assouad dimension of (X, d) by the conformal Assouad dimensions of closed balls of ultralimits of scaled subsets of X (see Corollary 4.1.8).

The points of the proofs of Theorems 1.2.1, 1.2.2 and 1.2.3 are to utilize the stability of the Assouad dimension under scaling of metrics, and to extract the arguments of Mackay and Tyson [76] in their lower estimation by tangent cones.

We say that a topological space X is an $(\omega_0 + 1)$*-space* if X is homeomorphic to the one-point compactification of the countable discrete topological space. This concept is named after the ordinal space $\omega_0 + 1$. The space $\omega_0 + 1$ can be regarded as the one-point compactification of the countable discrete space ω_0. Note that an $(\omega_0 + 1)$-space has a unique accumulation point.

By making use of universal metric spaces (see Section 2.5), we construct a metric $(\omega_0 + 1)$-space containing all compact metric spaces as its pseudo-cones.

Theorem 1.2.4 ([60])**.** *There exists a metric $(\omega_0 + 1)$-space X such that $\mathcal{PC}(X)$ contains all compact metric spaces.*

A metric space is said to be a *length space* if the distance of arbitrary two points in the metric space is equal to the infimum of lengths of arcs joining the two points. A metric space is said to be *proper* if all bounded closed sets in the metric space are compact.

Similarly to Theorem 1.2.4, by using universal metric spaces, we construct metric spaces containing all proper length space as its tangent or asymptotic cones.

Theorem 1.2.5 ([60])**.** *There exists a metric (ω_0+1)-space (X, d) for which every pointed proper length space (K, k, p) is a tangent cone of (X, d) at its unique accumulation point.*

Theorem 1.2.6 ([60])**.** *There exists a proper countable discrete metric space (X, d) for which every pointed proper length space (K, k, p) is an asymptotic cone of X at some point.*

The metric spaces stated in Theorems 1.2.4, 1.2.5 or 1.2.6 tell us that analogies of Theorem 1.2.1 for the topological dimension, the Hausdorff dimension and the conformal Hausdorff dimension are false (see Proposition 4.2.10). The author does not know whether an analogy of Theorem 1.2.1 for the Assouad–Nagata dimension holds true or not.

1.2.2 Tiling space

In this subsection, we develop Fraser–Yu's characterization [35] to more general metric spaces. To prove their characterization, Fraser and Yu essentially used the fact that the Euclidean spaces have a tiling structure of congruent cubes. While referring to their method, we introduce the notion of a tiling space. A tiling space is a pair of a metric space and a family called a tiling structure. The class of tiling spaces includes the Euclidean spaces, the p-adic numbers, the middle-third Cantor set, the Sierpiński gasket, and various self-similar spaces appearing in fractal geometry. For a doubling tiling space, we characterize metric subspaces possessing the same Assouad dimension as that of the whole space.

We first define covering pairs as follows: For a set S, we say that a family of subsets of S is a *covering* of S if the union of the family is equal to S. For a set X, we denote by $\mathrm{cov}(X)$ the set of all coverings of X. We call a map $\mathcal{P}$ from $\mathbb{N}$ or $\mathbb{Z}$ to $\mathrm{cov}(X)$ a *covering structure on* X. We denote by $\mathcal{P}_n$ the value of $\mathcal{P}$ at n. For $T \in \mathcal{P}_n$ and $k \in \mathbb{N}$, we put $[T]_k = \{ A \in P_{n+k} \mid A \subset T \}$. We call a pair $(X, \mathcal{P})$ of a set X and a covering structure $\mathcal{P}$ on X a *covering pair*. We denote by $\mathrm{dom}(\mathcal{P})$ the domain of the map $\mathcal{P}$. Note that $\mathrm{dom}(\mathcal{P})$ is either $\mathbb{N}$ or $\mathbb{Z}$. We next define tiling sets.

Definition 1.2.2 ([56])**.** Let $(X, \mathcal{P})$ be a covering pair. Let $N \in \mathbb{N}$. We say that $(X, \mathcal{P})$ is an *N-tiling set* if $(X, \mathcal{P})$ satisfies the following:

(S1) for every pair $n, m \in \mathrm{dom}(\mathcal{P})$ with $n < m$, and for every $A \in \mathcal{P}_n$, we have $\mathrm{card}([A]_{m-n}) = N^{m-n}$ and $A = \bigcup [A]_{m-n}$, where the symbol card stands for the cardinality;

(S2) for every $n \in \mathrm{dom}(\mathcal{P})$, and for every pair $A, B \in \mathcal{P}_n$, there exist $m \in \mathrm{dom}(\mathcal{P})$ and $C \in \mathcal{P}_m$ such that $A \cup B \subset C$ and $m < n$;

(S3) for every $n \in \mathrm{dom}(\mathcal{P})$, for all $l, m \in \mathbb{N}$, and for each $A \in \mathcal{P}_n$, we have

$$[A]_{m+l} = \bigcup_{T \in [A]_m} [T]_l.$$

We say that $(X, \mathcal{P})$ is a *tiling set* if it is an N-tiling set for some N. For a tiling set $(X, \mathcal{P})$, each member of $\mathcal{P}_n$ is called a *tile* of $(X, \mathcal{P})$.

We next specialize the notion of a tiling set for metric spaces.

Definition 1.2.3 ([56])**.** Let $N \in \mathbb{N}$ and $s \in (0, 1)$. Let (X, d) be a metric space. Let $\mathcal{P}$ be a covering map on X, and assume that $(X, \mathcal{P})$ is an N-tiling set. We say that the triple $(X, d, \mathcal{P})$ is an *(N, s)-pre-tiling space* if it satisfies the following:

(T1) there exist $D_1, D_2 \in (0, \infty)$ such that for every $n \in \mathrm{dom}(\mathcal{P})$, and for every $A \in \mathcal{P}_n$, we have $D_1 \leq \delta(A)/s^n \leq D_2$;

(T2) there exists $E \in (0, \infty)$ such that for every $n \in \mathrm{dom}(X)$, and for every $A \in \mathcal{P}_n$, there exists a point $p_A \in A$ satisfying that $U(p_A, Es^m) \subset A$, where $U(p_A, Es^m)$ stands for the open ball.

We also say that an (N, s)-pre-tiling space $(X, d, \mathcal{P})$ is an *(N, s)-tiling space* if it satisfies:

(U) for every countable sequence $\{A_i\}_{i \in \mathbb{N}}$ of tiles of $(X, d, \mathcal{P})$, there exists a subsequence $\{A_{\phi(i)}\}_{i \in \mathbb{N}}$ such that the sequence $\{(A_{\phi(i)}, (\delta(A_{\phi(i)}))^{-1} d)\}_{i \in \mathbb{N}}$ converges to the space $(T, (\delta(T))^{-1} d)$ for some tile T of $(X, d, \mathcal{P})$ in the sense of Gromov–Hausdorff.

A triple $(X, d, \mathcal{P})$ is said to be a *tiling* (resp. *pre-tiling*) space if it is an (N, s)-tiling (resp. (N, s)-pre-tiling) space for some N and s.

Two metric spaces (X, d) and (Y, e) are said to be *similar* if there exists $h \in (0, \infty)$ satisfying $\mathcal{GH}((X, hd), (Y, e)) = 0$. Similarity is an equivalence relation on metric spaces.

Let (X, d) be a metric space. Let $(X, \mathcal{P})$ be a tiling set. If the similarity classes of the tiles of $(X, \mathcal{P})$ is finite, then $(X, d, \mathcal{P})$ satisfies the condition (U). Thus the condition (U) is considered as a generalization of the finiteness of the similarity classes of tiles. There exist a pre-tiling space failing the condition (U) (see Example 5.4.2), and a tiling space whose tiles have infinite similarity classes (see Example 5.4.3).

To state our characterization of subsets of full Assouad dimension of tiling spaces, we introduce the specialized notion of a pseudo-cone for tiling spaces. Let $(X, d, \mathcal{P})$ be a pre-tiling space, and let F be a subset of X. We also denote by $\mathcal{TPC}(F, d)$ the class of all pseudo-cones approximated by $(\{A_i \cap F\}_{i\in\mathbb{N}}, \{u_i\}_{i\in\mathbb{N}})$, where $\{A_i\}_{i\in\mathbb{N}}$ is a sequence of tiles of $(X, \mathcal{P})$ and $\{u_i\}_{i\in\mathbb{N}}$ is a sequence in $(0, \infty)$.

Let $(X, d, \mathcal{P})$ be a tiling set. Let A be a tile of $(X, d, \mathcal{P})$. We say that a subset F of X satisfies the *asymptotic Steinhaus property for A* if for every $\epsilon \in (0, \infty)$, and for every finite subset S of A, there exist a finite subset T of F, and $\delta \in (0, \infty)$ satisfying that $\mathcal{GH}((T, d), (S, \delta d)) \leq \delta \cdot \epsilon$.

Our characterization is the following:

Theorem 1.2.7 ([56])**.** *Let $(X, d, \mathcal{P})$ be a doubling tiling space. Then for every subset F of X the following are equivalent:*

(1) $\dim_A(F, d) = \dim_A(X, d)$*;*

(2) *there exists a tile A of $(X, d, \mathcal{P})$ such that $A \in \mathcal{PC}(F, d)$;*

(3) *there exists a tile A of $(X, d, \mathcal{P})$ such that $A \in \mathcal{TPC}(F, d)$;*

(4) *there exists a tile A of $(X, d, \mathcal{P})$ such that $A \in \mathcal{KPC}(F, d)$;*

(5) *there exists a tile A of $(X, d, \mathcal{P})$ such that F satisfies the asymptotic Steinhaus property for A.*

In Theorem 1.2.7, the assumption of the doubling property for X is necessary. Note that the doubling property is equivalent to the finiteness of the Assouad dimension (see Section 2.3). There exists a tiling space that is not doubling (see Example 5.4.1).

Remark 1.2.2. Let (X, d, P) be a tiling space. If (X, d) is doubling, then for every tile $T \in P$ we have $\dim_A(T, d) = \dim_A(X, d)$ (see Corollary 5.2.3). If (X, d) is not doubling, then the equality does not necessarily hold. For example, the tiling space constructed in Example 5.4.1 has infinite Assouad dimension, and possesses a tile of finite Assouad dimension.

If a tiling space $(X, d, \mathcal{P})$ satisfies that the conformal dimensions of all the tiles of $(X, d, \mathcal{P})$ and X are equal to $\dim_A(X, d)$, then the condition $\mathrm{Cdim}_A(F, d) = \dim_A(X, d)$ is equivalent to the conditions (1)–(5) stated in Theorem 1.2.7. The assumption mentioned above seems to be quite strong. Indeed, the author does not know an example satisfying the assumption except the Euclidean spaces. Therefore we do not treat with the conformal dimensions of tiling spaces.

Attractors of iterated function systems on metric spaces are studied as canonical and classical examples of fractals, and their Hausdorff dimensions are investigated (see, for example, [34, Chapter 9], [100], and [101]).

Definition 1.2.4 ([56]). Let (X, d) be a complete metric space. Let $L \in (0, \infty)$. A map $f : X \to X$ is said to be an *L-similar transformation on X* if for all $x, y \in X$ we have $d_X(f(x), f(y)) = Ld_X(x, y)$. Let $N \geq 2$ and let $s \in (0, 1)$. We say that $\mathcal{S}$ is an *(N, s)-similar iterated function system on X* if $\mathcal{S}$ consists of N many s-similar transformations on X, say $\mathcal{S} = \{S_i\}_{i=0}^{N-1}$. A non-empty subset F of X is said to be an *attractor of the iterated function system $\mathcal{S}$* if F is compact and it satisfies $F = \bigcup_{i=0}^{N-1} S_i(F)$. Since X is complete, an attractor of $\mathcal{S}$ always uniquely exists (see [34, Chapter 9] for the Euclidean setting). We write $A_{\mathcal{S}}$ as the attractor of $\mathcal{S}$. The iterated function system $\mathcal{S}$ is said to satisfy the *strong open set condition* if there exists an open set V of X such that

(O1) $\bigcup_{i=0}^{N-1} S_i(V) \subset V$;

(O2) The family $\{S_i(V)\}_{i=0}^{N-1}$ is mutually disjoint;

(O3) $V \cap A_{\mathcal{S}} \neq \emptyset$.

We denote by $W_{\mathcal{S}}$ the set of all words generated by $\{0, \dots, N-1\}$. For every word $w = w_0 \cdots w_l$, we put $S_w = S_{w_l} \circ \cdots \circ S_{w_0}$, where each w_i belongs to $\{0, \dots, N-1\}$. We define a covering map $\mathcal{P}_{\mathcal{S}} : \mathbb{N} \to \operatorname{cov}(A_{\mathcal{S}})$ by

$$(\mathcal{P}_{\mathcal{S}})_n = \{\, S_w(A_{\mathcal{S}}) \mid w \in W_{\mathcal{S}} \text{ and } |w| = n \,\}, \tag{1.2.1}$$

where $|w|$ stands for the length of the word w. We denote by $d_{\mathcal{S}}$ the metric $d|_{(A_{\mathcal{S}})^2}$ on $A_{\mathcal{S}}$.

Similar iterated function systems provide us a plenty of tiling spaces.

Theorem 1.2.8 ([56]). *Let $N \in \mathbb{N}_{\geq 2}$ and $s \in (0, 1)$. Let $\mathcal{S}$ be an (N, s)-similar iterated function system on a complete metric space satisfying the strong open set condition. Let $A_{\mathcal{S}}$ be the attractor of $\mathcal{S}$, and $\mathcal{P}_{\mathcal{S}}$ the covering map defined by* (1.2.1)*. Then the triple $(A_{\mathcal{S}}, d_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}})$ is an (N, s)-tiling space.*

Due to Theorem 1.2.8, for instance, the middle-third Cantor set and the Sierpiński gasket are tiling spaces for some suitable covering structures induced from iterated function systems (see Subsection 5.3.3), and we can apply Theorem 1.2.7 to these metric spaces.

As a non-compact version of attractors, we introduce the notion of an extended attractor. We also prove a similar result to Theorem 1.2.8 for extended attractors (Theorem 5.3.3).

1.2.3 Spaces of metrics

In this subsection, we explain our interpolation theorem and theorems on topological distribution in spaces of metrics. Recall that, for a metrizable space X, the symbol $\mathrm{M}(X)$ stands for the set of all metrics on X that generate the same topology as the original one on X, and recall that the metric $\mathcal{D}_X$ on $\mathrm{M}(X)$ is defined by

$$\mathcal{D}_X(d, e) = \sup_{(x,y) \in X^2} |d(x, y) - e(x, y)|.$$

In the same way as ordinary metric spaces, the metric $\mathcal{D}_X$ generates the topology of $\mathrm{M}(X)$. In what follows, the space $\mathrm{M}(X)$ will be equipped with the topology induced from $\mathcal{D}_X$. We first generalize the Hausdorff extension theorem to an interpolation theorem of metrics preserving $\mathcal{D}_X$.

A family $\{S_i\}_{i \in I}$ of subsets of a topological space X is said to be *discrete* if for every $x \in X$ there exists a neighborhood of x intersecting at most single member of $\{S_i\}_{i \in I}$.

Theorem 1.2.9 ([59])**.** *Let X be a metrizable space, and let $\{A_i\}_{i\in I}$ be a discrete family of closed subsets of X. Then for every metric $d \in \mathrm{M}(X)$, and for every family $\{e_i\}_{i\in I}$ of metrics satisfying $e_i \in \mathrm{M}(A_i)$, there exists a metric $m \in \mathrm{M}(X)$ satisfying the following:*

(1) *for every $i \in I$ we have $m|_{A_i^2} = e_i$;*

(2) $\mathcal{D}_X(m, d) = \sup_{i\in I} \mathcal{D}_{A_i}(e_{A_i}, d|_{A_i^2})$.

Moreover, if X is completely metrizable, and if each $e_i \in \mathrm{M}(A_i)$ is a complete metric, then the metric $m \in \mathrm{M}(X)$ can be chosen as a complete one.

A central idea of the proof of Theorem 1.2.9 is a correspondence between a metric on a metrizable space and a topological embedding from a metrizable space into a Banach space. A metric $d \in \mathrm{M}(X)$ on a metrizable space X induces a topological embedding from X into a suitable Banach space such as the Kuratowski or the Arens–Eells isometric embedding (see Subsection 2.2.1). Conversely, a topological embedding $F : X \to V$ from a metrizable space X into a Banach space V with a norm $\|*\|_V$ induces a metric $m \in \mathrm{M}(X)$ on X defined by $m(x, y) = \|F(x) - F(y)\|_V$. To prove Theorem 1.2.9, we utilize this correspondence to translate the statement of Theorem 1.2.9 into an interpolation problem on topological embeddings into a Banach space. This idea is inspired by Hausdorff's method mentioned in Subsection 1.1.3. We then resolve such a problem on topological embeddings by using the Michael continuous selection theorem (Theorem 2.7.3), and a similar method to Kuratowski [70] (see also [46] and [1]) of converting a continuous function into a topological embedding by extending a codomain.

For a topological space X, a subset of X is said to be G_δ if it is represented as the intersection of countable family of open sets of X.

Our interpolation theorem (Theorem 1.2.9) enables us to investigate dense subsets in the topology of the space $(\mathrm{M}(X), \mathcal{D}_X)$ for a metrizable space X. To describe our next result precisely, we define a class of geometric properties that unify various properties defined by finite subsets of metric spaces. We denote by $\mathcal{P}^*(\mathbb{N})$ the set of all non-empty subsets of $\mathbb{N}$. For a topological space T, we denote by $\mathcal{F}(T)$ the set of all closed subsets of T. For $W \in \mathcal{P}^*(\mathbb{N})$, and for a set S, we denote by $\mathrm{Seq}(W, S)$ the set of all finite injective sequences $\{a_i\}_{i=1}^n$ in S satisfying $n \in W$.

Definition 1.2.5 ([59])**.** Let Q be an at most countable set, and let P be a topological space. Let $F : Q \to \mathcal{F}(P)$ and $G : Q \to \mathcal{P}^*(\mathbb{N})$ be maps. Let Z be a set. Let ϕ be a correspondence assigning a pair (q, X) of $q \in Q$ and a metrizable space X to a map

$$\phi^{q,X} : \mathrm{Seq}(G(q), X) \times Z \times \mathrm{M}(X) \to P.$$

We say that a sextuple (Q, P, F, G, Z, ϕ) is a *transmissible paremeter* if for every metrizable space X, for every $q \in Q$, and for every $z \in Z$ the following are satisfied:

(TP1) for every $a \in \mathrm{Seq}(G(q), X)$ the map $\phi^{q,X}(a, z) : \mathrm{M}(X) \to P$ defined by

$$\phi^{q,X}(a, z)(d) = \phi^{q,X}(a, z, d)$$

is continuous;

(TP2) for every $d \in \mathrm{M}(X)$, for every subset S of X and for every $a \in \mathrm{Seq}(G(q), S)$, we have $\phi^{q,X}(a, z, d) = \phi^{q,S}(a, z, d|_{S^2})$.

We now define a property determined by a transmissible parameter.

Definition 1.2.6 ([59])**.** Let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$ be a transmissible parameter. We say that a metric space (X, d) satisfies the $\mathfrak{G}$*-transmissible property* if there exists $q \in Q$ such that for every $z \in Z$ and for every $a \in \mathrm{Seq}(G(q), X)$ we have $\phi^{q,X}(a, z, d) \in F(q)$. We say that (X, d) satisfies the *anti-*$\mathfrak{G}$*-transmissible property* if (X, d) satisfies the negation of the $\mathfrak{G}$-transmissible property; namely, for every $q \in Q$ there exist an element $z \in Z$ and a sequence $a \in \mathrm{Seq}(G(q), X)$ satisfying $\phi^{q,X}(a, z, d) \in P \setminus F(q)$. A property on metric spaces is said to be a *transmissible property* (resp. *anti-transmissible property*) if it is equivalent to a $\mathfrak{G}$-transmissible property (resp. anti-$\mathfrak{G}$-transmissible property) for some transmissible parameter $\mathfrak{G}$.

Various properties appearing in metric geometry are transmissible properties.

Example 1.2.1. The following properties on metric spaces are transmissible properties (see Section 6.2).

(1) the doubling property;

(2) the uniform disconnectedness;

(3) satisfying the ultratriangle inequality;

(4) satisfying the Ptolemy inequality;

(5) the Gromov $\mathrm{Cycl}_m(0)$ condition;

(6) the Gromov hyperbolicity.

For our typicality theorems, we require the notion of singularity for transmissible properties similarly to other typicality theorems.

Definition 1.2.7 ([59])**.** Let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$ be a transmissible parameter. We say that $\mathfrak{G}$ is *singular* if for each $q \in Q$ and for every $\epsilon \in (0, \infty)$ there exist $z \in Z$ and a finite metrizable space (R, d_R) such that

(1) $\delta_{d_R}(R) \leq \epsilon$, where $\delta_{d_R}(R)$ stands for the diameter of R;

(2) $\mathrm{card}(R) \in G(q)$, where card stands for the cardinality;

(3) $\phi^{q,R}(R, z, d_R) \in P \setminus F(q)$.

Remark that not all transmissible parameters are singular; especially, the Gromov hyperbolicity does not have a singular transmissible parameter (see Proposition 6.3.15).

By Theorem 1.2.9, we obtain a theorem on dense G_δ subsets in spaces of metrics:

Theorem 1.2.10 ([59])**.** *Let* $\mathfrak{G}$ *be a singular transmissible parameter. Then for every non-discrete metrizable space* X*, the set of all* $d \in \mathrm{M}(X)$ *for which* (X, d) *satisfies the anti-*$\mathfrak{G}$*-transmissible property is dense* G_δ *in* $\mathrm{M}(X)$*.*

Theorem 1.2.10 can be considered as a theorem on typicality in spaces of metrics.

Remark 1.2.3. Theorem 1.2.10 holds true for the space $\mathrm{CM}(X)$ of all complete metrics in $\mathrm{M}(X)$ (see Theorems 6.2.7).

Our next result is based on the fact that for second countable locally compact space X the space $\mathrm{M}(X)$ is a Baire space (see Lemma 3.4.7). For a property P on metric spaces, we say that a metric space (X, d) satisfies *local* P if every non-empty open metric subspace of X satisfies the property P. As a local version of Theorem 1.2.10, we obtain the following:

Theorem 1.2.11 ([59])**.** *Let X be a second countable, locally compact locally non-discrete space. Then for every singular transmissible parameter $\mathfrak{G}$, the set of all metrics $d \in \mathrm{M}(X)$ for which (X, d) satisfies the local anti-$\mathfrak{G}$-transmissible property is dense G_δ in $\mathrm{M}(X)$.*

Note that all second countable locally compact spaces are metrizable, which is a consequence of the Urysohn metrization theorem.

We can apply Theorems 1.2.10 and 1.2.11 to the properties (1)–(5) mentioned in Example 1.2.1. Therefore we conclude that the set of all metrics not satisfying these properties are dense G_δ in spaces of metrics (see Corollaries 6.3.16 and 6.3.17). For a metric inequality, we can show a stronger statement (see also Corollary 6.3.10).

1.2.4 Spaces of ultrametrics

Let X be a set. A metric d on X is said to be an *ultrametric* or a *non-Archimedean metric* if for all $x, y, z \in X$ we have

$$d(x, y) \leq d(x, z) \vee d(z, y), \tag{1.2.2}$$

where the symbol $\vee$ stands for the maximum operator on $\mathbb{R}$. The inequality (2.1.1) is called the *strong triangle inequality*. We say that a set S is a *range set* if S is a subset of $[0, \infty)$ and $0 \in S$. For a range set S, we say that a metric $d : X^2 \to [0, \infty)$ on X is *S-valued* if $d(X^2)$ is contained in S. Note that $[0, \infty)$-valued ultrametrics are nothing but ultrametrics. The notion of an S-valued ultrametric is studied as a special case and a reasonable restriction of ultrametrics. For a countable range set R, Gao and Shao [38] studied R-valued universal ultrametric spaces of Urysohn-type and their isometry groups. Brodskiy, Dydak, Higes and Mitra [12] utilized $(\{0\} \cup \{\, 3^n \mid n \in \mathbb{Z}\,\})$-valued ultrametrics for their study on 0-dimensionality in categories of metric spaces.

The notion of an ultrametric can be considered as a 0-dimensional analogue of ordinary metrics, and it is often expected to prove ultrametric versions of theorems on metric spaces. For example, p-adic analysis and non-Archimedean functional analysis are ultrametric analogues of the ordinary analysis and functional analysis (see [67], [94], and [103]). Studies of constructions of Urysohn ultrametric spaces (see [120], [38], and [121]) can be considered as non-Archimedean analogues of Urysohn's study of universal spaces [118]. In this thesis, as realizations of such expectations, for every range set S, we provide S-valued ultrametric versions of the Arens–Eells isometric embedding theorem [2] of metric spaces, the Hausdorff extension theorem [45] of metrics, the Niemytzki–Tychonoff characterization [86] of compactness, and the interpolation theorem of metrics (Theorem 1.2.9) and theorems on dense G_δ subsets of spaces of metrics (Theorems 1.2.10 and 1.2.11).

Before stating our main results, we introduce some basic notions. Let R be a commutative ring, and let V be an R-module. A subset S of V is said to be *R-independent* if for every finite subset $\{f_1, \dots, f_n\}$ of S, and for all $N_1, \dots, N_n \in R$, the identity $\sum_{i=1}^{n} N_i f_i = 0$ implies $N_i = 0$ for all i. A function $\| * \| : V \to [0, \infty)$ is said to be an *ultra-norm on V* if the following are satisfied:

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) for every $x \in V$, we have $\| -x\| = \|x\|$;

(3) for all $x, y \in V$, we have $\|x + y\| \leq \|x\| \vee \|y\|$.

The pair $(V, \| * \|)$ is called an *ultra-normed R-module* (see also [122] and [94]). Note that ultra-norms induce invariant metrics under addition (see Subsection 3.3.2).

In 1956, Arens and Eells [2] established the result which today we call the Arens–Eells embedding theorem (see Theorem 2.2.2), stating that for every metric space (X, d), there exist a real normed linear space V and an isometric embedding $I : X \to V$ such that

(1) $I(X)$ is closed in V;

(2) $I(X)$ is linearly independent in V.

In the study on free non-Archimedean topological groups and Boolean groups with actions from topological groups, Megrelishvili and Shlossberg [78] proved an ultrametric version of the Arens–Eells embedding theorem, stating that every ultrametric space is isometrically embeddable into an ultra-normed Boolean group (a $\mathbb{Z}/2\mathbb{Z}$-module) as a closed set of it, which is a consequence of their isometric embedding theorem compatible with group actions. By introducing module structures into universal ultrametric spaces of Lemin–Lemin type [74], we obtain a more general S-valued ultrametric version of the Arens–Eells embedding theorem.

Theorem 1.2.12 ([58])**.** *Let S be a range set possessing at least two elements. Let R be an integral domain, and let (X, d) be an S-valued ultrametric space. Then there exist an S-valued ultra-normed R-module $(V, \| * \|)$, and an isometric embedding $I : X \to V$ such that*

(1) *$I(X)$ is closed in V;*

(2) *$I(X)$ is R-independent in V.*

Moreover, if (X, d) is complete, then $(V, \| * \|)$ can be chosen as a complete metric space.

Remark 1.2.4. Let R be an integral domain. Let t_R be the trivial absolute value on R defined by $t_R(x) = 1$ if $x \neq 0$; otherwise $t_R(x) = 0$. Let $(V, \| * \|)$ be an R-module constructed in the proof of Theorem 1.2.12. Then the ultra-norm $\| * \|$ on V is compatible with t_R, i.e., for every $r \in R$ and for every $x \in V$, we have $\|r \cdot x\| = t_R(r)\|x\|$. For every finite field, there exist no absolute values on it except the trivial valuation. Thus we can consider that Theorem 1.2.12 includes the Arens–Eells embedding theorem into normed spaces over all finite fields. The author does not know whether such an embedding theorem into normed spaces over all non-Archimedean valued fields holds true or not.

Remark 1.2.5. There are various isometric embeddings from an ultrametric space into a metric space with algebraic structures. For instance, Schikhof [102] constructed an isometric embedding from an ultrametric space into a non-Archimedean valued field. Timan and Vestfrid [111] proved the existence of an isometric embedding from an ultrametric space into a Hilbert space in a separable case. A. J. Lemin [73] proved the existence of such an isometric embedding in a general setting. The papers [112], [120] and [37] also contain results concerning this subject.

For a range set S, and for a topological space X, we denote by $\mathrm{UM}(X, S)$ the set of all S-valued ultrametrics on X generating the same topology as X. We also denote by $\mathrm{UM}(X)$ the set $\mathrm{UM}(X, [0, \infty))$. We say that a topological space X is *S-valued ultrametrizable* (resp. *ultrametrizable*) if $\mathrm{UM}(X, S) \neq \emptyset$ (resp. $\mathrm{UM}(X) \neq \emptyset$). We say that X is *completely S-valued ultrametrizable* (resp. *completely ultrametrizable*) if there exists a complete metric $d \in \mathrm{UM}(X, S)$ (resp. $d \in \mathrm{UM}(X)$).

We say that a range set S has *countable coinitiality* if there exists a non-zero strictly decreasing sequence $\{r_i\}_{i \in \mathbb{N}}$ in S convergent to 0 as $i \to \infty$.

Remark 1.2.6. For a range set S, and for a topological space X, it is worth clarifying a relation between the ultrametrizability and the S-valued ultrametrizability. Proposition 3.3.10 states that these two properties are equivalent to each other if S has countable coinitiality.

While referring to Toruńczyk's proof [113] of the Hausdorff extension theorem by the Arens–Eells embedding theorem, we prove an analogue of the Hausdorff extension theorem for ultrametric spaces by using Theorem 1.2.12.

Theorem 1.2.13 ([58])**.** *Let S be a range set. Let X be an S-valued ultrametrizable space, and let A be a closed subset of X. Then for every $e \in \mathrm{UM}(A, S)$, there exists $D \in \mathrm{UM}(X, S)$ with $D|_{A^2} = e$. Moreover, if X is completely metrizable and $e \in \mathrm{UM}(A, S)$ is complete, then $D \in \mathrm{UM}(X, S)$ can be chosen as a complete S-valued ultrametric.*

Remark 1.2.7. There are several studies on extending a partial or continuous ultrametrics (see [31], [107], [108], or [115]).

In 1928, Niemytzki and Tychonoff [86] proved that a metrizable space X is compact if and only if all metrics in $\mathrm{M}(X)$ are complete. Hausdorff [45] gave a simple proof of their characterization theorem by applying his extension theorem (Theorem 2.2.3). By using Hausdorff's argument and Theorem 1.2.13, we obtain an ultrametric version of the Niemytzki–Tychonoff theorem.

Corollary 1.2.14 ([58])**.** *Let S be a range set with the countable coinitiality. Then an S-valued ultrametrizable space X is compact if and only if all metrics in $\mathrm{UM}(X, S)$ are complete.*

To state our future results, for a topological space X, and for a range set S, we define a function $\mathcal{UD}_X^S : \mathrm{UM}(X, S)^2 \to [0, \infty]$ by assigning $\mathcal{UD}_X^S(d, e)$ to the infimum of $\epsilon \in S \sqcup \{\infty\}$ such that for all $x, y \in X$ we have $d(x, y) \leq e(x, y) \vee \epsilon$, and $e(x, y) \leq d(x, y) \vee \epsilon$. The function $\mathcal{UD}_X^S$ is an ultrametric on $\mathrm{UM}(X, S)$ valued in $\mathrm{CL}(S) \sqcup \{\infty\}$, where $\mathrm{CL}(S)$ stands for the closure of S in $[0, \infty)$. As is the case with $\mathcal{D}_X$, we can introduce a topology on $\mathrm{UM}(X, S)$ induced from $\mathcal{UD}_X^S$. In what follows, the space $\mathrm{UM}(X, S)$ will be equipped with the topology induced from $\mathcal{UD}_X^S$.

Remark 1.2.8. As a non-Archimedean analogue of the Gromov–Hausdorff distance, the non-Archimedean Gromov–Hausdorff distance was introduced by Zarichnyi [124]. Qiu [96] introduced the notion of a *strong ϵ-isometry* in the study on the non-Archimedean Gromov–Hausdorff distance. This concept is an analogue for ultrametric spaces of the notion of an ϵ-isometry in the study on the ordinary Gromov–Hausdorff distance (see [15]). The metric $\mathcal{UD}_X^S$ can be explained with the strong ϵ-isometries. Roughly speaking, for a range set S, for an S-valued ultrametrizable space X, and for S-valued ultrametrics $d, e \in \mathrm{UM}(X, S)$, the inequality $\mathcal{UD}_X^S(d, e) \leq \epsilon$ is equivalent to the statement that the identity maps $1_X : (X, d) \to (X, e)$ and $1_X : (X, e) \to (X, d)$ are strong ϵ-isometries.

By Theorems 1.2.12 and 1.2.13, and by tracing the proof of Theorem 1.2.9, we can prove an ultrametric version of Theorem 1.2.9. For a range set S, and for a subset E of S, we denote by $\sup E$ the supremum of E taken in $[0, \infty]$, not in S. For $C \in [1, \infty)$, we say that S is *C-quasi-complete* if for every bounded subset E of S, there exists $s \in S$ with $\sup E \leq s \leq C \cdot \sup E$. We say that S is *quasi-complete* if S is C-quasi-complete for some $C \in [1, \infty)$. For example, if a range set is closed or dense in $[0, \infty)$, then it is quasi-complete. Note that a range set is 1-quasi-complete if and only if it is closed under the supremum operator.

Theorem 1.2.15 ([58])**.** *Let $C \in [1,\infty)$, and let S be a C-quasi-complete range set. Let X be an S-valued ultrametrizable space, and let $\{A_i\}_{i\in I}$ be a discrete family of closed subsets of X. Then for every S-valued ultrametric $d \in \mathrm{UM}(X,S)$, and for every family $\{e_i\}_{i\in I}$ of S-valued ultrametrics satisfying $e_i \in \mathrm{UM}(A_i,S)$ for all $i \in I$, there exists an S-valued ultrametric $m \in \mathrm{UM}(X,S)$ with the following properties:*

(1) *for every $i \in I$ we have $m|_{A_i^2} = e_i$;*

(2) $\sup_{i\in I} \mathcal{UD}^S_{A_i}(e_i, d|_{A_i^2}) \leq \mathcal{UD}^S_X(m,d) \leq C \cdot \sup_{i\in I} \mathcal{UD}^S_{A_i}(e_i, d|_{A_i^2})$.

Moreover, if X is completely metrizable, and if each $e_i \in \mathrm{UM}(A_i,S)$ is complete, then the metric $m \in \mathrm{UM}(X,S)$ can be chosen as a complete one.

In the proof of Theorem 1.2.9, we use the Michael selection theorem for paracompact spaces. On the other hand, in order to prove Theorem 1.2.15, we use the 0-dimensional Michael selection theorem.

The following concept is an S-valued ultrametric version of the singularity of the transmissible parameters (compare with Definition 1.2.7).

Definition 1.2.8 ([58])**.** Let S be a range set. Let $\mathfrak{G} = (Q,P,F,G,Z,\phi)$ be a transmissible parameter. We say that $\mathfrak{G}$ is *S-ultra-singular* if for each $q \in Q$ and for every $\epsilon \in (0,\infty)$ there exist $z \in Z$, a finite S-valued ultrametric space (R,d_R), and an index $R = \{r_i\}_{i=1}^{\mathrm{card}(R)}$ such that

(1) $\delta_{d_R}(R) \leq \epsilon$;

(2) $\mathrm{card}(R) \in G(q)$;

(3) $\phi^{q,R}\left(\{r_i\}_{i=1}^{\mathrm{card}(R)}, z, d_R\right) \in P \setminus F(q)$.

Similarly to Theorems 1.2.10 and 1.2.11, Theorem 1.2.15 enables us to prove theorems on dense G_δ subsets of $\mathrm{UM}(X,S)$.

Theorem 1.2.16 ([58])**.** *Let S be a quasi-complete range set with the countable coinitiality. Let $\mathfrak{G}$ be an S-ultra-singular transmissible parameter. Let X be a non-discrete S-valued ultrametrizable space. Then the set of all $d \in \mathrm{UM}(X,S)$ for which (X,d) satisfies the anti-$\mathfrak{G}$-transmissible property is dense G_δ in the space $(\mathrm{UM}(X,S), \mathcal{UD}^S_X)$.*

Similarly to Theorem 1.2.11, in the proof of the following, we use the fact that for second countable locally compact space X, the space $\mathrm{UM}(X,S)$ is Baire (see Lemma 3.4.7).

Theorem 1.2.17 ([58])**.** *Let S be a quasi-complete range set with the countable coinitiality. Let X be a second countable, locally compact locally non-discrete S-valued ultrametrizable space. Then for every S-ultra-singular transmissible parameter $\mathfrak{G}$, the set of all $d \in \mathrm{UM}(X,S)$ for which (X,d) satisfies the local anti-$\mathfrak{G}$-transmissible property is a dense G_δ set in the space $(\mathrm{UM}(X,S), \mathcal{UD}^S_X)$.*

For example, the doubling property is a transmissible property with an ultra-singular transmissible parameter.

1.3 Organization

In Chapter 2, we prepare some notations, and review classical facts such as the Kuratowski embedding theorem, the Hausdorff metric extension theorem, the Baire category theorem, and the Michael continuous selection theorems. We also review basic properties of the classical concepts such as the Gromov–Hausdorff distance, universal metric spaces, ultralimits of sequences of metric spaces.

Chapter 3 is devoted to preparing basic statements concerning metrics or ultrametrics, including the telescope construction, amalgamations of (ultra)metrics, and the statement that $\mathrm{M}(X)$ and $\mathrm{UM}(X)$ are Baire if X is second countable locally compact.

In Chapter 4, we prove Theorems 1.2.1 and 1.2.2, which are lower estimations of the Assouad dimensions by pseudo-cones and ultralimits, respectively. We also provide a lower estimation of the conformal Assouad dimension (Theorem 1.2.3). By using universal metric spaces for separable metric spaces (see Section 2.5), we construct metric spaces containing large classes of metric spaces as pseudo-cones, tangent cones, asymptotic cones (Theorems 1.2.4, 1.2.5, and 1.2.6).

In Chapter 5, we prove Theorem 1.2.7, which is a generalization of Fraser–Yu's characterization of subsets of full Assouad dimension of the Euclidean spaces. We also prove Theorem 1.2.8, stating that attractors of iterated function systems are tiling spaces, and we introduce the notion of an extended attractor. We also provide counterexamples of tiling spaces related to the doubling property, bi-Lipschitz maps, and similarity classes of tiles.

In Chapter 6, we first prove an interpolation theorem of metrics preserving $\mathcal{D}_X$ (Theorem 1.2.9). As its application, we prove Theorem 1.2.10, which determines a topological distribution of metrics with transmissible properties in spaces of metrics. We also prove Theorem 1.2.11, which is a local version of Theorem 1.2.10.

In Chapter 7, we prove Theorem 1.2.12, which is an ultrametric version of the Arens–Eells isometric embedding theorem. As an application of Theorem 1.2.12, we prove Theorem 1.2.13, which is an extension theorem of ultrametrics. By using this extension theorem, we can develop an ultrametric analogue of the theory of dense G_δ subsets of spaces of metrics in Chapter 6. We prove an interpolation theorem of metrics preserving $\mathcal{UD}_X^S$ (Theorems 1.2.15), a theorem on dense G_δ subsets of spaces of ultrametrics (Theorem 1.2.16), and its local version (Theorem 1.2.17).

Acknowledgements

The author would like to express the deepest appreciation to the author's supervisor Professor Koichi Nagano for his advice and constant encouragement. The author would like to thank Professor Jun-ichi Inoguchi for his encouragement. The author would also like to thank Professor Kazuhiro Kawamura for his helpful comments. The author is deeply grateful to Professor Hajime Ono for his teaching as a supervisor in the author's undergraduate course at Saitama University.

The author would like to thank Enrico Le Donne, Tapio Rajala and Jerzy Dydak for their helpful comments on the references in [60].

The author is a Research Fellow (DC1) of Japan Society for Promotion of Science (JSPS) for 2018–2020. The author wishes to thank financial support by Garant-in-Aid for JSPS Fellows Grant Number 18J21300.

Chapter 2

Preliminaries

In this thesis, we denote by $\mathbb{N}$ the set of all non-negative integers. For a subset E of the set $\mathbb{R}$ of all real numbers (we intend that E is $\mathbb{N}$, $\mathbb{Z}$, or a range set), and for $n \in E$, we denote by $E_{\geq n}$ the set $\{ x \in E \mid x \geq n \}$. We also put $E_{+} = \{ x \in E \mid x > 0 \}$. For a set X, we denote by 1_X the identity map of X. For $n \in \mathbb{Z}_{+}$, we denote by X^n the Cartesian product of n many copies of the set X. Let X, Y be two sets. A map $f : X^2 \to Y$ is said to be *symmetric* if for all $x, y \in X$ we have $f(x, y) = f(y, x)$.

In this chapter, we prepare some notations and definitions, and we discuss basic or classical facts on metric spaces.

2.1 Metrics and ultrametrics

In this section, we introduce the notions of metrics and ultrametrics again, and we prepare some notations related to metric spaces.

Let X be a set. A function $d : X^2 \to [0, \infty)$ is said to be a *metric on* X if the following three conditions are satisfied:

(M1) for all $x, y \in X$, the equality $d(x, y) = 0$ holds if and only if $x = y$;

(M2) d is a symmetric map;

(M3) for all $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$.

The condition (M3) is called the *triangle inequality*.

Recall that a function $d : X^2 \to [0, \infty)$ is said to be an ultrametric or a non-Archimedean metric if d satisfies the conditions (M1) and (M2), and if for all $x, y, z \in X$ we have

$$d(x, y) \leq d(x, z) \vee d(z, y), \tag{2.1.1}$$

where the symbol $\vee$ stands for the maximum operator on $\mathbb{R}$. The inequality (2.1.1) is called the *strong triangle inequality*. Recall that a set S is called a range set if $S \subset [0, \infty)$ and $0 \in S$. For a range set S, we say that a metric $d : X^2 \to [0, \infty)$ on X is S-valued if $d(X^2)$ is contained in S. Note that $[0, \infty)$-valued ultrametrics are nothing but ordinary ultrametrics. Remark that the strong triangle inequality implies the triangle inequality. In particular, all ultrametrics are metrics.

Let (X, d) be a metric space. Let A be a subset of X. We denote by $\delta(A)$ the diameter of A, and we set $\alpha(A) = \inf\{ d(x, y) \mid x, y \in A \text{ and } x \neq y \}$. We denote by $B(x, r)$ (resp. $U(x, r)$) the closed (resp. open) ball centered at x with radius r. We also denote by

$B(A, r)$ the set $\bigcup_{a \in A} B(a, r)$. To emphasize a metric space under consideration, we often use symbols $\delta_d(A)$, $\alpha_d(A)$ instead of $\delta(A)$, $\alpha(A)$, respectively. To emphasize a metric or a underlying set under consideration, we often denote the closed (resp. open) ball by $B(x, r; d)$ or $B(x, r; X, d)$ (resp. $U(x, r; d)$ or $U(x, r; X, d)$), respectively. We also use this notation for the balls centered at subsets. A subset A of X is said to be *r-separated* if $\alpha(A) \geq r$. A subset A of X is said to be *separated* if it is r-separated for some r.

Let $p \in [1, \infty]$. For two metric spaces (X, d) and (Y, e), we denote by $d \times_p e$ the ℓ^p-product metric defined by

$$(d \times_p e)((a, b), (c, d)) = \begin{cases} (d(a, c)^p + e(b, d)^p)^{1/p} & \text{if } p \in [1, \infty); \\ d(a, c) \vee e(b, d) & \text{if } p = \infty. \end{cases}$$

It is well-known that $d \times_p e$ belongs to $\mathrm{M}(X \times Y)$ for any $p \in [0, \infty]$. In the case of ultrametrics, we obtain the following:

Lemma 2.1.1. *Let S be a range set. Let (X, d) and (Y, e) be S-valued ultrametric spaces. Then the metric $d \times_\infty e$ belongs to $\mathrm{UM}(X \times Y, S)$.*

Let (X, d) be a metric space. For $\epsilon \in (0, \infty)$, we define a function $d^\epsilon : X^2 \to [0, \infty)$ by $d^\epsilon(x, y) = (d(x, y))^\epsilon$. If d^ϵ is a metric, then the metric space (X, d^ϵ) is called a *snowflake of* (X, d).

Lemma 2.1.2. *Let (X, d) be a metric space. Then the following hold:*

(1) *for every $\epsilon \in (0, 1)$, the function d^ϵ is a metric.*

(2) *if d is an ultrametric, then for every $\epsilon \in (0, \infty)$ the function d^ϵ is an ultrametric.*

2.2 Basic statements on metric spaces

2.2.1 Isometric embedding theorems into a Banach space

For a metric space (X, d), we denote by $C_b(X)$ the Banach space of all bounded continuous functions on X equipped with the supremum norm. The following theorem is known as the Kuratowski embedding theorem [69], stating that every metric space can be isometrically embedded into the Banach space of all bounded continuous functions on the metric space. The proof of this theorem can be seen in, for example, [69], [32, Theorem 4.3.14], or [97, Theorem 5.33].

Theorem 2.2.1. *For every metric space (X, d), and for every point $o \in X$, the map $K : X \to C_b(X)$ defined by $K(x) = d_x - d_o$ is an isometric embedding, where $d_x : X \to \mathbb{R}$ is defined by $d_x(p) = d(x, p)$. Moreover, if (X, d) is bounded, then the map $L : X \to C_b(X)$ defined by $L(x) = d_x$ is an isometric embedding.*

The next is the Arens–Eells isometric embedding theorem [2], which states that every metric space can be isometrically embedded into a Banach space as a closed and independent set. The proof can be seen in [2], [113], or [83].

Theorem 2.2.2. *For every metric space (X, d), there exist a real normed linear space $(V, \| * \|)$ and an isometric embedding $I : X \to V$ such that*

(1) *$I(X)$ is closed in V;*

(2) *$I(X)$ is linearly independent in V.*

2.2.2 Extension theorems on metrics

The following celebrated theorem was first proven by Hausdorff [45] (cf., [46], [1], [9], [6], [113]).

Theorem 2.2.3. *Let X be a metrizable space, and let A be a closed subset of X. Then, for every $d \in \mathrm{M}(A)$, there exists $D \in \mathrm{M}(X)$ such that $D|_{A^2} = d$. Moreover, if X is completely metrizable, and if $d \in \mathrm{M}(A)$ is a complete metric on A, then $D \in \mathrm{M}(X)$ can be chosen as a complete metric on X.*

Hausdorff [45] utilized Theorem 2.2.3 in order to give a simple proof of the following Niemytzki–Tychonoff theorem.

Theorem 2.2.4. *A metrizable space X is compact if and only if all metrics in $\mathrm{M}(X)$ are complete.*

Remark 2.2.1. Nomizu and Ozeki [87] proved that a second countable connected differentiable manifold is compact if and only if all Riemannian metrics on the manifold are complete, as a consequence of their study on the existence of complete Riemannian metrics. This characterization can be considered as an analogue of the Niemytzki–Tychonoff theorem for Riemannian manifolds.

Nguyen Van Khue and Nguyen To Nhu [68] investigated a simultaneous extension of metrics as an improvement of the Hausdorff extension theorem (Theorem 2.2.3).

Theorem 2.2.5. *For every metrizable space X, and for every closed subset A of X, there exist maps $\Phi_1, \Phi_2 : \mathrm{M}(A) \to \mathrm{M}(X)$ such that*

(1) *Φ_1 and Φ_2 are extensors; namely, for every $d \in \mathrm{M}(A)$, we have $\Phi_1(d)|_{A^2} = d$ and $\Phi_2(d)|_{A^2} = d$.*

(2) *Φ_1 and Φ_2 are continuous with respect to the topologies induced from $\mathcal{D}_A$ and $\mathcal{D}_X$.*

(3) *Φ_1 is 20-Lipschitz with respect to the metrics $\mathcal{D}_A$ and $\mathcal{D}_X$.*

(4) *Φ_2 preserves orders; namely, if $d, e \in \mathrm{M}(A)$ satisfy $d \leq e$, then $\Phi_2(d) \leq \Phi_2(e)$.*

(5) *if X is completely metrizable, then Φ_1 and Φ_2 map any complete metric in $\mathrm{M}(A)$ into a complete metric in $\mathrm{M}(X)$.*

2.2.3 Baire spaces

A topological space X is said to be *Baire* if an intersection of every countable family of dense open subsets of X is dense in X.

The following is well-known as the Baire category theorem.

Theorem 2.2.6. *All completely metrizable spaces are Baire.*

The following is known as the Alexandroff theorem. The proof can be seen in [123, Theorem 24.12], [92, Theorem 12.1], or [105, Theorem 2.2.1].

Theorem 2.2.7. *All G_δ subspaces of a completely metrizable space are completely metrizable.*

By Theorem 2.2.7, we obtain the following:

Lemma 2.2.8. *All G_δ subspaces of a completely metrizable space are Baire.*

By the definition of a Baire space, we have:

Lemma 2.2.9. *A topological space is a Baire space if and only if for every countable family of dense G_δ subsets of the space, its intersection is dense.*

2.3 Assouad dimension

Let $N \in \mathbb{Z}_+$. A metric space (X, d) is said to be *N-doubling* if for every bounded subset S of X there exists a subset F of X satisfying that $S \subset B(F, \delta(S)/2)$ and $\mathrm{card}(F) \leq N$. Note that if a metric space is N-doubling, then so are all metric subspaces of the metric space. A metric space is said to be *doubling* if it is N-doubling for some N.

Let (X, d) be a metric space, and let S be a bounded subset of X. We denote by $\mathbf{Z}_{(X,d)}(S, r)$ the minimum integer N such that S can be covered by at most N many bounded subsets of X with diameters at most r. We denote by $\mathbf{A}(X, d)$ the set of all $\beta \in (0, \infty)$ for which there exists $C \in (0, \infty)$ such that for every bounded subset S of X, and for every positive number $r \in (0, \infty)$, we have $\mathbf{Z}_{(X,d)}(S, r) \leq C \cdot (\delta(S)/r)^{\beta}$.

The *Assouad dimension* $\dim_A(X, d)$ *of a metric space* (X, d) is defined as $\inf(\mathbf{A}(X, d))$ if the set $\mathbf{A}(X, d)$ is non-empty; otherwise, $\dim_A(X, d) = \infty$.

Let (X, d) be a metric space. We denote by $\mathbf{B}(X, d)$ the set of all $\beta \in (0, \infty)$ for which there exists $C \in (0, \infty)$ such that every finite subset A of X satisfies the inequality $\mathrm{card}(A) \leq C \cdot (\delta(A)/\alpha(A))^{\beta}$. We also denote by $\mathbf{C}(X, d)$ the set of all $\gamma \in (0, \infty)$ such that there exists $C \in (0, \infty)$ such that every bounded separated subset M of X satisfies the inequality $\mathrm{card}(M) \leq C \cdot (\delta(M)/\alpha(M))^{\gamma}$.

By definitions, we obtain the next two propositions.

Proposition 2.3.1. *For every metric space X, the following are equivalent:*

(1) (X, d) *is doubling;*

(2) $\mathbf{A}(X, d)$ *is non-empty;*

(3) $\mathbf{B}(X, d)$ *is non-empty;*

(4) $\mathbf{C}(X, d)$ *is non-empty;*

(5) $\dim_A(X, d) < \infty$.

Proposition 2.3.1 implies that a metric spaces is not doubling if and only if it has infinite Assouad dimension, and all bounded separated sets in a doubling space are finite.

The Assouad dimension can be calculated by $\mathbf{B}(X, d)$ and $\mathbf{C}(X, d)$.

Proposition 2.3.2. *For every metric space X, we have*

$$\dim_A(X, d) = \inf(\mathbf{B}(X, d)) = \inf(\mathbf{C}(X, d)).$$

The *lower Assouad dimension* $\dim_{LA}(X, d)$ *of* (X, d) is defined as the supremum of all $\beta \in (0, \infty)$ for which there exists $C \in (0, \infty)$ such that for every finite set S in X satisfies the inequality $\mathrm{card}(S) \geq C \cdot (\delta(S)/\alpha(S))^{\beta}$.

Proposition 2.3.3. *For every metric space X, we have $\dim_{LA}(X, d) \leq \dim_A(X, d)$.*

The following is known as the Assouad embedding theorem [5]. The proof can be seen in [48, Theorem 3.15].

Theorem 2.3.4. *Let (X, d) be a doubling metric space. For every $\epsilon \in (0, 1)$, there exist $N \in \mathbb{R}$, a map $f : X \to \mathbb{R}^N$ and $L \in (0, \infty)$ such that for all $x, y \in X$ we have*

$$L^{-1} \cdot d(x, y)^{\epsilon} \leq d_{\mathbb{R}^N}(f(x), f(y)) \leq L \cdot d(x, y)^{\epsilon},$$

where $d_{\mathbb{R}^N}$ is the Euclidean metric.

2.4 The Gromov–Hausdorff distance

In order to simplify our description, if there is no confusion, for a metic space (X, d), and for a subset E of X, we often denote the restricted metric $d|_{E^2}$ on E by the same symbol d as the ambient metric d. For a metric space (Z, h), and for subsets S, T of Z, we define the Hausdorff distance $\mathcal{H}(S, T; Z, h)$ between S and T in Z as the infimum of all $r \in (0, \infty)$ for which $S \subset B(T, r)$ and $T \subset B(S, r)$. For two metric spaces (X, d) and (Y, e), the *Gromov–Hausdorff distance* $\mathcal{GH}((X, d), (Y, e))$ between (X, d) and (Y, e) is defined as the infimum of all values $\mathcal{H}(i(X), j(Y); Z, h)$, where (Z, h) is a metric space and $i : (X, d) \to (Z, h)$ and $j : (Y, e) \to (Z, h)$ are isometric embeddings. For a metric space (X, d), and for subsets A, B of X, we use the notation $\mathcal{GH}(A, B)$ instead of $\mathcal{GH}((A, d), (B, d))$.

To treat the Gromov–Hausdorff distance in terms of geometric maps, we use the so-called approximation maps. For $\epsilon \in (0, \infty)$, and for metric spaces (X, d) and (Y, e), a pair (f, g) of maps $f : X \to Y$ and $g : Y \to X$ is said to be an *ϵ-approximation* if the following three conditions are satisfied:

(1) for all $x, y \in X$, we have $|d(x, y) - e(f(x), f(y))| < \epsilon$;

(2) for all $x, y \in Y$, we have $|e(x, y) - d(g(x), g(y))| < \epsilon$;

(3) for all $x \in X$ and $y \in Y$, we have $d(g \circ f(x), x) < \epsilon$ and $e(f \circ g(y), y) < \epsilon$.

By the definition of the Gromov–Hausdorff distance, we obtain:

Proposition 2.4.1. *Let (X, d) and (Y, e) be two metric spaces. Then for every $k \in (0, \infty)$ we have $\mathcal{GH}((X, kd), (Y, ke)) = k \cdot \mathcal{GH}((X, d), (Y, e))$.*

The next two claims can be seen in Sections 7.3 and 7.4 in [15].

Lemma 2.4.2. *Let (X, d) and (Y, e) be two metric spaces. Let $\epsilon \in (0, \infty)$. If we have $\mathcal{GH}((X, d), (Y, e)) \leq \epsilon$, then there exists a 2ϵ-approximation between them.*

Proposition 2.4.3. *For all bounded metric spaces (X, d) and (Y, e), we have*

$$|\delta_d(X) - \delta_e(Y)| \leq 2 \cdot \mathcal{GH}((X, d), (Y, e)).$$

A triple (X, d, p) of a set X, a metric d on X, and a point p in X is called a *pointed metric space*. We say that a sequence $\{(X_i, d_i, p_i)\}_{i \in \mathbb{N}}$ of pointed metric spaces *converges to (Y, e, q) in the pointed Gromov–Hausdorff topology* if there exist a sequence $\{\epsilon_i\}_{i \in \mathbb{N}}$ in $(0, \infty)$ convergent to 0, and a sequence $\{(f_i, g_i)\}_{i \in \mathbb{N}}$ of ϵ_i-approximation maps between X_i and Y such that $f_i(p_i) = q$ and $g_i(q) = p_i$.

The Gromov–Hausdorff distance of snowflakes of two metric spaces is a snowflake of the Gromov–Hausdorff distance of the original two spaces.

Proposition 2.4.4. *Let $\epsilon \in (0, 1)$, and let (X, d) and (Y, e) be two metric spaces. Then we have $\mathcal{GH}((X, d^\epsilon), (Y, e^\epsilon)) = \mathcal{GH}((X, d), (Y, e))^\epsilon$.*

Proof. Let (Z, h) be a metric space. For all subsets $A, B \subset Z$, we have

$$\mathcal{H}(A, B; Z, h^\epsilon) = \mathcal{H}(A, B; Z, h)^\epsilon.$$

This leads to the proposition. □

Let (X, d) be a metric space, and let $\epsilon \in (0, \infty)$. We say that a subset S of X is an *ϵ-net* if S is finite and $B(S, \epsilon) = X$.

A metric space is said to be *totally bounded* if for each $\epsilon \in (0, \infty)$ the metric space has an ϵ-net. A metric space is totally bounded if and only if it is approximated by its finite subset in the sense of Gromov–Hausdorff.

By the definitions of total boundedness and $\mathcal{GH}$, we have:

Proposition 2.4.5. *Let (X, d) be a totally bounded metric space, and let (Y, e) a metric space. If $\mathcal{GH}((X, d), (Y, e)) \leq \epsilon$, then there exists a finite subset E of Y satisfying that $\mathcal{GH}((X, d), (E, e)) \leq 2\epsilon$.*

We say that a sequence $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ of metric spaces is *uniformly precompact* if for every $\epsilon \in (0, \infty)$ there exists $M \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, every ϵ-separated set in (X_i, d_i) has at most M many elements. Note that if there exists $N \in \mathbb{N}$ such that for each $i \in \mathbb{N}$, the space (X_i, d_i) is N-doubling, then the sequence $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ is uniformly precompact. We say that a sequence $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ of metric spaces is *uniformly bounded* if there exists $M \in (0, \infty)$ such that $\sup_{i\in\mathbb{N}} \delta_{d_i}(X_i) \leq M$.

We now recall the following Gromov precompactness theorem (see Section 7.4 in [15], or [48, Theorem 2.3]). This precompactness theorem is used to guarantee the existence of pseudo-cones of doubling metric spaces (see Proposition 4.1.2).

Theorem 2.4.6. *If a sequence $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ of compact metric spaces is uniformly precompact and uniformly bounded, then there exists a subsequence of $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ convergent to a compact metric space in the Gromov–Hausdorff convergence.*

2.5 Universal metric spaces

Let $\mathcal{C}$ be a class of metric spaces. We say that a metric space (X, d) is *$\mathcal{C}$-universal*, or *universal for* $\mathcal{C}$ if every metic space (A, d_A) in the class $\mathcal{C}$ can be isometrically embedded into (X, d).

We denote by $\mathcal{S}$ the class of all separable metic spaces. We say that a metric space (X, d) is *injective for all finite metric spaces*, or *finitely injective* if for every finite metric space $(F \sqcup \{p\}, e)$, and for every isometric embedding $I : (F, e|_{F^2}) \to (X, d)$, there exists a isometric embedding $J : (F \sqcup \{p\}, e) \to (X, d)$ satisfying that $I|_F = J$. The so-called back-and-forth argument implies that all finitely injective separable complete metric spaces are isometric to each other. This unique metric space is called the *Urysohn universal metric space*, which in this thesis we denote by $(\mathbb{U}, d_{\mathbb{U}})$. The Urysohn universal metric space was first constructed by Urysohn [118], and other constructions can be seen in, for example, [48], [42], [65], [54], [79], or [51]. By the back-and-forth argument, we obtain:

Proposition 2.5.1. *The Urysohn universal metric space $(\mathbb{U}, d_{\mathbb{U}})$ is $\mathcal{S}$-universal.*

The proof of the following can be seen in [8] or [48].

Proposition 2.5.2. *The space $(C([0, 1]), \| * \|_\infty)$ of all real-valued continuous functions on $[0, 1]$ equipped with the supremum norm $\| * \|_\infty$ is $\mathcal{S}$-universal.*

A metric space (X, d) is said to be *homogeneous* if for all $x, y \in X$, there exists an isometric bijection $f : X \to X$ such that $f(x) = y$. Since $(C([0, 1]), \| * \|_\infty)$ is a Banach space and its metric is invariant under the addition, the metric space $(C([0, 1]), \| * \|_\infty)$ is homogeneous.

From the back-and-forth argument, it follows that the Urysohn universal metric space $(\mathbb{U}, d_{\mathbb{U}})$ satisfies a stronger homogeneous property called *ω-homogeneity* or *ultrahomogeneity*: For every finite metric subspace F of $(\mathbb{U}, d_{\mathbb{U}})$, and for every isometric embedding $I : F \to \mathbb{U}$, there exists an isometric bijection $J : \mathbb{U} \to \mathbb{U}$ such that $J|_F = I$. The proof of this stroger homogeneity of $(\mathbb{U}, d_{\mathbb{U}})$ can be seen in, for example, [79], or [51].

As a summary of argument discussed above, we obtain:

Corollary 2.5.3. *The metric spaces $(\mathbb{U}, d_{\mathbb{U}})$ and $(C([0,1]), \| * \|_\infty))$ are homogeneous $\mathcal{S}$-universal separable complete metric spaces.*

Remark 2.5.1. For every uncountable compact metrizable X, the space $C(X)$ of all real-valued continuous functions on X is $\mathcal{S}$-universal, which follows from the Dugundji extension theorem (see [28] or Theorem 2.7.10 in this thesis) and the facts that the Cantor set has this property (see the proof of Theorem 3.6 in [48]) and every uncountable Polish space has a subset homeomorphic to the Cantor set (see [66, Corollary 6.5] or [105, Theorem 2.6.3]).

By the definition of homogeneity, we obtain:

Proposition 2.5.4. *Let (U, u) be a homogeneous $\mathcal{S}$-universal metric space. Then for every $q \in U$, and for every pointed separable metric space (X, d, x), there exists an isometric embedding $f : X \to U$ such that $f(x) = q$.*

The following is known as Fréchet's embedding theorem (see [36], [48], [90], or [49]):

Theorem 2.5.5. *The Banach space $(\ell^\infty, \| * \|_\infty)$ of all bounded sequences in $\mathbb{R}$ equipped with the supremum norm $\| * \|$ is $\mathcal{S}$-universal.*

Remark that ℓ^∞ is not separable in contrast to the spaces $(\mathbb{U}, d_{\mathbb{U}})$ and $(C([0,1]), \| * \|_\infty)$.

There are several studies on constructions of universal spaces for not only ordinary metrics but also ultrametrics. For each cardinal τ, Lemin and Lemin [74] constructed a universal ultrametric space for the class of all ultrametric spaces of topological weight τ, which we use in Chapter 7 in this thesis. Vaughan [119] discussed the minimality of cardinals of universal ultrametric spaces for all ultrametric spaces of topological weight τ by using a subset of Lemin–Lemin's universal ultrametric spaces. Vestfrid [120], Gao and Shao [38], and Wan [121] studied universal ultrametric spaces of Urysohn-type; namely, universal ultrametric spaces satisfying the injectivity for all finite ultrametric spaces.

2.6 Ultralimits

Let μ be a set consisting of subsets of $\mathbb{N}$. We say that μ is a *filter on* $\mathbb{N}$ if the following are satisfied:

(1) $\emptyset \notin \mu$;

(2) $\mathbb{N} \in \mu$;

(3) for all $A, B \in \mu$, we have $A \cap B \in \mu$;

(4) if $A \in \mu$ and $B \subset \mathbb{N}$ satisfy $A \subset B$, then $B \in \mu$.

A set μ consisting of subsets of $\mathbb{N}$ is said to be an *ultrafilter on* $\mathbb{N}$ if it is a filter on $\mathbb{N}$ and if for every subset A of $\mathbb{N}$, either A or $\mathbb{N} \setminus A$ is contained in μ. An ultrafilter on $\mathbb{N}$ is said to be *non-principal* if it does not contain any finite subset of $\mathbb{N}$. Let $P(i)$ be a predicate with a single free variable i running in $\mathbb{N}$. We say that $P(i)$ *holds true for* μ*-almost all* i if we have $\{ i \in \mathbb{N} \mid P(i) \text{ holds true} \} \in \mu$.

The axiom of choice guarantees the existence of a non-principle ultrafilter on $\mathbb{N}$.

Lemma 2.6.1. *There exists a non-principle ultrafilter on* $\mathbb{N}$.

Let μ be a non-principal ultrafilter on $\mathbb{N}$. For a sequence $\{a_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}$, a real number u is said to be a *ultralimit of* $\{a_i\}_{i\in\mathbb{N}}$ *with respect to* μ if for every $\epsilon \in (0, \infty)$ we have $\{ i \in \mathbb{N} \mid |a_i - u| < \epsilon \} \in \mu$. In other words, a real number u is an ultralimit of a sequence $\{a_i\}_{i\in\mathbb{N}}$ if the sequence converges to u for μ-almost all $i \in \mathbb{N}$. In this case, we write $\lim_\mu a_i = u$. Note that an ultralimit of a bounded sequence in $\mathbb{R}$ always uniquely exists.

Let $\mathrm{S} = \{(X_i, d_i, p_i)\}_{i\in\mathbb{N}}$ be a sequence of pointed metric spaces. We put

$$B(\{(X_i, d_i, p_i)\}_{i\in\mathbb{N}}) = \left\{ \{x_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i \mid \sup_{i\in\mathbb{N}} d_i(p_i, x_i) < \infty \right\}.$$

Define an equivalence relation R_μ on $B(\{(X_i, d_i, p_i)\}_{i\in\mathbb{N}})$ in such a way that the relation $\{x_i\}_{i\in\mathbb{N}} R_\mu \{y_i\}_{i\in\mathbb{N}}$ holds if and only if $\lim_\mu d_{X_i}(x_i, y_i) = 0$. We denote by $[\{x_i\}_{i\in\mathbb{N}}]$ the equivalence class of $\{x_i\}_{i\in\mathbb{N}}$. Put

$$\lim_\mu (X_i, d_i, p_i) = B(\{(X_i, d_i, p_i)\}_{i\in\mathbb{N}})/R_\mu,$$

and $p_{\mu,\mathrm{S}} = [\{p_i\}_{i\in\mathbb{N}}]$. We define a metric $m_{\mu,\mathrm{S}}$ by

$$m_{\mu,\mathrm{S}}(x, y) = \lim_\mu d_{X_i}(x_i, y_i),$$

where $x = [\{x_i\}_{i\in\mathbb{N}}]$ and $y = [\{y_i\}_{i\in\mathbb{N}}]$. We call $(\lim_\mu (X_i, d_i, p_i), m_{\mu,\mathrm{S}}, p_{\mu,\mathrm{S}})$ the *ultralimit of the sequence* $\mathrm{S} = \{(X_i, d_i, p_i)\}_{i\in\mathbb{N}}$ *of pointed metric spaces with respect to* μ. Even if a limit space of a given sequence of pointed metric spaces does not exist in the pointed Gromov–Hausdorff topology, an ultralimit of the sequence always exists and behaves as a limit space of the sequence.

The following can be seen in [11, I.5.52] or [62, Proposition 3.2].

Lemma 2.6.2. *Let* $\mathrm{S} = \{(X_i, d_i, p_i)\}_{i\in\mathbb{N}}$ *be a sequence of pointed compact metric spaces. If the sequence* $\{(X_i, d_i, p_i)\}_{i\in\mathbb{N}}$ *converges to a pointed compact metric space* (X, D, p) *in the pointed Gromov–Hausdorff topology , then the metric space* $(\lim_\mu (X_i, d_i, p_i), m_{\mu,\mathrm{S}}, p_{\mu,\mathrm{S}})$ *is isometric to* (X, D, p).

For every uniformly bounded sequence $\{X_i\}_{i\in\mathbb{N}}$, and for every choice $\{p_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i$ of base points, we have $B(\{(X_i, p_i)\}_{i\in\mathbb{N}}) = \prod_{i\in\mathbb{N}} X_i$. Therefore Lemma 2.6.2 implies:

Lemma 2.6.3. *Let* $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ *be a uniformly bounded sequence of compact metric spaces. If the sequence* $\{(X_i, d)\}_{i\in\mathbb{N}}$ *converges to a compact metric space* (L, D) *in the Gromov–Hausdorff sense, then for every choice* $\{p_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i$ *of base points, and for every ultrafilter* μ *on* $\mathbb{N}$*, the limit metric space* (L, D) *is isometric to the ultralimit of* $\{(X_i, d_i, p_i)\}_{i\in\mathbb{N}}$ *with respect to the ultrafilter* μ.

2.7 Continuous functions on metric spaces

A family consisting of subsets of a topological space is said to be *locally finite* if every point in the space has a neighborhood intersecting at most finitely many members of the family. A topological space is said to be *paracompact* if every open covering of the space has a refinement which is a locally finite open cover of the space.

The following is known as the Stone theorem on paracompactness of metric spaces, proven in [109].

Theorem 2.7.1. *All metrizable spaces are paracompact.*

A topological space X is said to be 0-*dimensional* if for every pair of disjoint two closed subsets A and B of X, there exists a clopen subset Q of X with $A \subset Q$ and $Q \cap B = \emptyset$. Such a space is sometimes also said to be *ultranormal*. Note that a metric space is 0-dimensional if and only if every finite open covering of the space has a refinement which is a covering consisting of mutually disjoint finite open subsets. This characterization follows from the Katětov–Morita theorem stating that for every metric space, the large inductive dimension coincides with the covering dimension This coincidence theorem was originally proven by Katětov [64] and Morita [84] independently (see also [93, Theorem 5.4], [17, Theorem 18.7], or [85, Theorem II.7]).

The following was proven by de Groot [43] (see also [21]).

Proposition 2.7.2. *All ultrametrizable spaces are* 0*-dimensional.*

2.7.1 The Michael continuous selection theorems

In the mid-1950s, Michael generalized the Tietze–Uryson extension theorem of functions on normal topological spaces as theorems on existence of selections of lower semi-continuous set-valued maps on topological spaces satisfying various topological properties such as normality, collectionwise normality, paracompactness, and ultraparacompactness (see, for example, [80], [81], [82]. These papers are only a few parts of Michael's numerous works). Developments of selection theorems can be seen in [97].

Let X and Y be two topological spaces. Let $\mathcal{S}$ be a set consisting of subsets of Y. A map $\phi : X \to \mathcal{S}$ is said to be *lower semi-continuous* if for every open subset O of Y the set $\{ x \in X \mid \phi(x) \cap O \neq \emptyset \}$ is open in X. A map $f : X \to Y$ is said to be a *selection of* ϕ if $f(x) \in \phi(x)$ for all $x \in X$. For a Banach space V, we denote by $\mathcal{CC}(V)$ the set of all non-empty closed convex subsets of V. Let Z be a metrizable space. We denote by $\mathcal{C}(Z)$ the set of all non-empty closed subsets of Z.

The following two theorems are utilized for our studies on interpolation theorems of metrics and ultrametrics. Theorem 2.7.3 is known as one of the Michael continuous selection theorems proven in [80]. Theorem 2.7.4 is known as the 0-dimensional Michael continuous selection theorem, which was stated in [82], essentially in [81]. The statements of the two theorems in this thesis seem to be slightly different from the original ones. These forms of the statements follows from Proposition 1.4 in [80], which states that problems on existence of a selection of a set-valued map and problems of extending a selection on a closed subset are equivalent.

Theorem 2.7.3. *Let X be a paracompact Hausdorff space, and A a closed subsets of X. Let V be a Banach space. Let $\phi : X \to \mathcal{CC}(V)$ be a lower semi-continuous map. If a continuous map $f : A \to V$ satisfies $f(x) \in \phi(x)$ for all $x \in A$, then there exists a continuous map $F : X \to V$ satisfying that $F|_A = f$ and $F(x) \in \phi(x)$ for all $x \in X$.*

Remark 2.7.1. The conclusion in Theorem 2.7.3 characterizes paracompactness (see [80]).

By Theorem 2.7.1, we can apply Theorem 2.7.3 to all metrizable spaces.

Theorem 2.7.4. *Let X be a 0-dimensional paracompact space, and let A be a closed subsets of X. Let Z be a completely metrizable space. Let $\phi : X \to \mathcal{C}(Z)$ be a lower semi-continuous map. If a continuous map $f : A \to Z$ satisfies $f(x) \in \phi(x)$ for all $x \in A$, then there exists a continuous map $F : X \to Z$ satisfying tha $F|_A = f$ and $F(x) \in \phi(x)$ for all $x \in X$.*

By Theorem 2.7.1 and Proposition 2.7.2, we can apply Theorem 2.7.4 to all ultra-metrizable spaces.

Let V be an Abelian group. We say that a metric d on V is *invariant*, or *invariant under the addition* if for all $x, y, a \in V$ we have

$$d(x + a, y + a) = d(x, y).$$

For example, for a normed or ultra-normed space $(B, \| * \|_B)$, the (ultra)metric d_B defined by $d_B(x, y) = \|x - y\|_B$ is invariant under the addition.

We now prove statements on the lower semi-continuity of concrete set-valued maps.

Proposition 2.7.5. *Let (V, d) be a pair of an Abelian group and an invariant metric under the addition. Let $x, y \in V$. Then for every $r \in (0, \infty)$ we have*

$$\mathcal{H}(B(x, r), B(y, r); V, d) \leq d(x, y).$$

Proof. Since the metric d is invariant under the addition, for every point $w \in B(y, r)$ we obtain $x+w-y \in B(x, r)$ and $d(w, x+w-y) = d(x, y)$. Therefore, we also obtain the inclusion $B(y, r) \subset B(B(x, r), d(x, y))$. In a similar way, we have $B(x, r) \subset B(B(y, r), d(x, y))$. Thus, we conclude that $\mathcal{H}(B(x, r), B(y, r)) \leq d(x, y)$. □

Proposition 2.7.6. *Let (V, d) be a pair of an Abelian group and an invariant metric under the addition. Let X be a topological space. Let $H : X \to V$ be a continuous map and $r \in (0, \infty)$. Then a map $\phi : X \to \mathcal{C}(V)$ defined by $\phi(x) = B(H(x), r)$ is lower semi-continuous.*

Proof. For every open subset O of V, and for every point $a \in X$ with $\phi(a) \cap O \neq \emptyset$, choose $u \in \phi(a) \cap O$ and $l \in (0, \infty)$ with $U(u, l) \subset O$. From Proposition 2.7.5, it follows that there exists a neighborhood N of a such that every $x \in N$ satisfies

$$\mathcal{H}(\phi(x), \phi(a); V, d) \leq d(H(x), H(a)) < l.$$

Then for every $x \in N$, we have $\phi(x) \cap U(u, l) \neq \emptyset$. Hence $\phi(x) \cap O \neq \emptyset$. Therefore the set $\{ x \in X \mid \phi(x) \cap O \neq \emptyset \}$ is open in X. This leads to the proposition. □

By letting V be a Banach or ultra-normed space in the statement of Proposition 2.7.6, we obtain the following two corollaries. Note that all closed balls in a Banach space is closed convex.

Corollary 2.7.7. *Let X be a topological space. Let V be a Banach space. Let $H : X \to V$ be a continuous map and $r \in (0, \infty)$. Then a set-valued map $\phi : X \to \mathcal{CC}(V)$ defined by $\phi(x) = B(H(x), r)$ is lower semi-continuous.*

Corollary 2.7.8. *Let X be a topological space. Let R be a commutative ring, and let $(V, \| * \|)$ be an ultra-normed R-module. Let $H : X \to V$ be a continuous map, and let $r \in (0, \infty)$. Then a set-valued map $\phi : X \to \mathcal{C}(V)$ defined by $\phi(x) = B(H(x), r)$ is lower semi-continuous.*

2.7.2 The Dugundji extension theorem

Dugundji [28] proved an extension theorem on continuous maps from metric spaces to locally convex linear spaces as a generalization of the Tietze–Urysohn extension theorem.

Theorem 2.7.9. *Let X be a metrizable space, and let A be a closed subset of X. Let E be a locally convex linear space. Then every continuous map $f : A \to E$ extends to a continuous map $F : X \to E$.*

This theorem has applications for extensions of metrics. Arens [1] and Toruńczyk [113] gave other proofs of the Hausdorff extension theorem (Theorem 2.2.3) by using the Dugundji extension theorem. Nguyen Van Khue and Nguyen To Nhu [68] used the Dugundji extension theorem to prove their extension theorem of metrics (Theorem 2.2.5). In the previous research of extension of metrics, the Dugundji extension theorem was frequently used. In this thesis, we use the Michael selection theorems instead of the Dugundji extension theorem.

For a topological space X, we denote by $C_b(X)$ the Banach space of all bounded continuous real-valued functions on X equipped with the supremum norm. The following is a form of the Dugundji extension theorem (see [28, Theorem 5.1]):

Theorem 2.7.10. *Let X be a metrizable space, and let A be a closed subset of X. Then there exists a linear map $\phi : C_b(A) \to C_b(X)$ such that for every $f \in C_b(A)$ we have $\phi(f)|_A = f$ and $\|f\| = \|\phi(f)\|$.*

2.7.3 A retraction theorem for ultrametrizable spaces

For a topological space X, we say that a closed subset A of X is a *retraction of* X if there exists a continuous function $f : X \to A$ such that $f|_A = 1_A$.

The following theorem was stated in [24, Theorem 1.1], and a Lipschitz version of it was proven in [12, Theorem 2.9] with a simple proof.

Theorem 2.7.11. *Every non-empty closed subset of an ultrametrizable space is a retraction of the whole space.*

Let X be an ultrametrizable space, and let A be a closed subset of X. Let Y be a topological space. By Theorem 2.7.11, there exists a retraction $r : X \to A$. For every map $f : A \to Y$, if we put $F = f \circ r$, then F is a continuous extension of f.

Corollary 2.7.12. *Let X be an ultrametrizable space, and let A be a non-empty closed subset of X. Then, for every topological space Y, every continuous map from A to Y can be extended to a continuous map from X to Y.*

Chapter 3

Basic statements on metrics and ultrametrics

In this chapter, we review basic statements on metrics and ultrametrics.

3.1 Telescope construction

In this section, we introduce the telescope construction, originally defined in the author's paper [57]. In this thesis, this construction is used to show the existence of a metric $(\omega_0 + 1)$-space with certain geometric properties.

In this thesis, we sometimes use the disjoint union $\coprod_{i\in I} A_i$ of non-disjoint family $\{A_i\}_{i\in I}$. Whenever we consider the disjoint union $\coprod_{i\in I} A_i$ of a family $\{A_i\}_{i\in I}$ of sets (this family is not necessarily disjoint), we identify the family $\{A_i\}_{i\in I}$ with its disjoint copy unless otherwise stated.

Definition 3.1.1 ([57])**.** We say that a triple $\mathbf{B} = (B, d_B, b)$ is a *telescope base* if (B, d_B) is a metric space homeomorphic to the one-point compactification of $\mathbb{N}$, and if b is a bijective map $b : \mathbb{N} \cup \{\infty\} \to B$ such that b_∞ is the unique accumulation point of B. Let $\mathbf{B} = (B, d_B, b)$ be a telescope base. For $n \in \mathbb{N}$ we put

$$R_n(\mathbf{B}) = \sup\{\, r \in (0, \infty) \mid U(b_n, r) = \{b_n\}\}.$$

Note that $R_n(\mathbf{B})$ is equal to the distance from b_n to $B \setminus \{b_n\}$ with respect to d_B.

Definition 3.1.2. Define a metric on $\mathbb{N} \cup \{\infty\}$ by

$$D(n,m) = \begin{cases} 2^{-\min\{n,m\}} & n \neq m; \\ 0 & n = m, \end{cases}$$

where we consider that $n < \infty$ for all $n \in \mathbb{N}$. The triple $\mathbf{A} = \big(\mathbb{N} \cup \{\infty\}, D, 1_{\mathbb{N}\cup\{\infty\}}\big)$ is a telescope base. Remark that $R_n(\mathbf{A}) = 2^{-n}$ for all $n \in \mathbb{N}$.

Let $\mathbf{X} = \{(X_i, d_i)\}_{i\in\mathbb{N}}$ be a countable family of metric spaces, and let $\mathbf{B} = (B, d_B, b)$ be a telescope base. We say that $\mathcal{P} = (\mathbf{X}, \mathbf{B})$ is a *compatible pair* if for each $n \in \mathbb{N}$ we have $\operatorname{diam}(X_n) \le R_n(\mathbf{B})$. Let $\mathcal{P} = (\mathbf{X}, \mathbf{B})$ be a compatible pair. Put

$$T(\mathcal{P}) = \{\infty\} \sqcup \coprod_{i\in\mathbb{N}} X_i,$$

and define a function $d_{\mathcal{P}}$ on $T(\mathcal{P})$ by

$$d_{\mathcal{P}}(x, y) = \begin{cases} d_i(x, y) & \text{if } x, y \in X_i \text{ for some } i, \\ d_B(b_i, b_j) & \text{if } x \in X_i, y \in X_j \text{ for some } i \neq j, \\ d_B(b_\infty, b_i) & \text{if } x = \infty, y \in X_i \text{ for some } i, \\ d_B(b_i, b_\infty) & \text{if } x \in X_i, y = \infty \text{ for some } i. \end{cases}$$

Then, we have:

Lemma 3.1.1. *The function $d_{\mathcal{P}}$ is a metric.*

Proof. We first put $X_\infty = \{\infty\}$. By the definition, the function $d_{\mathcal{P}}$ satisfies the conditions (M1) and (M2) in the definition of a metric, thus it suffices to show that $d_{\mathcal{P}}$ satisfies the triangle inequality.

Let $i, j, k \in \mathbb{N} \sqcup \{\infty\}$ be distinct numbers. In the case where $x, y, z \in X_i$, the triangle inequality is satisfied since d_i is a metric. In the case where $x \in X_i$ and $y, z \in X_j$, we have

$$d_{\mathcal{P}}(x, y) = d_B(b_i, b_j) \leq d_B(b_i, b_j) + d_j(y, z) = d_{\mathcal{P}}(x, z) + d_{\mathcal{P}}(z, y).$$

In the case where $x, y \in X_i$ and $z \in X_j$, we have

$$d_{\mathcal{P}}(x, y) = d_i(x, y) \leq \delta_{d_i}(X_i) \leq R_i(\mathbf{B}) \leq d_B(b_i, b_j) \leq d_{\mathcal{P}}(x, z) + d_{\mathcal{P}}(z, y).$$

In the case where $x \in X_i$, $y \in X_j$, and $z \in X_k$, we have

$$d_{\mathcal{P}}(x, y) = d_B(b_i, b_j) \leq d_B(b_i, b_k) + d_B(b_k, b_j) = d_{\mathcal{P}}(x, z) + d_{\mathcal{P}}(z, y).$$

Since i, j, k are arbitrary, we conclude that $d_{\mathcal{P}}$ is a metric. □

Based on Lemma 3.1.1, we call the metric space $(T(\mathcal{P}), d_{\mathcal{P}})$ the *telescope metric space of* $\mathcal{P}$. Note that this construction was first defined in [57].

By replacing "+" with "$\vee$" in the proof of Lemma 3.1.1, we obtain the proof of the following lemma first stated in [57, Lemma 3.1]:

Lemma 3.1.2. *Let $\mathcal{P} = (\mathbf{X}, \mathbf{B})$ be a telescope pair. If $\mathbf{X}$ and $\mathbf{B}$ consist of ultrametric spaces, then the space $(T(\mathcal{P}), d_{\mathcal{P}})$ is an ultrametric space.*

In the case where $\mathbf{X}$ consists of finite metric spaces, we obtain:

Lemma 3.1.3. *Let $\mathcal{P} = (\mathbf{X}, \mathbf{B})$ be a telescope pair. If each member in $\mathbf{X}$ is a finite metric space, then the space $(T(\mathcal{P}), d_{\mathcal{P}})$ is a metric $(\omega_0 + 1)$-space.*

3.2 Amalgamation methods

Amalgamation methods of metrics are ways to glue metrics together under a certain condition. Amalgamation statements such as Propositions 3.2.1 and 3.2.2 are folklore in the theory of metric spaces.

3.2.1 Amalgamations of metrics

The following two propositions are known as amalgamations of metrics. The proofs can be seen in, for example, [10] (cf., [118], [44]). For the sake of self-containedness, we give proofs of these statements.

Proposition 3.2.1. *Let (X, d_X) and (Y, d_Y) be metric spaces. If $X \cap Y \neq \emptyset$ and $d_X = d_Y$ on $(X \cap Y)^2$, then there exists a metric h on $X \cup Y$ such that*

(1) $h|_{X^2} = d_X$;

(2) $h|_{Y^2} = d_Y$.

Proof. We define a symmetric function $h : (X \cup Y)^2 \to [0, \infty)$ by

$$h(x, y) = \begin{cases} d_X(x, y) & \text{if } x, y \in X; \\ d_Y(x, y) & \text{if } x, y \in Y; \\ \inf_{z \in X \cap Y}(d_X(x, z) + d_Y(z, y)) & \text{if } (x, y) \in X \times Y. \end{cases}$$

The assumption $d_X|_{(X \cap Y)^2} = d_Y|_{(X \cap Y)^2}$ implies that the function h is well-defined. By the definition of h, the conditions (1) and (2) in the proposition are satisfied.

We next prove that h satisfies the triangle inequality. In the case where $x, y \in X$ and $z \in Y$, by the definition of h, for all $a, b \in X \cap Y$ we have

$$\begin{aligned} h(x, y) = d_X(x, y) &\leq d_X(x, a) + d_X(a, b) + d_X(b, y) = d_X(x, a) + d_Y(a, b) + d_X(b, y) \\ &\leq (d_X(x, a) + d_Y(a, z)) + (d_Y(z, b) + d_X(b, y)). \end{aligned}$$

Thus we obtain $h(x, y) \leq h(x, z) + h(z, y)$. In the case where $x, z \in X$ and $y \in Y$, every point $a \in X \cap Y$ satisfies

$$h(x, y) \leq d_X(x, a) + d_Y(a, y) \leq d_X(x, z) + d_X(z, a) + d_Y(a, y).$$

Thus we have $h(x, y) \leq h(x, z) + h(z, y)$. By replacing the role of X with that of Y, we conclude that h satisfies the triangle inequality, and hence it is a metric on $X \cup Y$. □

For a mutually disjoint family $\{T_i\}_{i \in I}$ of topological spaces, we consider that the space $\coprod_{i \in I} T_i$ is always equipped with the direct sum topology.

Proposition 3.2.2. *Let (X, d_X) and (Y, d_Y) be metric spaces. If $X \cap Y = \emptyset$, then for every $r \in (0, \infty)$ there exists a metric $h \in \mathrm{M}(X \sqcup Y)$ such that*

(1) $h|_{X^2} = d_X$;

(2) $h|_{Y^2} = d_Y$;

(3) *for all $x \in X$ and $y \in Y$ we have $r \leq h(x, y)$.*

Proof. Take two fixed points $a \in X$ and $b \in Y$. Take $r \in (0, \infty)$. We define a symmetric function $h : (X \cup Y)^2 \to [0, \infty)$ by

$$h(x, y) = \begin{cases} d_X(x, y) & \text{if } x, y \in X; \\ d_Y(x, y) & \text{if } x, y \in Y; \\ d_X(x, a) + r + d_Y(b, y). & \text{if } (x, y) \in X \times Y. \end{cases}$$

By this definition, we see that the conditions (1), (2) and (3) are satisfied.

We prove that the function h satisfies the triangle inequality. In the case where $x, y \in X$ and $z \in Y$, we have

$$\begin{aligned} h(x,y) = d_X(x,y) &\le d_X(x,a) + d_X(a,y) \\ &\le (d_X(x,a) + r + d_Y(b,z)) + (d_X(y,a) + r + d_Y(b,z) = h(x,z) + h(z,y). \end{aligned}$$

Thus we obtain $h(x,y) \le h(x,z) + h(z,y)$. In the case where $x, z \in X$ and $y \in Y$, we have

$$\begin{aligned} h(x,y) &= d_X(x,a) + r + d_Y(b,y) \\ &\le d_X(x,z) + d_X(z,a) + r + d_Y(b,y) = h(x,z) + h(z,y). \end{aligned}$$

Thus we obtain $h(x,y) \le h(x,z) + h(z,y)$.

By replacing the role of X with that of Y, we conclude that h satisfies the triangle inequality, and hence it belongs to $\mathrm{M}(X \sqcup Y)$. □

Let X and Y be sets, and let $\tau : X \to Y$ be a bijection. For a metric d on Y, we denote by $\tau^* d$ the metric on X defined by $(\tau^* d)(x,y) = d(\tau(x), \tau(y))$. This metric is sometimes called a *pullback metric of d by τ*. Note that the map τ is an isometry between $(X, \tau^* d)$ and (Y, d).

The following proposition can be considered as a specific case of the realization of the Gromov–Hausdorff distance of two metric spaces (see [15, Chapter 7]).

Proposition 3.2.3. *Let X be a metrizable space. Assume that $r \in (0, \infty)$ and metrics $d, e \in \mathrm{M}(X)$ satisfy $\mathcal{D}_X(d,e) \le r$. Put $X_0 = X$, and let X_1 be a set satisfying that $\mathrm{card}(X_1) = \mathrm{card}(X_0)$ and $X_0 \cap X_1 = \emptyset$. Let $\tau : X_0 \to X_1$ be a bijection. Then there exists a metric $h \in \mathrm{M}(X_0 \sqcup X_1)$ such that*

(1) $h|_{X_0^2} = d$;

(2) $h|_{X_1^2} = (\tau^{-1})^* e$;

(3) *for every $x \in X_0$ we have $h(x, \tau(x)) = r/2$.*

Proof. We define a symmetric function $h : (X_0 \sqcup X_1)^2 \to [0, \infty)$ by

$$h(x,y) = \begin{cases} d(x,y) & \text{if } x, y \in X_0; \\ (\tau^{-1})^* e(x,y) & \text{if } x, y \in X_1; \\ \inf_{a \in X_0}(d(x,a) + r/2 + (\tau^{-1})^* e(\tau(a), y)) & \text{if } (x,y) \in X_0 \times X_1. \end{cases}$$

By the definition, every point $x \in X$ satisfies $h(x, \tau(x)) \ge r/2$, and

$$h(x, \tau(x)) \le d(x,x) + r/2 + (\tau^{-1})^* e(\tau(x), \tau(x)) = r/2.$$

Therefore every point $x \in X$ satisfies $h(x, \tau(x)) = r/2$.

We next prove that h satisfies the triangle inequality. In the case where $x, y \in X_0$ and $z \in X_1$, all points $a, b \in X_0$ satisfy

$$\begin{aligned} h(x,y) = d(x,y) &\le d(x,a) + d(a,b) + d(b,y) \\ &\le d(x,a) + r + (\tau^{-1})^* e(\tau(a), \tau(b)) + d(b,y) \\ &\le d(x,a) + r + (\tau^{-1})^* e(\tau(a), z) + (\tau^{-1})^* e(\tau(b), z) + d(b,y) \\ &\le (d(x,a) + r/2 + (\tau^{-1})^* e(\tau(a), z)) + (d(y,b) + r/2 + (\tau^{-1})^* e(\tau(b), z)). \end{aligned}$$

Thus we obtain $h(x, y) \leq h(x, z) + h(z, y)$. In the case where $x, z \in X_0$ and $y \in X_1$, every point $a \in X_0$ satisfies

$$h(x, y) \leq d(x, a) + r/2 + (\tau^{-1})^* e(\tau(a), y) \leq d(x, z) + (d(z, a) + r/2 + (\tau^{-1})^* e(\tau(a), y)).$$

Thus $h(x, y) \leq h(x, z) + h(z, y)$. By replacing the role of X_0 with that of X_1, we conclude that the metric h satisfies the triangle inequality, and hence it is a metric as required. □

Lemma 3.2.4. *Let $\{(A_i.e_i)\}_{i \in I}$ be a family consisting of mutually disjoint metric spaces. Then there exists a metric $h \in \mathrm{M}(\coprod_{i \in I} A_i)$ such that for every $i \in I$ we have $h|_{A_i^2} = e_i$.*

Proof. We may assume that I is an ordinal. By transfinite induction, we construct a desired metric h as follows: Let $a \in I + 1$. Assume that for every $b < a$ we already define metrics $\{h_b\}_{b<a}$ such that

(1) for every $b < a$, we have $h_b \in \mathrm{M}(\coprod_{i<b} A_i)$;

(2) if $i < j < a$, then for all $x, y \in A_i$ we have $h_j(x, y) = h_i(x, y)$;

(3) if $i \neq j$, $i, j < b$, and $x \in A_i$ and $y \in A_j$, then we have $1 \leq h_b(x, y)$.

If $a = b + 1$, then we can define a metric $h_a \in \mathrm{M}(\coprod_{i<a} A_i)$ by using Proposition 3.2.2 for $X = \coprod_{i<b} A_i$, $Y = A_a$ and $r = 1$. Next assume that a is a limit ordinal. We define a function h_a on $\left(\coprod_{i<a} A_i\right)^2$ by

$$h_a(x, y) = h_i(x, y),$$

where $i < a$ is the first ordinal with $x, y \in \coprod_{k<i} A_k$. By the inductive hypothesis (1), the function h_a is well-defined. From the hypotheses (2) and (3), it follows that h_a is a metric with $h_a \in \mathrm{M}(\coprod_{i<a} A_i)$. Put $h = h_I$, then the lemma is proven. □

Lemma 3.2.5. *Let X be a metrizable space, and let $\{A_i\}_{i \in I}$ be a discrete family of closed subsets of the space X. Let $d \in \mathrm{M}(X)$, and let $\{e_i\}_{i \in I}$ be a family of metrics such that $e_i \in M(A_i)$. Put*

$$\eta = \sup_{i \in I} \mathcal{D}_{A_i}(e_{A_i}, d|_{A_i^2})$$

and assume that $\eta < \infty$. Let $\{B_i\}_{i \in I}$ be a family of mutually disjoint sets such that $\mathrm{card}(B_i) = \mathrm{card}(A_i)$ and $X \cap B_i = \emptyset$ for all $i \in I$. For each $i \in I$, let $\tau_i : A_i \to B_i$ be a bijection. Let $\tau : \coprod_{i \in I} A_i \to \coprod_{i \in I} B_i$ be a natural bijective map induced from $\{\tau_i\}_{i \in I}$. Then there exists a metric h on $X \sqcup \coprod_{i \in I} B_i$ satisfying the following conditions:

(1) *for every $i \in I$ we have $h|_{B_i^2} = (\tau_i^{-1})^* e_i$;*

(2) *$h|_{X^2} = d$;*

(3) *for every $x \in \coprod_{i \in I} A_i$ we have $h(x, \tau(x)) = \eta/2$.*

Proof. By Proposition 3.2.3, for all $i \in I$, we find a metric $l_i \in \mathrm{M}(A_i \sqcup B_i)$ such that

(1) $l_i|_{A_i^2} = d|_{A_i^2}$;

(2) $l_i|_{B_i^2} = (\tau_i^{-1})^* e_i$;

(3) for all $x \in A_i$ we have $l_i(x, \tau(x)) = \eta/2$.

By Lemma 3.2.4, we obtain a metric $k \in \mathrm{M}(\coprod_{i\in I}(A_i \sqcup B_i))$ such that for each $i \in I$ we have

$$k|_{(A_i \sqcup B_i)^2} = l_i.$$

Since we have

$$X \cap \left(\coprod_{i\in I}(A_i \sqcup B_i)\right) = \coprod_{i\in I} A_i,$$

by Proposition 3.2.1, we obtain a metric h on $X \sqcup \coprod_{i\in I} B_i$ such that

(1) $h|_{X^2} = d$;

(2) $h|_{(\coprod_{i\in I} B_i)^2} = k|_{(\coprod_{i\in I} B_i)^2}$.

By the definitions of metrics l_i and k, we see that h is a metric as required. □

3.2.2 Amalgamations of ultrametrics

By replacing the symbol "+" with the symbol "∨" in the proofs of the statements in Subsection 3.2.1, we can prove ultrametric analogues of the amalgamation statements stated in the previous subsection.

The following is an ultrametric analogue of Proposition 3.2.1. The proof also can be seen in, for example, [10, Theorem 2.2].

Proposition 3.2.6. *Let S be a range set. Let (X, d_X) and (Y, d_Y) be two S-valued ultrametric spaces. If $X \cap Y = \emptyset$, then for every $r \in S_+$ there exists an S-valued ultrametric $h \in \mathrm{UM}(X \sqcup Y, S)$ such that*

(1) *$h|_{X^2} = d_X$;*

(2) *$h|_{Y^2} = d_Y$;*

(3) *for all $x \in X$ and $y \in Y$ we have $r \leq h(x, y)$.*

Proof. Take two fixed points $a \in X$ and $b \in Y$. Take $r \in (0, \infty)$. We define a symmetric function $h : (X \cup Y)^2 \to [0, \infty)$ by

$$h(x, y) = \begin{cases} d_X(x, y) & \text{if } x, y \in X; \\ d_Y(x, y) & \text{if } x, y \in Y; \\ d_X(x, a) \vee r \vee d_Y(b, y). & \text{if } (x, y) \in X \times Y. \end{cases}$$

Similarly to the proof of Theorem 3.2.1, then the conditions (1), (2) and (3) are satisfied, and h is an S-valued ultrametric on $X \sqcup Y$. □

As a consequence of Proposition 3.2.6, we can construct a one-point extension of an S-valued ultrametric space.

Corollary 3.2.7. *Let S be a range set possessing at least two elements. Let (X, d) be an S-valued ultrametric space, and let $o \notin X$. Then there exists an S-valued ultrametric $D \in \mathrm{UM}(X \sqcup \{o\}, S)$ with $D|_{X^2} = d$.*

The following lemma is a specialized version of Proposition 3.2.2 for our study on S-valued ultrametrics.

Lemma 3.2.8. *Let S be a range set. Let (X, d_X) and (Y, d_Y) be S-valued ultrametric spaces. Assume that*

(A) $Z \neq \emptyset$;

(B) $d_X = d_Y$ *on* $(X \cap Y)^2$;

(C) *there exists* $s \in S_+$ *such that for every* $x \in X \setminus (X \cap Y)$ *we have* $\inf_{z \in X \cap Y} d_X(x, z) = s$.

Then there exists an S*-valued ultrametric* h *on* $X \cup Y$ *such that*

(1) $h|_{X^2} = d_X$;

(2) $h|_{Y^2} = d_Y$.

Proof. We define a symmetric function $h : (X \cup Y)^2 \to [0, \infty)$ by

$$h(x,y) = \begin{cases} d_X(x,y) & \text{if } x, y \in X; \\ d_Y(x,y) & \text{if } x, y \in Y; \\ \inf_{z \in X \cap Y}(d_X(x,z) \vee d_Y(z,y)) & \text{if } (x,y) \in X \times Y. \end{cases}$$

Since $d_X|_{(X \cap Y)^2} = d_Y|_{(X \cap Y)^2}$, the function h is well-defined. By the definition, h satisfies the conditions (1) and (2) in the statement. By a similar argument to Proposition 3.2.2, we see that h satisfies the strong triangle inequality.

We now prove that h takes values in S. It suffices to show that all points $x \in X \setminus (X \cap Y)$ and $y \in Y \setminus (X \cap Y)$ satisfy $h(x, y) \in S$. By the assumption (C) and the definition of h, we obtain $s \leq h(x, y)$. If $s = h(x, y)$, then $h(x, y)$ is in S. If $s < h(x, y)$, then by the assumption (C), there exists $z \in X \cap Y$ with $h(x, z) < h(x, y)$. Lemma 3.3.7 implies that $h(x, y) = h(z, y)$. Since $h(z, y) = d_Y(z, y)$, we have $h(x, y) \in S$. □

The following proposition is an ultrametric version of Proposition 3.2.3.

Proposition 3.2.9. *Let* S *be a range set. Let* X *be an ultrametrizable space. Assume that* $r \in S_+$ *and* $d, e \in \mathrm{UM}(X, S)$ *satisfy* $\mathcal{UD}_X^S(d, e) \leq r$. *Put* $X_0 = X$, *and let* X_1 *be a set satisfying that* $\mathrm{card}(X_1) = \mathrm{card}(X_0)$ *and* $X_0 \cap X_1 = \emptyset$. *Let* $\tau : X_0 \to X_1$ *be a bijective map. Then there exists an ultrametric* $h \in \mathrm{UM}(X_0 \sqcup X_1, S)$ *such that*

(1) $h|_{X_0^2} = d$;

(2) $h|_{X_1^2} = (\tau^{-1})^* e$;

(3) *for every* $x \in X_0$ *we have* $h(x, \tau(x)) = r$.

Proof. We define a symmetric function $h : (X_0 \sqcup X_1)^2 \to [0, \infty)$ by

$$h(x,y) = \begin{cases} d(x,y) & \text{if } x, y \in X_0; \\ (\tau^{-1})^* e(x,y) & \text{if } x, y \in X_1; \\ \inf_{a \in X_0}(d(x,a) \vee r \vee (\tau^{-1})^* e(\tau(a), y)) & \text{if } (x,y) \in X_0 \times X_1. \end{cases}$$

By the definition of h, every point $x \in X$ satisfies $h(x, \tau(x)) \geq r$, and we have

$$h(x, \tau(x)) \leq d(x,x) \vee r \vee (\tau^{-1})^* e(\tau(x), \tau(x)) = r.$$

Therefore every point $x \in X$ satisfies $h(x, \tau(x)) = r$.

We now prove that h satisfies the strong triangle inequality. In the case where $x, y \in X_0$ and $z \in X_1$, by $\mathcal{UD}_X^S(d, e) \le r$ all points $a, b \in X_0$ satisfy

$$\begin{aligned} h(x, y) = d(x, y) &\le d(x, a) \vee d(a, b) \vee d(b, y) \\ &\le d(x, a) \vee r \vee (\tau^{-1})^* e(\tau(a), \tau(b)) \vee d(b, y) \\ &\le d(x, a) \vee r \vee (\tau^{-1})^* e(\tau(a), z) \vee (\tau^{-1})^* e(\tau(b), z) \vee d(b, y) \\ &\le (d(x, a) \vee r \vee (\tau^{-1})^* e(\tau(a), z)) \vee (d(y, b) \vee r \vee (\tau^{-1})^* e(\tau(b), z)). \end{aligned}$$

Thus we obtain $h(x, y) \le h(x, z) \vee h(z, y)$. In the case where $x, z \in X_0$ and $y \in X_1$, every point $a \in X_0$ satisfies

$$\begin{aligned} h(x, y) &\le d(x, a) \vee r \vee (\tau^{-1})^* e(\tau(a), y) \\ &\le d(x, z) \vee (d(z, a) \vee r \vee (\tau^{-1})^* e(\tau(a), y)). \end{aligned}$$

Thus $h(x, y) \le h(x, z) \vee h(z, y)$. By replacing the role of X_0 with that of X_1, we see that h satisfies the strong triangle inequality. By the property (3), we also see that $h \in \mathrm{UM}(X_0 \sqcup X_1)$.

We next prove that h takes values in S. It suffices to show that for all $(x, y) \in X_0 \times X_1$, we have $h(x, y) \in S$. The definition of h yields $r \le h(x, y)$. If $r = h(x, y)$, then $h(x, y)$ is in S. If $r < h(x, y)$, then by $h(x, \tau(x)) = r$, we have $h(x, \tau(x)) < h(x, y)$. From Lemma 3.3.7, it follows that $h(x, y) = h(\tau(x), y)$. Since $h(\tau(x), y) = (\tau^{-1})^* e(\tau(x), y)$, we conclude that h is S-valued. □

The next lemma is an ultrametric version of Lemma 3.2.4.

Lemma 3.2.10. *Let S be a range set, and let $s \in S_+$. Let $\{(A_i, e_i)\}_{i \in I}$ be a mutually disjoint family of S-valued ultrametric spaces. Then there exists an ultrametric $h \in \mathrm{UM}(\coprod_{i \in I} A_i, S)$ such that*

(1) *for every $i \in I$ we have $h|_{A_i^2} = e_i$;*

(2) *for all distinct $i, j \in I$, and for all $x \in A_i$ and $y \in A_j$, we have $s \le h(x, y)$.*

Proof. We may assume that I is an ordinal. Similarly to the proof of Lemma 3.2.4, by transfinite induction, Proposition 3.2.6 guarantees the existence of a sequence $\{h_i\}_{i < I+1}$ of S-valued ultrametrics such that

(1) for every $a < I + 1$ we have $h_a \in \mathrm{UM}(\coprod_{i < a} A_i, S)$;

(2) if $i < j < a$, then for all $x, y \in A_i$ we have $h_j(x, y) = h_i(x, y)$;

(3) if $i \ne j$ and $x \in A_i$ and $y \in A_j$, then for every $a < I + 1$ with $i < a$ and $j < a$, we have $s \le h_a(x, y)$.

Put $h = h_I$, then it is a desired S-valued ultrametric. □

The following is an ultrametric analogue of Lemma 3.2.5. This lemma has a key role in the proof of Theorem 1.2.15.

Lemma 3.2.11. *Let S be a range set. Let X be an ultrametrizable space, and let $\{A_i\}_{i \in I}$ be a discrete family of closed subsets of X. Let $d \in \mathrm{UM}(X, S)$, and let $\{e_i\}_{i \in I}$ be a family of* *ultrametrics with* $e_i \in \mathrm{UM}(A_i, S)$ *for all* $i \in I$. *Assume that* $\sup_{i\in I} \mathcal{UD}^S_{A_i}(e_{A_i}, d|_{A_i^2}) < \infty$. *Let* η *be a member in* S_+ *such that*

$$\sup_{i\in I} \mathcal{UD}^S_{A_i}(e_{A_i}, d|_{A_i^2}) \leq \eta.$$

Let $\{B_i\}_{i\in I}$ *be a family of mutually disjoint sets with* $\mathrm{card}(B_i) = \mathrm{card}(A_i)$ *and* $X \cap B_i = \emptyset$ *for all* $i \in I$. *For each* $i \in I$, *let* $\tau_i : A_i \to B_i$ *be a bijection. Let* $\tau : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ *be a natural bijective map induced from* $\{\tau_i\}_{i\in I}$. *Then there exists an* S*-valued ultrametric* h *on* $X \sqcup \coprod_{i\in I} B_i$ *such that*

(1) *for every* $i \in I$ *we have* $h|_{B_i^2} = (\tau_i^{-1})^* e_i$;

(2) $h|_{X^2} = d$;

(3) *for every* $x \in \coprod_{i\in I} A_i$ *we have* $h(x, \tau(x)) = \eta$.

Proof. By using Proposition 3.2.9, for every $i \in I$, we first obtain an S-valued ultrametric $l_i \in \mathrm{UM}(A_i \sqcup B_i, S)$ such that

(1) $l_i|_{A_i^2} = d|_{A_i^2}$;

(2) $l_i|_{B_i^2} = (\tau_i^{-1})^* e_i$;

(3) for every $x \in A_i$ we have $l_i(x, \tau(x)) = \eta$.

By Lemma 3.2.10, we obtain an S-valued ultrametric $k \in \mathrm{UM}(\coprod_{i\in I}(A_i \sqcup B_i), S)$ such that

(1) for each $i \in I$ we have $k|_{(A_i \sqcup B_i)^2} = l_i$;

(2) for all distinct $i, j \in I$, and for all $x \in A_i \sqcup B_i$ and $y \in A_j \sqcup B_j$, we have $\eta \leq h(x, y)$.

We see that

$$X \cap \left(\coprod_{i\in I} (A_i \sqcup B_i) \right) = \coprod_{i\in I} A_i.$$

By the property (3) of the definition of h, the ultrametric k satisfies the assumptions stated in Lemma 3.2.8. Therefore we obtain an S-valued ultrametric h on $X \sqcup \coprod_{i\in I} B_i$ such that

(1) $h|_{X^2} = d$;

(2) $h|_{(\coprod_{i\in I} B_i)^2} = k|_{(\coprod_{i\in I} B_i)^2}$.

By the definitions of l_i and k, we conclude that h is a desired S-valued ultrametric. □

3.3 Basic properties of S-valued ultrametric spaces

In this section, we investigate basic properties on S-valued ultrametrics for a range set S.

3.3.1 Modification of ultrametrics

A function $\psi : [0, \infty) \to [0, \infty)$ is said to be *amenable* if $\psi^{-1}(\{0\}) = \{0\}$. Pongsriiam and Termwuttipong [95] proved the following (see [95, Theorem 9], and see also [27, Theorem 2.9]):

Theorem 3.3.1. *Let $\psi : [0, \infty) \to [0, \infty)$ be a function. Then the following statements are equivalent:*

(1) *ψ is increasing and amenable;*

(2) *for every set X, and for every ultrametric d on X, the function $\psi \circ d$ is an ultrametric on X.*

(3) *for every set X with $\mathrm{card}(X) = 3$, and for every ultrametric d on X, the function $\psi \circ d$ is an ultrametric on X.*

Remark 3.3.1. The condition (3) in Theorem 3.3.1 does not appear explicitly in the statement of [95, Theorem 9]; however, the proof of [95, Theorem 9] contains it.

The following lemma is an application of Pongsriiam and Termwuttipong's result for topologically compatible ultrametrics, and it can be considered as an ultrametric analogue of [23, Theorem 3.2].

Lemma 3.3.2. *Let $\psi : [0, \infty) \to [0, \infty)$ be a function. Then the following statements are equivalent:*

(1) *ψ is increasing, amenable, and continuous at the point 0.*

(2) *for every topological space X, and for every $d \in \mathrm{UM}(X)$, we have $\psi \circ d \in \mathrm{UM}(X)$.*

Proof. We first prove the implication (1) $\Longrightarrow$ (2). Take $d \in \mathrm{UM}(X)$. By Theorem 3.3.1, we see that $\psi \circ d$ is an ultrametric on X. We now prove that $\psi \circ d$ induces the same topology on X. Take $x \in X$ and $r \in (0, \infty)$. Since ψ is continuous at 0, there exists $l \in (0, \infty)$ with $\psi(l) < r$. Then, we have $U(x, l; d) \subset U(x, r; \psi \circ d)$. Since ψ is increasing, we have $U(x, \psi(r)/2; \psi \circ d) \subset U(x, r; d)$. Since $x \in X$ and $r \in (0, \infty)$ are arbitrary, we conclude that $\psi \circ d \in \mathrm{UM}(X)$.

We next prove the implication (2) $\Longrightarrow$ (1). By the equivalence of the conditions (1) and (3) in Theorem 3.3.1, we see that ψ is increasing and amenable. In order to show that ψ is continuious at 0, take an arbitrary strictly sequence $\{r(n)\}_{n \in \mathbb{N}}$ with $\lim_{n\to\infty} r(n) = 0$. Put $r(\infty) = 0$. Put $X = \mathbb{N} \sqcup \{\infty\}$, and we consider that X is the one-point compactification of $\mathbb{N}$. Define an ultrametric d on X by

$$d(n, m) = \begin{cases} r(n) \vee r(m) & \text{if } n \neq m; \\ 0 & \text{if } n = m. \end{cases}$$

Since $d \in \mathrm{UM}(X)$, the condition (2) implies that $\psi \circ d \in \mathrm{UM}(X)$. Since the point ∞ is the unique accumulation point of X, and since X is compact, for every $\epsilon \in (0, \infty)$ the set $X \setminus U(\infty, \epsilon; \psi \circ d)$ is finite. Then, by $d(\infty, m) = r(m)$ for all $m \in \mathbb{N}$, we conclude that for all sufficiently large $n \in \mathbb{N}$ we have $\psi(r(n)) < \epsilon$. Since $\{r(n)\}_{n\in\mathbb{N}}$ is arbitrary, the function ψ is continuous at 0. □

Lemma 3.3.3. *Let S be a range set. Let (X, d) be an S-valued ultrametric space. Let $\epsilon \in S_+$. Then the function $e : X^2 \to [0, \infty)$ defined by $e = \min\{d, \epsilon\}$ belongs to $\mathrm{UM}(X, S)$.*

Proof. Applying Lemma 3.3.2 to the map $\psi : [0, \infty) \to [0, \infty)$ defined by $\psi(x) = \min\{x, \epsilon\}$, we obtain the lemma. □

3.3.2 Invariant metrics on modules

We here discuss ultra-norms on modules. Ultra-norms and invariant metrics are identical:

Lemma 3.3.4. *Let R be a commutative ring and let V be an R-module. If $\| * \|$ is an ultra-normed on V, then the metric d on V defined by $d(x, y) = \|x - y\|$ is an invariant ultrametric on V. Conversely, if d is an invariant ultrametric on V, then the function $\| * \| : V \to [0, \infty)$ defined by $\|x\| = d(x, 0)$ is an ultra-norm on V.*

Based on Lemma 3.3.4, in what follows, we will use a pair (V, d) of V and an invariant ultrametric d on V, rather than a pair $(V, \| * \|)$ of V and an ultra-norm $\| * \|$ on V.

By the definition of an ultra-norm, we obtain:

Lemma 3.3.5. *Let R be a commutative ring, and let (V, d) be an ultra-normed R-module. Then the addition $+ : V \times V \to V; (x, y) \mapsto x + y$ and the inversion $m : V \to V$ defined as $m(x) = -x$ are continuous maps with respect to the topology induced from d.*

The next lemma is utilized in the proof of Theorem 1.2.12.

Lemma 3.3.6. *Let R be a commutative ring, and let (V, d) be an ultra-normed R-module. If for every non-zero $r \in R$ and for every $v \in V$ we have $d(r \cdot v, 0) = d(v, 0)$, then there exists an ultra-normed R-module (W, D) which contains V as an R-submodule such that $d = D|_{V^2}$, and the metric space (W, D) is complete, and V is a dense subset of (W, D).*

Proof. Let (W, D) be the completion of (V, d). We introduce an R-module structure into the completion (W, D) of (V, d). For all $x, y \in W$, take sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in V such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then we define an addition on W by

$$x + y = \lim_{n\to\infty} (x_n + y_n).$$

We now check the well-definedness of this addition. Let $\{z_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{w\in\mathbb{N}}$ be sequences such that $z_n \to x$ and $w_n \to y$ as $n \to \infty$. Then we have

$$\|(x_n + y_n) - (z_n + w_n)\| \leq \|x_n - z_n\| \vee \|y_n - w_n\|.$$

where $\| * \|$ is an ultra-norm induced from d. Since $\|x_n - z_n\| \vee \|y_n - w_n\| \to 0$ as $n \to \infty$, we see that the addition is well-defined.

For every $r \in R$, we define a scalar multiplication on W by

$$r \cdot x = \lim_{n\to\infty} r \cdot x_n.$$

Take a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \to x$ as $n \to \infty$. Then

$$d(r \cdot x_n, r \cdot y_n) = d(r \cdot (x_n - y_n), 0) = d(x_n - y_n, 0),$$

and hence $d(r \cdot x_n, r \cdot y_n) \to 0$ as $n \to \infty$. Thus this scalar multiplication is well-defined. By these definitions, (W, D) becomes an ultra-normed R-module which contains V as an R-submodule. This finishes the proof. □

3.3.3 Basic properties of S-valued ultrametric spaces

The next lemma states that every triangle in an ultrametric space is isosceles, and the side-length of the legs of the isosceles triangle is equal to or greater than the side-length of the base.

Lemma 3.3.7. *Let X be a set, and let $w : X^2 \to \mathbb{R}$ be a symmetric map. Then w satisfies the strong triangle inequality if and only if for all points $x, y, z \in X$ the inequality $w(x, z) < w(y, z)$ implies the equality $w(y, z) = w(x, y)$.*

Proof. We first assume that w satisfies the strong triangle inequality. If $w(x, z) < w(y, z)$, then the inequality $w(y, z) \le w(y, x) \vee w(x, z)$ implies $w(y, z) \le w(x, y)$. By using $w(x, y) \le w(x, z) \vee w(z, y)$ again, we have $w(x, z) = w(x, y)$.

We next assume that w satisfies the condition stated in the lemma. Suppose that there exist $x, y, z \in X$ such that $w(y, z) > w(y, x) \vee w(x, z)$. Then by the assumption, we have $w(y, z) = w(x, y)$. This contradicts $w(y, z) > w(y, x)$. □

The strong triangle inequality implies the following (see (12) in [21, Theorem 1.6]):

Proposition 3.3.8. *Let S be a range set, and let (X, d) be an S-valued ultrametric space. Then the completion of (X, d) is an S-valued ultrametric space.*

Remark 3.3.2. By Proposition 3.3.8, we see that for every separable ultrametric space (X, d), the set $\{ d(x, y) \mid x, y \in X \}$ of values of the metric d is countable. This phenomenon is a reason why we consider S-valued ultrametrics for a range set S.

We now prove that for every range set S with the countable coinitiality, ultrametrizability and S-valued ultrametrizability are equivalent to each other. Recall that a range set S has countable coinitiality if there exists a non-zero strictly decreasing sequence $\{r_i\}_{i\in\mathbb{N}}$ in S convergent to 0.

Lemma 3.3.9. *Let S be a range set with the countable coinitiality. Let $\{r(i)\}_{i\in\mathbb{N}}$ be a strictly decreasing sequence in S such that $\lim_{i\to\infty} r(i) = 0$. Put $T = \{0\} \cup \{ r(i) \mid i \in \mathbb{N} \}$. Then, for every topological space X, from $\mathrm{UM}(X, S) \neq \emptyset$ it follows that $\mathrm{UM}(X, T) \neq \emptyset$.*

Proof. Take $d \in \mathrm{UM}(X, S)$. Define a function $\psi : [0, \infty) \to [0, \infty)$ by

$$\psi(x) = \begin{cases} r(1) & \text{if } r(1) < x; \\ r(n) & \text{if } r(n+1) < x \le r(n); \\ 0 & \text{if } x = 0. \end{cases}$$

Thus, ψ is increasing, amenable and continuous at 0. Put $e = \psi \circ d$. Since $\psi([0, \infty)) = T$, by Lemma 3.3.2 we have $e \in \mathrm{UM}(X, T)$. □

Proposition 3.3.10. *Let S be a range set with the countable coinitiality, and let X be a topological space. Then X is ultrametrizable if and only if X is S-valued ultrametrizable.*

Proof. It suffices to show that if X is ultrametrizable, then we have $\mathrm{UM}(X, S) \neq \emptyset$. Let $\{r(i)\}_{i\in\mathbb{N}}$ be a strictly decreasing sequence in S such that $r(i) \to 0$ as $i \to \infty$. Put $T = \{0\} \cup \{ r(i) \mid i \in \mathbb{N} \}$, and put $A = \{0\} \cup \{ 2^{-i} \mid i \in \mathbb{N} \}$. Then there exists an increasing amenable function $\psi : [0, \infty) \to [0, \infty)$ which is continuous at 0 and satisfies $\psi(A) = T$. Since $\mathrm{UM}(X) \neq \emptyset$, Lemma 3.3.9 implies that $\mathrm{UM}(X, A) \neq \emptyset$. Thus, by Lemmas 3.3.2 and $\psi(A) = T$, we have $\mathrm{UM}(X, T) \neq \emptyset$. From $\mathrm{UM}(X, T) \subseteq \mathrm{UM}(X, S)$, the proposition follows. □

Remark 3.3.3. If S does not have countable coinitiality, Proposition 3.3.10 does not hold true. In this case, all S-valued ultrametrizable spaces are discrete (see Lemma 3.3.13).

We now clarify a relation between complete S-valued ultrametrizability for a range set S and complete metrizability. The proofs of the following lemma and proposition are adapted from [123, Theorem 24.12] or [105, Theorem 2.2.1] for the case of S-valued ultrametrics.

Lemma 3.3.11. *Let S be a range set with the countable coinitiality. Let X be a completely S-valued ultrametrizable, and let G be an open subset of X. Then G is completely S-valued ultrametrizable.*

Proof. Sine X is 0-dimensional, and since all open sets of metric spaces are F_σ in the whole space, there exists a sequence $\{O_n\}_{n\in\mathbb{N}}$ of clopen sets of X such that

(1) for each $n \in \mathbb{N}$, we have $O_n \subset O_{n+1}$;

(2) $G = \bigcup_{n\in\mathbb{N}} O_n$.

Take a sequence of $\{a_n\}_{n\in\mathbb{N}}$ in the field $\mathbb{Q}_2$ of all 2-adic numbers such that for each $m \in \mathbb{N}$ the sum $\sum_{i=m}^{\infty} a_i$ converges to a non-zero 2-adic number (for example, we can take $a_n = 2^n - 2^{n+1}$). Define a function $F : X \to \mathbb{Q}_2$ by $F(x) = \sum_{i=1}^{\infty} a_i \cdot \chi_{O_i}(x)$, where χ_{O_i} is the characteristic function of O_i. From the definition of $\{O_n\}_{n\in\mathbb{N}}$ and the assumption on $\{a_n\}_{n\in\mathbb{N}}$, it follows that the function F is continuous. By the assumption on $\{a_n\}_{n\in\mathbb{N}}$, for every $x \in G$, we have $F(x) \neq 0$ and $F|_{X\setminus G} = 0$. By using the assumption on X, take a complete S-valued ultrametric $d \in \mathrm{UM}(X, S)$. We denote by $v_2 : \mathbb{Q}_2 \to \mathbb{Z}\sqcup\{\infty\}$ the 2-adic valuation on $\mathbb{Q}_2$. Take a non-zero strictly decreasing sequence $\{r(i)\}_{i\in\mathbb{N}}$ in S convergent to 0 as $i \to \infty$. We put $r(\infty) = 0$. Then a metric $W : (\mathbb{Q}_2)^2 \to S$ defined by

$$W(x, y) = r(v_2(x - y) \vee 0)$$

belongs to $\mathrm{UM}(\mathbb{Q}_2, S)$, and it is complete. Define a metric D on G by

$$D(x, y) = W\left(\frac{1}{F(x)}, \frac{1}{F(y)}\right) \vee d(x, y).$$

Since the function $1/F$ is continuous on G, we have $D \in \mathrm{UM}(G, S)$. We next show that D is complete. Assume that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in (G, D). Then $\{1/F(x_n)\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ are Cauchy in $(\mathbb{Q}_2, W)$ and (X, d), respectively. Thus, there exist $A \in \mathbb{Q}_2$ and $B \in X$ such that $1/F(x_n) \to A$ in $(\mathbb{Q}_2, W)$, and $x_n \to B$ in (X, d) as $n \to \infty$. If $B \notin G$, then we have $F(x_n) \to 0$, and hence we also have $F(x_n)(1/F(x_n)) \to 0 \cdot B = 0$ as $n \to \infty$. This contradicts to $0 \neq 1$ in $\mathbb{Q}_2$. Thus $B \in G$, and hence the metric D is complete. Therefore we conclude that G is completely S-valued ultrametrizable. □

By using the amalgamation lemma (Lemma 3.2.10), we obtain another proof of Lemma 3.3.11. Remark that, in the following proof, the assumption of the countable coinitiality of S is not used.

Another proof of Lemma 3.3.11. Similarly to the previous proof (put $Q_n = O_n \setminus O_{n-1}$), we obtain a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of clopen subsets of X with

(1) if $n \neq m$, then $Q_n \cap Q_m = \emptyset$;

(2) $G = \bigcup_{n\in\mathbb{N}} Q_n$.

Note that by (2) the space G is homeomorphic to the direct sum space $\coprod_{i\in I} Q_i$. Let $d \in \mathrm{UM}(X, S)$ be a complete ultrametric on X. Then, for each $n \in \mathbb{N}$, since Q_n is closed, the restricted metric $d|_{Q_n^2}$ is complete. By Lemma 3.2.10, we obtain a complete S-valued ultrametric in $\mathrm{UM}(G, S)$. □

Proposition 3.3.12. *Let S be a range set with the countable coinitiality. A topological space X is completely S-valued ultrametrizable if and only if X is completely metrizable and S-valued ultrametrizable.*

Proof. It suffices to show that if X is completely metrizable and S-valued ultrametrizable, then X is completely S-valued ultrametrizable.

Take a non-zero strictly decreasing sequence $\{r(i)\}_{i\in\mathbb{N}}$ in S convergent to 0 as $i \to \infty$. We put $T = \{0\} \cup \{\, r(i) \mid i \in \mathbb{N}\,\}$. Then T is a range set with $T \subset S$. Note that T has countable coinitiality and it is a closed set of $[0, \infty)$. By Proposition 3.3.10, we can take an ultrametric $d \in \mathrm{UM}(X, T)$. Let (Y, D) be a completion of (X, d). From Proposition 3.3.8, it follows that the space (Y, D) is a T-valued ultrametric space. Since X is completely metrizable, X is G_δ in Y (see [32, Theorem 4.3.24], or [92, Theorem 12.3]). Thus there exists a sequence $\{G_n\}_{n\in\mathbb{N}}$ of open sets in Y such that $X = \bigcap_{n\in\mathbb{N}} G_n$. By Lemmas 3.3.3 and 3.3.11, we can take a sequence $\{e_n\}_{n\in\mathbb{N}}$ of complete T-valued ultrametrics such that $e_n \in \mathrm{UM}(G_n, T)$ and $e_n(x, y) \leq r(n)$ for all $x, y \in G_n$ and for all $n \in \mathbb{N}$. Define an S-valued ultrametric $h \in \mathrm{UM}(X, S)$ by $h(x, y) = \sup_{n\in\mathbb{N}} e_n(x, y)$. Then h is complete. Since T is a closed set of $[0, \infty)$, we have $h \in \mathrm{UM}(X, T)$. Since $\mathrm{UM}(X, T) \subset \mathrm{UM}(X, S)$, we obtain a complete S-valued ultrametric $h \in \mathrm{UM}(X, S)$. □

If a range set does not have countable coinitiality, we obtain:

Lemma 3.3.13. *Let S be a range set without the countable coinitiality. Then every S-valued ultrametric space (X, d) is a complete metric generating the discrete topology.*

Proof. Since S does not have countable coinitiality, there exists $r \in [0, \infty)$ such that $[0, r) \cap S = \{0\}$. Thus for every $x \in X$ we have $U(x, r) = \{x\}$. This implies the lemma. □

Lemma 3.3.13 and Proposition 3.3.12 yields the following (see [58, Proposition 2.17]):

Proposition 3.3.14 ([58])**.** *For a range set S, a topological space is completely S-valued ultrametrizable if and only if it is completely metrizable and S-valued ultrametrizable.*

Since every G_δ subset of every completely metrizable space is complete metrizable (see Theorem 2.2.7), Proposition 3.3.14 implies:

Corollary 3.3.15. *Let S be a range set. If X is a completely S-valued ultrametrizable space, then so is every G_δ subset of X.*

3.4 Function spaces

In this section, we investigate function spaces which contain spaces of metrics or ultrametrics.

We first discuss the basic properties on a specific ultrametric u_S on a range set S. By using these properties, we prove Lemma 3.4.7, stating that spaces of metrics and ultrametrics are Baire spaces.

We define an ultrametric u_S on a range set S by assigning $u_S(x, y)$ to the infimum of $\epsilon \in (0, \infty)$ such that $x \leq y \vee \epsilon$ and $y \leq x \vee \epsilon$. We denote by d_E the Euclidean metric on S defined by $d_E(x, y) = |x - y|$. By the definition of u_S, we obtain:

Lemma 3.4.1. *Let S be a range set. Then for all distinct $x, y \in S$ we have $u_S(x, y) = x \vee y$. Hence u_S is an S-valued ultrametric on S.*

By the definitions of d_E and u_S, we have:

Lemma 3.4.2. *Let S be a range set. Then for all $a, b \in S$ we have*

$$d_E(a, b) \leq u_S(a, b).$$

Moreover, the identity map $1_S : (S, u_S) \to (S, d_E)$ is continuous.

Lemma 3.4.3. *For every range set S, the space (S, u_S) is complete.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in (S, u_S). Assume that there exists $a \in S$ satisfying that the set $\{\, n \in \mathbb{N} \mid a_n = a \,\}$ is infinite. Since $\{a_n\}_{n\in\mathbb{N}}$ is Cauchy, it converges to the point a. Assume next that for every $a \in S$, the set $\{\, n \in \mathbb{N} \mid a_n = a \,\}$ is finite. For every $\epsilon \in (0, \infty)$, we can take $N \in \mathbb{N}$ such that for all $n, m > N$, we have $u_S(a_n, a_m) \leq \epsilon$. By the assumption on finiteness, for every $n \in N$, there exists $m > N$ with $a_n \neq a_m$. Thus by Lemma 3.4.1 we have $a_n \leq \epsilon$. This implies that $a_n \to 0$ as $n \to \infty$. Therefore the space (S, u_S) is complete. □

Let S be a range set. Let H be a topological space, and let $C(H, S)$ be the set of all continuous functions from H into S, where S is equipped with the relative Euclidean topology. We define an ultrametric $\mathcal{U}_H^S$ on $C(H, S)$ by $\mathcal{U}_H^S(f, g) = \min\{1, \sup_{x\in H} u_S(f(x), g(x))\}$. We also define a metric $\mathcal{E}_H$ on $C(T, [0, \infty))$ by $\mathcal{E}_H(f, g) = \min\{1, \sup_{x\in H} |f(x) - g(x)|\}$.

Remark 3.4.1. Let S be a range set. The spaces $\left(\mathrm{UM}(X, S), \mathcal{UD}_X^S\right)$ and $(\mathrm{M}(X), \mathcal{D}_X)$ can be considered as topological subspaces of the spaces $\left(C(X^2, S), \mathcal{U}_{X^2}^S\right)$ and $(C(X^2, [0, \infty)), \mathcal{E}_{X^2})$, respectively, by the definitions of these metrics. Namely,

(1) for every S-valued ultrametrizable space X, the set $\mathrm{UM}(X, S)$ is a subset of $C(X^2, S)$, and the metrics $\mathcal{U}_{X^2}^S$ and $\mathcal{UD}_X^S$ generate the same topology on $\mathrm{UM}(X, S)$.

(2) for every metrizable space X, we have $\mathrm{M}(X) \subset C(X^2, [0, \infty))$, and the metric $\mathcal{E}_{X^2}$ on $\mathrm{M}(X)$ generates the same topology as that induced from $\mathcal{D}_X$.

By the definitions of $\mathcal{E}_H$ and $\mathcal{U}_H^S$, and by Lemma 3.4.2, we have:

Lemma 3.4.4. *Let S be a range set. Let H be a topological space. Then for all maps $f, g \in C(H, S)$ we have*

$$\mathcal{E}_H(f, g) \leq \mathcal{U}_H^S(f, g).$$

Moreover, the inclusion map $(C(H, S), \mathcal{U}_H^S) \to (C(H, [0, \infty)), \mathcal{E}_H)$ is continuous.

Lemma 3.4.5. *For every topological space H, the space $(C(H, [0, \infty)), \mathcal{E}_H)$ is complete.*

Proof. Take a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ in $C(H, [0, \infty))$. For every $x \in H$, the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy in $[0, \infty)$. Thus, it has a limit, say $F(x)$. By the definition, for every $x \in H$ we have $\lim_{n\to\infty} \sup_{x\in H} |F(x) - f_n(x)| = 0$. We now prove that the function $F : H \to [0, \infty)$ is continuous. Fix $a \in H$. For every $\epsilon \in (0, \infty)$, take $N \in \mathbb{N}$ with $\sup_{x\in H} |F(x) - f_N(x)| < \epsilon$, and take a neighborhood V of a such that every point $p \in V$ satisfies the condition $|f_N(p) - f_N(a)| < \epsilon$. Then, for every $p \in V$ we have

$$\begin{aligned} |F(p) - F(a)| &\leq |F(p) - f_N(p)| + |f_N(p) - f_N(a)| + |f_N(a) - F(a)| \\ &\leq |f_N(p) - f_N(a)| + 2 \sup_{x\in H} |F(x) - f_N(x)| \leq 3\epsilon. \end{aligned}$$

Therefore $F : H \to [0, \infty)$ is continuous, and hence $F \in C(H, [0, \infty))$, which means that the metric space $(C(H, [0, \infty)), \mathcal{E}_H)$ is complete. □

Similarly to Lemma 3.4.3, Lemmas 3.4.4 and 3.4.5 lead to the following:

Lemma 3.4.6. *Let S be a range set. Let H be a topological space. Then the ultrametric space $(C(H, S), \mathcal{U}_H^S)$ is complete.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $C(H,S)$. Then for every $x \in H$, we find that $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy in (S, u_S). By Lemma 3.4.3, $\{f_n(x)\}_{n\in\mathbb{N}}$ have a limit, say $F(x)$. By Lemma 3.4.4, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is also Cauchy in $(C(H,[0,\infty)), \mathcal{E}_H)$, and it has a limit $G \in (C(H,[0,\infty)), \mathcal{E}_H)$. Note that G is continuous. Lemma 3.4.3 yields $F = G$. Therefore $\{f_n\}_{n\in\mathbb{N}}$ has a limit in $C(H,S)$. This finishes the proof. □

The following is a summarized statement of [59, Lemma 5.1] and [58, Lemma 7.6].

Lemma 3.4.7 ([59, 58])**.** *Let X be a second countable locally compact space. Then the following hold:*

(1) *the space* $\mathrm{M}(X)$ *is a Baire space;*

(2) *if S is a range set, then the space* $\mathrm{M}(X,S)$ *is a Baire space.*

Proof. We first prove the former part. Lemma 3.4.5 implies that $(C(X^2,[0,\infty)), \mathcal{E}_{X^2})$ is completely metrizable. Thus, by Lemma 2.2.8, in order to prove the lemma, it suffices to show that the space $\mathrm{M}(X)$ is a G_δ subset of the space $(C(X^2,[0,\infty)), \mathcal{E}_{X^2})$.

We denote by P the set of all $f \in C(X^2,[0,\infty))$ such that

(1) for every $x \in X$ we have $f(x) \geq 0$ and $f(x,x) = 0$;

(2) for all $x, y \in X$ we have $f(x,y) = f(y,x)$;

(3) for all $x, y, z \in X$ we have $f(x,y) \leq f(x,z) + f(z,y)$.

Namely, P is the set of all continuous pseudo-metrics on X. Note that P is a closed subset in the metric space $(C(X^2,[0,\infty)), \mathcal{E}_{X^2})$. Since all closed subsets of a metric space are G_δ in the whole space, the set P is G_δ in the metric space $(C(X^2,[0,\infty)), \mathcal{E}_{X^2})$.

Now we take a sequence $\{D_n\}_{n\in\mathbb{N}}$ of compact subsets of X^2 with $\bigcup_{n\in\mathbb{N}} D_n = X^2 \setminus \Delta_X$, where $\Delta_X = \{\, (x,x) \in X^2 \mid x \in X \,\}$, and take a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets of X satisfying that $K_n \subset \mathrm{INT}(K_{n+1})$ and $\bigcup_{n\in\mathbb{N}} K_n = X$, where INT means the interior. For every $n \in \mathbb{N}$, let L_n be the set of all $f \in C(X^2,[0,\infty))$ for which there exist $c \in (0,\infty)$ and $N \in \mathbb{N}$ such that for each $k > N$ we have

$$\inf_{x\in K_n} \inf_{y\in X\setminus K_k} f(x,y) > c.$$

For each $n \in \mathbb{N}$, let E_n be the set of all $f \in C(X^2,[0,\infty))$ such that for each $(x,y) \in D_n$ we have $0 < f(x,y)$. Note that each L_n and each E_n are open in $(C(X^2,[0,\infty)), \mathcal{E}_{X^2})$.

We now prove that

$$\mathrm{M}(X) = P \cap \left(\bigcap_{n\in\mathbb{N}} L_n\right) \cap \left(\bigcap_{n\in\mathbb{N}} E_n\right).$$

Take $d \in \mathrm{M}(X)$. Since d is a metric, we have $d \in P$ and $d \in \bigcap_{n\in\mathbb{N}} E_n$. By the inclusion $K_n \subset \mathrm{INT}(K_{n+1})$, and by $d \in \mathrm{M}(X)$, for every $N \in \mathbb{N}$ and for every $k > N$ we have

$$0 < d(K_N, X \setminus \mathrm{INT}(K_{N+1})) \leq d(K_N, X \setminus K_{N+1}) \leq d(K_N, X \setminus K_k).$$

Thus $d \in \bigcap_{n\in\mathbb{N}} L_n$, and hence we obtain

$$\mathrm{M}(X) \subset P \cap \left(\bigcap_{n\in\mathbb{N}} L_n\right) \cap \left(\bigcap_{n\in\mathbb{N}} E_n\right).$$

Next take $d \in P \cap \left(\bigcap_{n\in\mathbb{N}} L_n\right) \cap \left(\bigcap_{n\in\mathbb{N}} E_n\right)$. Since $d \in P \cap \left(\bigcap_{n\in\mathbb{N}} E_n\right)$, the function d is continuous on X^2 and it satisfies all the conditions of the definition of a metric on X. We show that for every metric $e \in \mathrm{M}(X)$, the metric d is topologically equivalent to e. Since d is continuous on X^2, the metric d generates a weaker topology than that of (X, e). Namely, if $x_n \to a$ in (X, e), then $x_n \to a$ in (X, d). Assume next that $x_n \to a$ in (X, d). Since $\{K_n\}_{n\in\mathbb{N}}$ is a covering of X, there exists $M \in \mathbb{N}$ such that $a \in K_M$. If there exist infinitely many i satisfying $(X \setminus K_i) \cap \{\, x_n \mid n \in \mathbb{N}\,\} \neq \emptyset$, then we have

$$\liminf_{i\to\infty} d(K_M, X \setminus K_i) \leq \lim_{j\to\infty} d(a, x_j) = 0.$$

This contradicts $d \in \bigcap_{n\in\mathbb{N}} L_n$. Hence there exists $m \in \mathbb{N}$ such that $\{x_n\}_{n\in\mathbb{N}}$ is contained in K_m. If there exists $r \in (0, \infty)$ such that infinitely many n satisfying $e(x_n, a) \geq r$, then by the compactness of the subset K_m, there exists a convergent subsequence $\{x_{\psi(n)}\}_{n\in\mathbb{N}}$ in (X, e) with $e(x_{\psi(n)}, a) \geq r$. Since d generates a weaker topology than that of (X, e), and since $x_n \to a$ in (X, d), the limit point of $\{x_{\psi(n)}\}_{n\in\mathbb{N}}$ in (X, e) coincides with the point a. This is a contradiction. Thus d generates the same topology as e, and hence we obtain

$$\mathrm{M}(X) \supset P \cap \left(\bigcap_{n\in\mathbb{N}} L_n\right) \cap \left(\bigcap_{n\in\mathbb{N}} E_n\right).$$

Thus $\mathrm{M}(X)$ is G_δ in $(C(X^2, [0, \infty)), \mathcal{E}_{X^2})$. This finishes the proof of the former part.

We next prove the latter part. By Lemma 3.4.6, the metric space $(C(X^2, S), \mathcal{U}^S_{X^2})$ is completely metrizable. Thus by Lemma 2.2.8, to prove the lemma, it suffices to show that $\mathrm{UM}(X, S)$ is G_δ in $(C(X^2, S), \mathcal{U}^S_{X^2})$. Let Q be the set of all $f \in C(X^2, [0, \infty))$ such that

(1) for every $x \in X$ we have $f(x) \geq 0$ and $f(x, x) = 0$;

(2) for all $x, y \in X$ we have $f(x, y) = f(y, x)$;

(3) for all $x, y, z \in X$ we have $f(x, y) \leq f(x, z) \vee f(z, y)$.

Namely, Q is the set of all continuous pseudo-ultrametrics on X. The set Q is a closed subset in the metric space $(C(X^2, [0, \infty)), \mathcal{E}_{X^2})$. Since all closed subsets of a metric space are G_δ in the whole space, the set Q is G_δ in the metric space $(C(X^2, [0, \infty)), \mathcal{E}_{X^2})$.

Similarly to the proof of the former part, we obtain

$$\mathrm{UM}(X) = Q \cap \left(\bigcap_{n\in\mathbb{N}} L_n\right) \cap \left(\bigcap_{n\in\mathbb{N}} E_n\right)$$

as subsets of $(C(X^2, [0, \infty)), \mathcal{E}_{X^2})$; namely, $\mathrm{UM}(X)$ is a G_δ subset of $(C(X^2, [0, \infty)), \mathcal{E}_{X^2})$. Since the inclusion map from $(C(X^2, S), \mathcal{U}^S_{X^2})$ into $(C(X^2, [0, \infty)), \mathcal{E}_{X^2})$ is continuous (Lemma 3.4.4), and since $\mathrm{UM}(X, S) = \mathrm{UM}(X) \cap C(X^2, S)$, we conclude that $\mathrm{UM}(X, S)$ is G_δ in the space $(C(X^2, S), \mathcal{U}^S_{X^2})$. This completes the proof of the lemma. □

Remark 3.4.2. Let X be a metrizable space. The metric $\mathcal{D}_X$ on $\mathrm{M}(X)$ is not complete. There exists a sequence of metrics convergent to the zero function on X^2. Similarly, the ultrametric $\mathcal{UD}^S_X$ on $\mathrm{UM}(X)$ is not complete.

Remark 3.4.3. The definition of ultrametrics u_S and $\mathcal{UD}^S_X$ are inspired by the notion of the difference-completeness (see [22, Definition 4.8]).

Chapter 4

Pseudo-cones and Assouad dimensions

In this chapter, we prove lower estimations of the Assouad dimensions (Theorems 1.2.1, 1.2.2 and 1.2.3), and prove the existence of metric spaces containing large classes of metric spaces as its pseudo-cones, tangent or asymptotic cones (Theorems 1.2.4, 1.2.5 and 1.2.6).

4.1 Lower estimations of the Assouad dimensions

In this section we prove Theorems 1.2.1, 1.2.2 and 1.2.3.

4.1.1 Basic properties of pseudo-cones

We here observe some properties of pseudo-cones of metric spaces.

By Proposition 2.4.1, we have:

Proposition 4.1.1. *Let (X, d) be a metric space. If $(A, a) \in \mathcal{PC}(X, d)$, then for every $k \in (0, \infty)$ the metric space (A, ka) is in $\mathcal{PC}(X, d)$.*

Applying Gromov's precompactness theorem (Theorem 2.4.6), we obtain:

Proposition 4.1.2. *Let (X, d) be a doubling metric space. Let $\{(A_i, d)\}_{i\in\mathbb{N}}$ be a sequence of compact sets of X. If $\{(A_i, u_i d)\}_{i\in\mathbb{N}}$ is uniformly bounded for a sequence $\{u_i\}_{i\in\mathbb{N}}$ in $(0, \infty)$, then there exists a convergent subsequence $\{(A_{\phi(i)}, u_{\phi(i)}d\}_{i\in\mathbb{N}})$ of $\{(A_i, d)\}_{i\in\mathbb{N}}$ in the sense of Gromov–Hausdorff.*

Let (X, d) be a proper metric space, and $p \in X$. A pointed metric space (Y, e, y) is said to be a *tangent* (resp. *asymptotic*) *cone of X at p* if there exist a sequence $\{p_i\}_{i\in\mathbb{N}}$ in X convergent to p as $i \to \infty$, and a sequence $\{r_i\}$ in $(0, \infty)$ convergent to 0 (resp. ∞) as $i \to \infty$ such that for every $R \in (0, \infty)$ we have $(B(p_i, R/r_i), r_i d, p_i) \to (B(y, R), e, y)$ as $i \to \infty$ in the pointed Gromov–Hausdorff topology (see Section 8.1 in [15]).

By the definitions of a tangent cone and an asymptotic cone, we obtain:

Proposition 4.1.3. *Let (X, d) be a proper metric space, and let (Y, e, y) be a tangent or asymptotic cone of X. Then for every $R \in (0, \infty)$ we have $(B(y, R), e) \in \mathcal{PC}(X, d)$.*

4.1.2 Theorems on lower estimations

We first prove Theorem 1.2.1.

Proof of Theorem 1.2.1. Let (X, d) be a metic space, and let $(P, e) \in \mathcal{PC}(X, d)$. We assume that the space (P, e) is approximated by $(\{(A_i, d)\}_{i\in\mathbb{N}}, \{u_i\}_{i\in\mathbb{N}})$. Suppose that $\dim_A(X, d) < \dim_A(P, e)$. Take $\beta \in \mathbf{B}(X, d)$ such that $\dim_A(X, d) < \beta < \dim_A(P, e)$. Since $\beta \in \mathbf{B}(X, d)$, there exists $M \in (0, \infty)$ such that every finite set T of X satisfies the inequality $\operatorname{card}(T) \leq M \cdot (\delta_d(T)/\alpha_d(T))^\beta$.

Put $C = 4^\beta(M+1)$. From $\beta < \dim_A(P, e)$, it follws that $\beta \notin \mathbf{B}(P, e)$. Thus there exists a finite set S of P with $\operatorname{card}(S) > C \cdot (\delta_e(S)/\alpha_e(S))^\beta$. Since $\mathcal{GH}((A_i, u_i d), (P, e)) \to 0$ as $i \to \infty$, we can take $N \in \mathbb{N}$ such that $\mathcal{GH}((A_N, u_N d), (P, e)) < \alpha_e(S)/20$. By Lemma 2.4.2, there exists an $(\alpha_e(S)/10)$-approximation (f, g) between $(A_N, u_N d)$ and (P, e). For each $x \in S$, take $t_x \in A_N$ such that $t_x \in B(g(x), \alpha_e(S)/10; A_N, u_N d)$. Note that if $x \neq y$, then $t_x \neq t_y$ by the definition. Put $T = \{ t_x \mid x \in S \}$. For all $x, y \in S$, we obtain

$$u_N d(t_x, t_y) \leq e(x, y) + 3\alpha_e(S)/10 \leq 2\delta_e(S),$$

and

$$u_N d(t_x, t_y) \geq e(x, y) - 3\alpha_e(S)/10 \geq 2^{-1}\alpha_e(S).$$

Thus, we have $\delta_d(T) \leq 2u_N^{-1}\delta_e(S)$ and $\alpha_d(T) \geq 2^{-1}u_N^{-1}\alpha_e(S)$, and hence

$$\begin{aligned}\operatorname{card}(T) = \operatorname{card}(S) &> C(\delta_e(S)/\alpha_e(S))^\beta \\ &= 4^{-\beta}C(2u_N^{-1}\delta_e(S)/2^{-1}u_N^{-1}\alpha_e(S))^\beta \geq 4^{-\beta}C(\delta_d(T)/\alpha_d(T))^\beta.\end{aligned}$$

On the other hand, we also have $\operatorname{card}(T) \leq M(\delta_d(T)/\alpha_d(T))^\beta$. These inequalities imply $4^{-\beta}C < M$. This is a contradiction. This finishes the proof of Theorem 1.2.1. □

Since every metric space (X, d) belongs to $\mathcal{PC}(X, d)$, by Theorem 1.2.1 we obtain:

Corollary 4.1.4. *Let (X, d) and (Y, e) be two metric spaces. If $\mathcal{GH}(((X, d), (Y, e)) = 0$, then $\dim_A(X, d) = \dim_A(Y, e)$.*

This corollary slightly generalizes the fact that the Assouad dimensions of a metric space and its completion are identical.

By a similar method to Theorem 1.2.1, we obtain the following. The definition of the lower Assouad dimension $\dim_{LA}$ can be seen in Section 2.3.

Theorem 4.1.5 ([60])**.** *Let (X, d) be a metric space. Then every $(P, e) \in \mathcal{PC}(X, d)$ satisfies*

$$\dim_{LA}(X, d) \leq \dim_{LA}(P, e).$$

Next we prove Theorem 1.2.2.

Proof of Theorem 1.2.2. Let (X, d) be a metric space. Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of subsets of X, and let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $(0, \infty)$. Let μ be a non-principal ultrafilter on $\mathbb{N}$. Take $a_i \in A_i$. Put $\mathrm{T} = \{(A_i, u_i d, a_i)\}$, and $P = \lim_\mu(A_i, u_i d, a_i)$ and $D = m_{\mu,\mathrm{T}}$.

Assume that $\dim_A(X, d) < \dim_A(P, D)$. Let $\beta \in \mathbf{B}(X, d)$, $M \in (0, \infty)$, $C = 4^\beta(M+1)$ and $S \subset P$ be the same objects as in the proof of Theorem 1.2.1. We may assume that $S = \{[x_{1,i}], [x_{1,i}] \dots, [x_{n,i}]\}$. Put $S_i = \{x_{1,i}, \dots, x_{n,i}\} \subset A_i$ for each $i \in \mathbb{N}$. By the definition of an ultralimit, for μ-almost all $i \in \mathbb{N}$, and for all $k, l \in \{1, \dots, n\}$ we have

$$|u_i d(x_{k,i}, x_{l,i}) - D([x_{k,i}], [x_{l,i}])| < \alpha_D(S)/2.$$

Then for such μ-almost all $i \in \mathbb{N}$ we also have $\delta_d(S_i) \leq 2u_i^{-1}\delta_D(S)$ and $\alpha_d(S_i) \geq 2^{-1}u_i^{-1}$. Since $\operatorname{card}(S_i) = \operatorname{card}(S)$, by a similar argument to the proof of Theorem 1.2.1, we obtain $4^{-\beta}C < M$. This is a contradiction. This finishes the proof of Theorem 1.2.2. □

By a similar method to the proof of Theorem 1.2.2, we obtain:

Theorem 4.1.6 ([60])**.** *Let X be a metric space. Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of subsets of X, and let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $(0,\infty)$. Take $a_i \in A_i$ for each $i \in \mathbb{N}$. Put $\mathrm{S} = \{(A_i, u_i d, a_i)\}$. Then for every non-principal ultrafilter μ on $\mathbb{N}$ we have*

$$\dim_{LA}(X,d) \leq \dim_{LA}\left(\lim_{\mu}(A_i, u_i d, a_i), m_{\mu,\mathrm{S}}\right).$$

4.1.3 Conformal Assouad dimension

Let $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism. Let (X,d) and (Y,e) be two metric spaces. A homeomorphism $f : X \to Y$ is said to be *η-quasi-symmetric* if the following holds:

(QS) for all $x,y,z \in X$ and for all $t \in [0,\infty)$, the inequality $d_X(x,y) \leq t d_X(x,z)$ implies that $d_Y(f(x,f(y))) \leq \eta(t) d_Y(f(x),f(z))$.

A homeomorphism $f : X \to Y$ is *quasi-symmetric* if it is η-quasi-symmetric for some homeomorphism $\eta : [0,\infty) \to [0,\infty)$. Note that the inverse of a quasi-symmetric map is also quasi-symmetric.

For a metric space X, the *conformal Assouad dimension* $\mathrm{Cdim}_A X$ of X is defined as the infimum of all the Assouad dimensions of all quasi-symmetric images of X.

In the proof of Theorem 1.2.3, we use the following theorem due to Tukia and Väisälä (see [114, Theorem 2.21]).

Theorem 4.1.7. *If a map $f : (X,d) \to (Y,e)$ between metric spaces satisfies the condition* (QS)*, then f is either a constant map or a quasi-symmetric embedding.*

We now prove Theorem 1.2.3.

Proof of Theorem 1.2.3. Let (X,d) be a metric space, and $(P,h) \in \mathcal{KPC}(X)$. Since the doubling property is invariant under quasi-symmetric maps, every non-doubling space has infinite conformal Assouad dimension. Thus we may assume that X is doubling. Take a metric space (Y,e) and an η-quasi-symmetric map $f : X \to Y$. We may assume that P is compact and P has at least two elements. We assume that P is approximated by $(\{A_i\}_{i\in\mathbb{N}}, \{u_i\}_{i\in\mathbb{N}})$, where $\{A_i\}_{i\in\mathbb{N}}$ is a sequence of compact sets in X. By Proposition 2.4.3, we have $\sup_i \delta_{u_i d}(A_i) < \infty$. For each $i \in \mathbb{N}$, put $B_i = f(A_i)$ and $v_i = (\delta_e(B_i))^{-1}$. By Proposition 4.1.2, by choosing a suitable subsequence if necessary, we find a limit compact metric space $(Q,l) \in \mathcal{KPC}(Y,e)$ of the sequence $\{(B_i, v_i e)\}_{i\in\mathbb{N}}$.

Let μ be a non-principal ultrafilter on $\mathbb{N}$. For each $i \in \mathbb{N}$ take $a_i \in A_i$. Put $b_i = f(a_i)$, and put $\mathrm{S} = \{(A_i, u_i d, a_i)\}$ and $\mathrm{T} = \{(B_i, v_i e, b_i)\}_{i\in\mathbb{N}}$.

By applying Lemma 2.6.3 to the sequences $\{(A_i, u_i d, a_i)\}_{i\in\mathbb{N}}$ and $\{(B_i, v_i e, b_i)\}_{i\in\mathbb{N}}$, we see that $(P,h) = (\lim_\mu(A_i, u_i d, a_i), m_{\mu,\mathrm{S}})$ and $(Q,l) = (\lim_\mu(B_i, v_i e, b_i), m_{\mu,\mathrm{T}})$. Since f is continuous and $\delta_{v_i e}(B_i) = 1$ for all $i \in \mathbb{N}$, the map $f : X \to Y$ induces a map $F : P \to Q$ defined by $F([\{x_i\}_{i\in\mathbb{N}}]) = [\{f(x_i)\}_{i\in\mathbb{N}}]$. Replacing the role of f with that of f^{-1}, we obtain the inverse of F. Thus F is a bijection.

To prove that F satisfies the condition (QS), we assume that $h(x,y) \leq t h(x,z)$, where $x = [\{x_i\}_{i\in\mathbb{N}}]$, $y = [\{y_i\}_{i\in\mathbb{N}}]$, $z = [\{z_i\}_{i\in\mathbb{N}}]$. By the definition of an ultralimit, for every $\epsilon \in (0,\infty)$, and for μ-almost all $i \in \mathbb{N}$, we have

$$u_i d(x_i, y_i) < (t+\epsilon) u_i d(x_i, z_i).$$

Thus, since f is η-quasi-symmetric, for μ-almost all $i \in \mathbb{N}$ we obtain

$$v_i e(f(x_i), f(y_i)) < \eta(t+\epsilon) v_i e(f(x_i), f(z_i)).$$

Then we also obtain

$$l(F(x), F(z)) < \eta(t+\epsilon) l(F(x), F(z)).$$

By letting $\epsilon \to 0$, we obtain $l(F(x), F(z)) \leq \eta(t) l(F(x), F(z))$. Since F is bijective and non-constant, by Theorem 4.1.7 we see that F is an η-quasi-symmetric map. Thus $\mathrm{Cdim}_A(P, h) \leq \dim_A(Q, l)$. Theorem 1.2.1 implies $\dim_A(Q, l) \leq \dim_A(Y, e)$, and hence $\mathrm{Cdim}_A(P, h) \leq \dim_A(Y, e)$. Since the metric space (Y, e) is arbitrary, we conclude that $\mathrm{Cdim}_A(P, h) \leq \mathrm{Cdim}_A(X, d)$. This leads to Theorem 1.2.3. □

As a corollary of Theorem 1.2.3, we obtain:

Corollary 4.1.8 ([60])**.** *Let (X, d) be a metric space. Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of subsets of X, and let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $(0, \infty)$. Let μ be a non-principal ultrafilter on $\mathbb{N}$. Take $p_i \in A_i$ for each $i \in \mathbb{N}$. Put $\mathrm{U} = \{(A_i, u_i d, p_i)\}_{i\in\mathbb{N}}$, and put*

$$(Y, e, q) = (\lim_{\mu}(A_i, u_i d, p_i), m_{\mu,\mathrm{U}}, p_{\mu,\mathrm{U}}).$$

Then for every $R \in (0, \infty)$ we have

$$\mathrm{Cdim}_A(B(q, R; Y, e), e) \leq \mathrm{Cdim}_A(X, d).$$

Proof. Since the Assouad dimension of the completion of X coincides with that of the original space, we may assume that X is doubling and complete. Note that $\lim_\mu(A_i, u_i d, p_i)$ is isometric to $\lim_\mu(\mathrm{CL}(A_i), u_i d, p_i)$ with respect to the canonical metrics on the ultralimits, where CL stands for the closure operator of X. Thus, we may assume that each A_i is closed. We also note that each A_i is doubling and complete, and hence it is proper.

Put

$$S = \left\{ \{x_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} A_i \mid u_i d(p_i, x_i) < 2R \right\} / R_\mu,$$

and

$$T = \left\{ \{x_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} A_i \mid u_i d(p_i, x_i) \leq 2R \right\} / R_\mu.$$

By the definition of an ultralimit, we have $B(p, R; m_{\mu,\mathrm{U}}) \subset S$ and $S \subset T$ and

$$T = \lim_{\mu}(B(p_i, 2R; A_i, u_i d), u_i d, p_i).$$

Since (A_i, d) is proper, the ball $B(p_i, R; A_i, u_i d)$ is compact. Thus Theorem 1.2.3 implies

$$\mathrm{Cdim}_A(T, d_T) \leq \mathrm{Cdim}_A(X, d),$$

where d_T is the canonical metric on T as an ultralimit. By the monotonicity of the conformal Assouad dimension, we have $\mathrm{Cdim}_A(B(p_i, R; A_i, u_i d), d_T) \leq \mathrm{Cdim}_A(T, d_T)$, and hence

$$\mathrm{Cdim}_A(B(p_i, R; A_i, u_i d), d_T) \leq \mathrm{Cdim}_A(X, d).$$

Since d_T and e coincide on $B(p_i, R; A_i, u_i d)$, we obtain the corollary. □

4.2 Examples of metric spaces with large cones

In this section, we prove Theorems 1.2.4, 1.2.5 and 1.2.6. We construct examples containing large classes of metric space as their pseudo-cones, tangent or asymptotic cones.

4.2.1 Proof of Theorem 1.2.4

We construct a metric $(\omega_0 + 1)$-space containing all compact metric spaces as its pseudo-cones. Recall that the symbol $\mathcal{S}$ stands for the class of all separable metric spaces. By the virtue of the telescope construction (see Section 3.1), and by the existence of separable $\mathcal{S}$-universal metric space (see Section 2.5), we can prove Theorem 1.2.4.

Proof of Theorem 1.2.4. In order to construct a desired space, we use the telescope construction (see Section 3.1). Let (U, u) be a separable $\mathcal{S}$-universal metric space. Let Q be a countable dense set of U, and let $\{K_i\}_{i\in\mathbb{N}}$ be the set of all finite subsets of Q. Put $\mathbf{J} = \{(K_i, (2^i\delta_u(K_i))^{-1}u)\}$. Then $\mathcal{J} = (\mathbf{J}, \mathbf{A})$ is a compatible pair, where $\mathbf{A}$ is a telescope base defined in Definition 3.1.2. Put $(X, d) = \big(T(\mathcal{J}), d_{T(\mathcal{J})}\big)$. Then, Lemma 3.1.3 implies that the metric space (X, d) is an $(\omega_0 + 1)$-metric space, and the point ∞ is its unique accumulation point. Let (K, k) be any compact metric space. Since (K, k) can be isometrically embedded into (U, u), there exists a monotone injective map $\phi : \mathbb{N} \to \mathbb{N}$ such that $\mathcal{H}(K_{\phi(i)}, K; U, u) \to 0$ as $i \to 0$. By the definition of a telescope space, for each $i \in \mathbb{N}$ we have $K_{\phi(i)} \subset X$. Thus (K, k) is a pseudo-cone of (X, d) approximated by $(\{(K_{\phi(i)})\}_{i\in\mathbb{N}}, \{2^{\phi(i)}\delta_u(K_{\phi(i)})\}_{i\in\mathbb{N}})$. This finishes the proof of Theorem 1.2.4. □

By a similar argument, we also obtain:

Proposition 4.2.1. *Let (U, u) be a separable $\mathcal{S}$-universal metric space. If Q is a countable dense set of U, then we have $\mathcal{PC}(Q, u) = \mathcal{S}$.*

4.2.2 Proofs of Theorems 1.2.5 and 1.2.6

By an argument using arcs in a length space, we obtain the following estimation of the Hausdorff distance between concentric balls in a length space.

Proposition 4.2.2. *Let (X, d) be a length space. Then all $p \in X$ and $r, R \in (0, \infty)$ satisfy*

$$\mathcal{H}(B(p, r), B(p, R); X, d) \leq |r - R|.$$

The following proposition plays a key role in the proof of Theorems 1.2.5 and 1.2.6.

Proposition 4.2.3. *Let (U, u) be a separable metric space, and let Q be a countable dense subset of U. Let (K, e) be a length metric subspace of (U, u), and let $p \in K$. For all $i, k \in \mathbb{N}$, put $l_{k,i} = k \cdot 2^{-i}$. Assume that a sequence $\{A_i\}_{i\in\mathbb{N}}$ of subsets of Q satisfies the following for all $i \in \mathbb{N}$:*

(A1) *$p \in A_i$;*

(A2) *for each $k \in \{0, \dots, 2^{2i}\}$ we have $\mathcal{H}(B(p, l_{k,i}; K, e), B(p, l_{k,i}; A_i, u); U, u) \leq 2^{-i}$.*

Then for every $R \in (0, \infty)$, the sequence $\{(B(p, R; A_i, u), u, p)\}_{i\in\mathbb{N}}$ converges to the metric space $(B(p, R; K), p)$ in the pointed Gromov–Hausdorff topology.

Proof. Take $N \in \mathbb{N}$ with $R < 2^{2N}$. Then for each $i \geq N$, we can take $k \in \{0, \dots, 2^{2i}\}$ with

$$l_{k,i} \leq R < l_{k+1,i}.$$

By the condition (A2), for $m \in \{k, k+1\}$, we see that

$$\mathcal{H}(B(p, l_{m,i}; A_i.u), B(p, l_{m,i}; K, e); U, u) \leq 2^{-i}.$$

Thus, we have

$$B(p, l_{k,i}; K, e) \subset B(B(p, l_{k,i}; A_i, u), 2^{-i}; U, u), \tag{4.2.1}$$

and

$$B(p, l_{k+1,i}; A_i, u) \subset B(B(p, l_{k+1,i}; K, e), 2^{-i}; U, u). \tag{4.2.2}$$

Since for $m \in \{k, k+1\}$ we have $|R - l_{m,i}| \leq 2^{-i}$, by Proposition 4.2.2, for $m \in \{k, k+1\}$ we have

$$\mathcal{H}(B(p, R; K, e), B(p, l_{m,i}; K, e); U, u) \leq 2^{-i}.$$

Thus we obtain

$$B(p, R; K, e) \subset B(B(p, l_{k,i}; K, e), 2^{-i}; U, u), \tag{4.2.3}$$

and

$$B(p, l_{k+1,i}; K, e) \subset B(B(p, R; K, e), 2^{-i}; U, u). \tag{4.2.4}$$

Since $B(p, l_{k,i}; A_i, u) \subset B(p, R; A_i, u)$, by (4.2.1) and (4.2.3), we obtain

$$B(p, R; K, e) \subset B(B(p, R; A_i, u), 2^{-i+1}; U, u). \tag{4.2.5}$$

Since $B(p, R; A_i, u) \subset B(p, l_{k+1,i}; A_i, u)$, by (4.2.2) and (4.2.4), we obtain

$$B(p, R; A_i, u) \subset B(B(p, R; K, e), 2^{-i+1}; U, u). \tag{4.2.6}$$

Then, the inclusions (4.2.5) and (4.2.6) imply that

$$\mathcal{H}(B(p, R; K, e), B(p, R; A_i, u); U, u) \leq 2^{-i+1}.$$

Hence we conclude that the sequence $\{(B(p, R; A_i), u, p)\}_{i \in \mathbb{N}}$ converges to $(B(p, R; K), e, p)$ in the pointed Gromov–Hausdorff topology. □

We now prove Theorem 1.2.5.

Proof of Theorem 1.2.5. Let (K, k, p) be a pointed proper length space. We may assume that (K, k) has at least two elements. Let (U, u) be a separable homogeneous $\mathcal{S}$-universal metric space. For instance, the space $(C([0, 1]), \| * \|_\infty)$ or $(\mathbb{U}, d_{\mathbb{U}})$ can be chosen as (U, u) (see Corollary 2.5.3). Let Q be a countable dense subset of U. Let $I = \{F_i\}_{i \in \mathbb{N}}$ be a sequence consisting of all finite subsets of Q, and we may assume that for every finite subset A of Q there exists infinitely many $n \in \mathbb{N}$ such that $F_n = A$. By Proposition 2.5.4, we may assume that $K \subset U$, $u|_{K^2} = k$, and $p \in Q$.

For each $i \in \mathbb{N}$, set $r_i = (i+1)! \cdot \delta(F_i)$. Put $\mathbf{J} = \{(F_i, (r_i)^{-1} u)\}_{i \in \mathbb{N}}$. Then $\mathfrak{J} = (\mathbf{J}, \mathbf{A})$ is a compatible pair, where $\mathbf{A}$ is a telescope base in Definition 3.1.2. Put $(X, d) = (T(\mathfrak{J}), d_{T(\mathfrak{J})})$. The space X is a metric $(\omega_0 + 1)$-space, and ∞ is its unique accumulation point.

Since (K, k) is proper, we can take a sequence $\{A_i\}_{i \in \mathbb{N}}$ of finite subsets of Q satisfying the conditions (A1) and (A2) in Proposition 4.2.3. By the definitions of $I = \{F_i\}_{i \in \mathbb{N}}$ and the space $(T(\mathfrak{J}), d_{T(\mathfrak{J})})$, there exists a strictly increasing map $\phi : \mathbb{N} \to \mathbb{N}$ such that the metric subspace $(F_{\phi(i)}, r_{\phi(i)}d)$ of $(X, r_{\phi(i)}d)$ is isometric to (A_i, u) for each $i \in \mathbb{N}$. Let $q_i \in F_{\phi(i)}$ be a point corresponding to $p \in A_i$. Note that $r_{\phi(i)} \cdot 2^{-\phi(i)+1} \to \infty$ as $i \to \infty$.

To prove that (K, k, p) is a tangent cone of (X, d), we show that for each $R \in (0, \infty)$, the sequence $\{(B(q_i, R/r_i; X, d), r_i d, q_i)\}_{i\in\mathbb{N}}$ converges to the space $(B(p, R; K, k), k, p)$ in the pointed Gromov–Hausdorff topology. By the definition of $\{r_i\}_{i\in\mathbb{N}}$, we can take $N \in \mathbb{N}$ such that if $i > N$, then we have $R < r_{\phi(i)} \cdot 2^{-\phi(i)+1}$. By the definition of (X, d), and by $R < r_{\phi(i)} \cdot 2^{-\phi(i)+1}$ for each $i > N$, the pointed metric space $(B(q_i, R/r_{\phi(i)}; X, d), r_{\phi(i)}d, q_i)$ is isometric to $(B(p, R; F_{\phi(i)}, u), u, p)$. From Proposition 4.2.3, it follows that the sequence $\{(B(q_i, R/r_{\phi(i)}; X), r_{\phi(i)}d, q_i)\}_{i\in\mathbb{N}}$ converges to $(B(p, R; K), k, p)$ in the pointed Gromov–Hausdorff topology. Since $q_i \to \infty$ in X as $i \to \infty$, we conclude that (K, k, p) is a tangent cone of X at ∞. This completes the proof of Theorem 1.2.5. □

We next prove Theorem 1.2.6. As a core part to construct a metric space mentioned in Theorem 1.2.6, we begin with the following elementary lemma on a surjective map between countable sets, which states the existence of a polite indexing of a countable set.

Lemma 4.2.4. *There exists a surjective map $C : \mathbb{N} \to \mathbb{N}^2 \times \mathbb{Z}$ satisfying the following:*

(B1) $C(0) = (0, 0, 0)$*;*

(B2) *for every $n \in \mathbb{N}$ and for every $i \in \{1, 2, 3\}$, we have $|\pi_i(C(n)) - \pi_i(C(n + 1))| \le 1$, where π_i is the i-th projection;*

(B3) *for every $(x, y, z) \in \mathbb{N}^2 \times \mathbb{Z}$, the set $C^{-1}(x, y, z)$ is infinite.*

Proof. Take a surjective map $A : \mathbb{N} \to \mathbb{N}^2 \times \mathbb{Z}$ satisfying the conditions (B1) and (B2). Define a map $H : \mathbb{N} \to \mathbb{N}$ by $H(n) = \min_{k\in\mathbb{N}} |n - k^2|$. Then H satisfies the following:

(1) $H(0) = 0$;

(2) for every $n \in \mathbb{N}$, the set $H^{-1}(n)$ is infinite;

(3) for every $n \in \mathbb{N}$, we have $|H(n) - H(n + 1)| \le 1$.

Then the map $A \circ H : \mathbb{N} \to \mathbb{N}^2 \times \mathbb{Z}$ satisfies the conditions (B1), (B2) and (B3). □

By the conditions (B1) and (B2), we inductively obtain:

Lemma 4.2.5. *If a map $C : \mathbb{N} \to \mathbb{N}^2 \times \mathbb{Z}$ satisfies the conditions* (B1) *and* (B2)*, then we have $|\pi_i(C(n))| \le n$ for all $n \in \mathbb{N}$ and for all $i \in \{1, 2, 3\}$.*

We now show the existence of a metric space containing all proper length space as its asymptotic cones. Such a space is constructed as follows: Let (U, u) be a separable homogeneous $\mathcal{S}$-universal metric space (see Corollary 2.5.3), and let Q be a countable dense subset of U. For each $(j, k) \in \mathbb{N} \times \mathbb{Z}$, let $I_{(j,k)} = \{F_{(i,j,k)}\}_{i\in\mathbb{N}}$ be a sequence consisting of all finite subsets of Q satisfying the following three conditions for all $(i, j, k) \in \mathbb{N}^2 \times \mathbb{Z}$:

(C1) $q \in F_{(i,j,k)}$;

(C2) $2^{-k} \le \delta_u(F_{(i,j,k)}) < 2^{-k+1}$;

(C3) $2^{-j} \le \alpha_u(F_{(i,j,k)})/\delta_u(F_{(i,j,k)}) < 2^{-j+1}$.

Take a surjective map $C : \mathbb{N} \to \mathbb{N}^2 \times \mathbb{Z}$ stated in Lemma 4.2.4. For each $i \in \mathbb{N}$, define $G_i = F_{C(i)}$. Put $J = \{G_i\}_{i\in\mathbb{N}}$. Then J consists of all finite subsets of Q containing q.

For each $i \in \mathbb{N}$, let $a_i = (\alpha(G_i))^{-1} \cdot 2^{i^2}$. Put

$$X = \{q\} \sqcup \coprod_{i\in\mathbb{N}} (G_i \setminus \{q\}),$$

and define a metric $d_X : X \times X \to [0, \infty)$ on X by

$$d_X(x, y) = \begin{cases} a_i d_{G_i}(x, y) & \text{if } x, y \in G_i \text{ for some } i \in \mathbb{N}; \\ a_i d_{G_i}(x, q) + a_j d_{G_j}(q, y) & \text{if } x \in G_i \text{ and } y \in G_j \text{ for some } i \neq j. \end{cases}$$

This construction can be considered as a specialized version of amalgamation methods (see Section 3.2). Note that the metric space (X, d_X) is countable, proper, and discrete.

We are going to prove that every pointed proper length space is an asymptotic cone of X. To simplify our notation, for $R \in (0, \infty)$, and for $i \in \mathbb{N}$, put $B_i(R) = B(q, a_iR; X, d_X)$. By the definition of d_X, the space $(B_i(R), a_i^{-1}d_X)$ contains an isometric copy of the metric space $(B(q, R; G_i, u), u)$ containing p, say $S_i(R)$. We also put $T_i(R) = B_i(R) \setminus S_i(R)$. Note that $S_i(R) \subset G_i$ and $p \in S_i(R)$. We next prove some properties of (X, d_X).

Lemma 4.2.6. *Let $R \in (0, \infty)$. If $i \in \mathbb{N}$ satisfies $2^{i+1}\delta(G_i) > R$, then for every $k > i$ we have $B_i(R) \cap G_k = \emptyset$.*

Proof. For every $x \in G_k$, we see that $d_X(q, x) \geq 2^{k^2} \geq 2^{(i+1)^2}$. Lemma 4.2.5 implies

$$2^{(i+1)^2}/a_i = 2^{(i+1)^2-i^2}\alpha(G_i) \geq 2^{2i+1}2^{-\pi_2(C(i))}\delta(G_i) \geq 2^{i+1}\delta(G_i) > R.$$

Hence $a_iR < 2^{(i+1)^2}$. This leads to the conclusion. □

By Lemma 4.2.6 and by the definition of $T_i(R)$, we obtain:

Corollary 4.2.7. *For every $R \in (0, \infty)$, if $i \in \mathbb{N}$ satisfies $2^{i+1}\delta(G_i) > R$, we have*

$$T_i(R) \subset \bigcup_{j=0}^{i-1} G_j.$$

Lemma 4.2.8. *For every $i \geq 1$, we have $\alpha(G_i)/\alpha(G_{i-1}) < 16$.*

Proof. By the conditions (B2), (C2) and (C3), we obtain

$$\begin{aligned} \alpha(G_i)/\alpha(G_{i-1}) &< 2^{-\pi_2(C(i))+1+\pi_2(C(i-1))}\delta(G_i)/\delta(G_{i-1}) \leq 4 \cdot \delta(G_i)/\delta(G_{i-1}) \\ &< 4 \cdot 2^{-\pi_3(C(i))+1+\pi_3(C(i-1))} \leq 16. \end{aligned}$$

This proves the lemma. □

Lemma 4.2.9. *Assume that $i \in \mathbb{N}$ and $R \in (0, \infty)$ satisfy $2^{i+1}\delta(G_i) > R$. Then for all $x \in T_i(R)$, we have $(a_i)^{-1}d_X(q, x) < 32 \cdot 2^{-i}$. In particular, we have*

$$\mathcal{H}(B_i(R), S_i(R); B_i(R), (a_i)^{-1}d_X) \leq 32 \cdot 2^{-i}.$$

Proof. By Corollary 4.2.7, we have $(a_i)^{-1}d_X(p, x) \leq a_{i-1}\delta(G_{i-1})/a_i$. Thus, Lemmas 4.2.8 and 4.2.5 imply

$$a_{i-1}\delta(G_{i-1})/a_i = 2^{(i-1)^2-i^2}\delta(G_{i-1})(\alpha(G_i)/\alpha(G_{i-1})) < 16 \cdot 2^{-\pi_3(C(i-1))+1-2i+1} \leq 32 \cdot 2^{-i}.$$

This shows the former part of the lemma. From the former part and the facts that $q \in S_i(R)$ and $S_i(R) \subset B_i(R)$, the latter part follows. □

We now prove Theorem 1.2.6.

Proof of Theorem 1.2.6. We first show that the metric space (X, d_X) constructed above is a desired space. Let (K, k, p) be a pointed proper length space. By Proposition 2.5.4, we may assume that $K \subset U$, $u|_{K^2} = k$, and $p = q$. Since (K, k) is proper, we can take a sequence $\{A_i\}_{i\in\mathbb{N}}$ of finite subsets of Q satisfying the conditions (A1) and (A2) in Proposition 4.2.3. By the definition of $J = \{G_i\}_{i\in\mathbb{N}}$, and by the condition (B3), there exists a strictly increasing map $\phi : \mathbb{N} \to \mathbb{N}$ such that $G_{\phi(i)} = A_i$ for every $i \in \mathbb{N}$.

We next show that for each $R \in (0, \infty)$, the sequence $\{(B_{\phi(i)}(R), (a_{\phi(i)})^{-1}d_X, q)\}_{i\in\mathbb{N}}$ converges to $(B(q, R; K, k), k, q)$ in the pointed Gromov–Hausdorff topology. Note that, by the conditions (C2) and (C3) and Lemma 4.2.5, we have $a_i \to \infty$ as $i \to \infty$.

By the definition of d_X, we have $\delta(G_{\phi(i)}) \cdot 2^{\phi(i)+1} \to \infty$ as $i \to \infty$. Hence we can take $N \in \mathbb{N}$ such that for every $i \geq N$, we have $R < \delta(G_{\phi(i)}) \cdot 2^{\phi(i)+1}$. Lemma 4.2.9 yields

$$\mathcal{H}(B_{\phi(i)}(R), S_{\phi(i)}(R); B_{\phi(i)}(R), (a_\phi)^{-1}d_X) < 32 \cdot 2^{-\phi(i)} \leq 32 \cdot 2^{-i}. \tag{4.2.7}$$

Since the metric space $(S_{\phi(i)}(R), d_X)$ is isometric to $(B(q, R; A_i; u), u)$, by Proposition 4.2.3, the sequence $\{(B_{\phi(i)}(R), (a_{\phi(i)})^{-1}d_X, q)\}_{i\in\mathbb{N}}$ converges to the pointed metric space $(B(q, R; K, k), k, q)$. By this convergence, and by the inequality (4.2.7), the sequence $\{(B_{\phi(i)}(R), (a_{\phi(i)})^{-1}d_X, q)\}_{i\in\mathbb{N}}$ converges to the pointed metric space $(B(q, R; K, k), k, q)$ in the pointed Gromov–Hausdorff topology. Therefore (K, k, p) is an asymptotic cone of X. This completes the proof of Theorem 1.2.6. □

Remark 4.2.1. Let (X, d) be a metric space mentioned in Theorems 1.2.4, 1.2.5, 1.2.6 or Proposition 4.2.1. By Theorem 1.2.1 and Proposition 4.1.3, we obtain $\dim_A(X, d) = \infty$.

All metric (ω_0+1)-spaces and all countable metic spaces have the topological dimension 0, and have the Hausdorff dimension 0. Thus, Theorems 1.2.4, 1.2.5, 1.2.6 or Proposition 4.2.1 tells us that analogues of Theorem 1.2.1 for the topological dimension, the Hausdorff dimension and the conformal Hausdorff dimension are false. More precisely, we have:

Proposition 4.2.10. *There exists a metric space X such that for some $(P, h) \in \mathcal{PC}(X, d)$ we have*

(1) $\dim_T(X, d) < \dim_T(P, h)$;

(2) $\dim_H(X, d) < \dim_H(P, h)$;

(3) $\mathrm{Cdim}_H(X, d) < \mathrm{Cdim}_H(P, h)$,

where $\dim_T$, $\dim_H$ and Cdim_H stand for the topological dimension, the Hausdorff dimension and the conformal Hausdorff dimension, respectively.

Remark 4.2.2. Chen–Rossi [18] studied a metric space containing a vast class of metric spaces as tangent cones of it. They constructed a compact subset X of $\mathbb{R}^N$ with $\dim_H(X, d_{\mathbb{R}^N}) = 0$ that contains all similarity classes of compact subsets of $[0, 1]^N$ as tangent cones at countable dense subset of X (see [18, Corollary 5.2]). Erdös–Kakutani [33] constructed a perfect subset of $\mathbb{R}$ of measure 0 containing some similarity copies of all finite subsets of $\mathbb{R}$. Holsztynski [52] constructed, for all $n \in \mathbb{N}$, a metric $d \in \mathrm{M}(\mathbb{R}^n)$ such that all finite metric spaces can be isometrically embedded into $(\mathbb{R}^n, d)$ (see [91] for a detailed construction). The metric spaces mentioned above are examples failing an analogy of Theorem 1.2.1 for the Hausdorff dimension and the conformal Hausdorff dimension.

Chapter 5

Tiling spaces

In this chapter, we investigate basic geometric properties of spaces with tiling structure, and we prove Theorem 1.2.7. We prove Theorem 1.2.8, stating that attractors of iterated function systems are tiling spaces, and by using this, we give some examples of tiling spaces. We also provide counterexamples related to the assumption on Theorem 1.2.7, bi-Lipschitz images of tiling spaces, and similarity classes of tiles.

5.1 Properties of spaces with tiling structures

We discuss basic properties of tiling sets, and (pre-)tiling spaces.

5.1.1 Basic properties of tiling spaces

Proposition 5.1.1 ([56])**.** *Let $(X, \mathcal{P})$ be a tiling set. Then for all $n, m \in \mathrm{dom}(\mathcal{P})$ with $n < m$, and for every tile $A \in \mathcal{P}_m$, there uniquely exists a tile $B \in \mathcal{P}_n$ with $A \subset B$.*

Proof. Let $(X, d, \mathcal{P})$ be an N-tiling set for some N. Suppose that there exist two distinct tiles $B, C \in \mathcal{P}_n$ such that $A \subset B \cap C$. By the condition (S2), there exist $k \in \mathrm{dom}(\mathcal{P})$ and $D \in \mathcal{P}_k$ such that $B \cup C \subset D$ and $k < n$. The condition (S1) yields $\mathrm{card}([D]_{m-k}) = N^{m-k}$.

Put $[D]_{n-k} = \{T_i\}_{i=1}^{N^{n-k}}$. We may assume that $T_1 = B$ and $T_2 = C$. The condition (S3) yields $[D]_{m-k} = \bigcup_{i=1}^{N^{n-k}} [T_i]_{m-n}$. By $A \subset B \cap C$, we have $A \in [T_1]_{m-n} \cap [T_2]_{m-n}$, and

$$[D]_{m-k} = ([T_1]_{m-n} \setminus \{A\}) \cup \bigcup_{i=2}^{N^{n-k}} [T_i]_{m-n}.$$

Since the condition (S1) yields $\mathrm{card}([T_i]_{m-n}) = N^{m-n}$ for all $i \in \{1, \dots, N^{n-k}\}$, we obtain

$$\mathrm{card}([D]_{m-k}) \leq (\mathrm{card}([T_1]_{m-n}) - 1) + \sum_{i=2}^{N^{n-k}} \mathrm{card}([T_i]_{m-n} = N^{m-k} - 1 < N^{m-k}.$$

This is a contradiction. Then we obtain the proposition. □

By the conditions (S2) and (T1), we have the following propositions:

Proposition 5.1.2 ([56])**.** *If a pre-tiling space $(X, d, \mathcal{P})$ is bounded as a metric space, then we have $\mathrm{dom}(\mathcal{P}) = \mathbb{N}$ and $\mathcal{P}_0 = \{X\}$.*

Proof. By the condition (T1) and the boundedness of X, the tiling index $\mathrm{dom}(\mathcal{P})$ must be $\mathbb{N}$. Suppose that $\mathcal{P}_0$ has two distinct elements. Then this contradicts the condition (S2) and $\mathrm{dom}(\mathcal{P}) = \mathbb{N}$. Therefore $\mathcal{P}_0 = \{X\}$. $\square$

Proposition 5.1.3 ([56])**.** *Let $(X, d, \mathcal{P})$ be a pre-tiling space. Then the metric space (X, d) is bounded if and only if* $\mathrm{dom}(\mathcal{P}) = \mathbb{N}$. *Equivalently, the metric space (X, d) is unbounded if and only if* $\mathrm{dom}(\mathcal{P}) = \mathbb{Z}$.

Proof. By Proposition 5.1.2, it suffices to show that if $\mathrm{dom}(\mathcal{P}) = \mathbb{N}$, then X is bounded. This holds true by the condition (T1). $\square$

For a subset A of a metric space X, we denote by $\mathrm{INT}(A)$ the interior of A in X.

Lemma 5.1.4 ([56])**.** *For every pre-tiling space $(X, d, \mathcal{P})$, and for every distinct pair of $A, B \in \mathcal{P}_n$ we have* $\mathrm{INT}(A) \cap \mathrm{INT}(B) = \emptyset$.

Proof. If for some distinct $A, B \in P_n$ we have $\mathrm{INT}(A) \cap \mathrm{INT}(B) \neq \emptyset$, then, by the condition (T1), there exist $k \in \mathrm{dom}(\mathcal{P})$ and $C \in \mathcal{P}_k$ such that $C \subset \mathrm{INT}(A) \cap \mathrm{INT}(B)$. This contradicts Proposition 5.1.1. $\square$

Lemma 5.1.5 ([56])**.** *All bounded pre-tiling spaces are totally bounded.*

Proof. Let $(X, d, \mathcal{P})$ be a bounded (N, s)-pre-tiling space, and let D_2 be a constant appeared in the condition (T1) for $(X, d, \mathcal{P})$. Proposition 5.1.3 implies $\mathrm{dom}(\mathcal{P}) = \mathbb{N}$. For each $n \in \mathbb{N}$, and for each $A \in [X]_n$, take a point $q_A \in A$. The conditions (S1) and (T1) for $(X, d, \mathcal{P})$ imply that the set $\{\, q_A \in X \mid A \in [X]_n \,\}$ is a $(D_2 s^n)$-net of X. $\square$

Since a totally bounded complete metric space is compact, we have:

Corollary 5.1.6 ([56])**.** *All bounded complete pre-tiling spaces are compact.*

Since a totally bounded metric space is separable, we obtain:

Corollary 5.1.7 ([56])**.** *All bounded pre-tiling spaces are separable.*

We next show the countability of tiling structures.

Proposition 5.1.8 ([56])**.** *For every pre-tiling space $(X, d, \mathcal{P})$, each $\mathcal{P}_n$ is countable.*

Proof. By Propositions 5.1.2 and 5.1.3, we may assume that X is unbounded. Take a sequence $\{T_i\}_{i\in\mathbb{N}}$ of tiles of $(X, d, \mathcal{P})$ such that for all $i \in \mathbb{N}$ we have $T_i \in \mathcal{P}_{-i}$ and $T_i \subset T_{i+1}$. From the condition (S2) and Proposition 5.1.1, it follows that $X = \bigcup_{i\in\mathbb{N}} T_i$. Then we obtain $\mathcal{P}_n = \bigcup_{-i\leq n} [T_i]_{n+i}$. This shows the proposition. $\square$

By Propositions 5.1.7 and 5.1.8, we obtain:

Corollary 5.1.9 ([56])**.** *All pre-tiling spaces are separable.*

5.1.2 Bi-Lipschitz maps and tiling spaces

Let (X, d_X) and (Y, d_Y) are metric spaces. Let $f : (X, d_X) \to (Y, d_Y)$ be a map. Let $L \in [1, \infty)$ and $\gamma \in (0, \infty)$. We say that f is *(L, γ)-homogeneously bi-Hölder* if for all $x, y \in X$ we have

$$L^{-1} \cdot d_X(x, y)^\gamma \leq d_Y(f(x), f(y)) \leq L \cdot d_X(x, y)^\gamma.$$

A map is said to be *homogeneously bi-Hölder* if it is an (L, γ)-homogeneously bi-Hölder map for some L and γ. A map is *L-bi-Lipschitz* if it is $(L, 1)$-homogeneously bi-Hölder.

Remark 5.1.1. Let (X, d) be an ultrametric space. For every $\gamma \in (0, \infty)$, the function d^γ is also an ultrametric. Thus the identity map $1_X : (X, d) \to (X, d^\gamma)$ is $(1, \gamma)$-homogeneously bi-Hölder for any $\gamma \in (0, \infty)$.

Lemma 5.1.10 ([56])**.** *Let $f : X \to Y$ be a surjective (L, γ)-homogeneously bi-Hölder map between metric spaces. Then every point $x \in X$ and every number $r \in (0, \infty)$ satisfy*

$$f(B(x, r)) \subset B(f(x), Lr^\gamma) \subset f(B(x, L^{2/\gamma} r)).$$

Lemma 5.1.10 implies that a pre-tiling structure is invariant under homogeneously bi-Hölder maps.

Proposition 5.1.11 ([56])**.** *Every homogeneously bi-Hölder image of an arbitrary pre-tiling space is a pre-tiling space. More precisely, the image of an arbitrary (N, s)-pre-tiling space under an (L, γ)-homogeneously bi-Hölder map is an (N, s^γ)-pre-tiling space.*

Since bi-Lipschitz maps are homogeneously bi-Hölder, we have:

Corollary 5.1.12 ([56])**.** *Every bi-Lipschitz image of an arbitrary (N, s)-pre-tiling space is an (N, s)-pre-tiling space.*

In spite of the virtue of Proposition 5.1.11, a homogeneously bi-Hölder image of a tiling space is not always a tiling space (see Example 5.4.2).

Proposition 2.4.4 implies that specific bi-Hölder images of tiling spaces are tiling spaces.

Proposition 5.1.13 ([56])**.** *Let $(X, d, \mathcal{P})$ be an (N, s)-tiling space and let $\epsilon \in (0, 1)$. Then $(X, d^\epsilon, \mathcal{P})$ is an (N, s^ϵ)-tiling space.*

Let (X, d) be a metric space, and let $\mathcal{P} : \mathrm{dom}(\mathcal{P}) \to \mathrm{cov}(X)$ be a map. Define a map $\mathcal{P}^C : \mathrm{dom}(\mathcal{P}) \to \mathrm{cov}(X)$ by $\mathcal{P}^C_n = \{ \mathrm{CL}(A) \mid A \in \mathcal{P}_n \}$, where CL is the closure operator in X. The following proposition allows us to assume that tiles of pre-tiling spaces are closed.

Proposition 5.1.14 ([56])**.** *Let $(X, d, \mathcal{P})$ be an (N, s)-pre-tiling space. Then $(X, d, \mathcal{P}^C)$ is also an (N, s)-pre-tiling space. Moreover, if $(X, d, \mathcal{P})$ satisfies the condition* (U)*, then so does $(X, d, \mathcal{P}^C)$.*

Proof. By the definition of $\mathcal{P}^C$, the condition (S3) is satisfied. From Lemma 5.1.4 and the condition (T2), it follows that for each pair $n, m \in \mathrm{dom}(\mathcal{P})$ with $n < m$ and for each $A \in \mathcal{P}_n$, if two tiles $S, T \in [A]_{m-n}$ satisfy $S \neq T$, then $\mathrm{CL}(S) \neq \mathrm{CL}(T)$. Thus the condition (S1) is satisfied. By the equality $\mathrm{CL}(A \cup B) = \mathrm{CL}(A) \cup \mathrm{CL}(B)$, the condition (S2) is satisfied. Then the pair $\left(X, \mathcal{P}^C\right)$ is an N-tiling set. By the facts that $\delta(A) = \delta(\mathrm{CL}(A))$ and that if $A \subset B$, then $\mathrm{CL}(A) \subset \mathrm{CL}(B)$, we conclude that the triple $\left(X, d, \mathcal{P}^C\right)$ is a pre-tiling space. Since for every subset A of X we have $\mathcal{GH}(A, \mathrm{CL}(A)) = 0$, we obtain the latter part of the proposition. $\square$

Let (X, d) be a metric space. We say that a covering pair $(X, \mathcal{P})$ is *self-similar* if there exists $s \in (0, 1)$ such that for each $n \in \mathrm{dom}(\mathcal{P})$, for each $A \in \mathcal{P}_n$ and for each $B \in \mathcal{P}_{n+1}$, we have $\mathcal{GH}((A, sd), (B, d)) = 0$. By the definition of the self-similarity, we have:

Lemma 5.1.15 ([56]). *Let (X, d) be a metric space. If a covering pair $(X, \mathcal{P})$ is self-similar, then $(X, d, \mathcal{P})$ satisfies the conditions* (T1) *and* (U).

For a product of pre-tiling spaces, we obtain:

Proposition 5.1.16 ([56]). *Let $p \in [1, \infty]$. Let $(X, d, \mathcal{P})$ and $(Y, e, \mathcal{Q})$ be (N, s)-pre-tiling spaces with $\mathrm{dom}(\mathcal{P}) = \mathrm{dom}(\mathcal{Q})$. Define a covering structure $\mathcal{R} : \mathrm{dom}(\mathcal{P}) \to \mathrm{cov}(X)$ by*

$$\mathcal{R}_n = \{ A \times B \mid A \in P_n, \ B \in \mathcal{Q}_n \}.$$

Then the triple $(X \times Y, d \times_p e, \mathcal{R})$ is a (N^2, s)-pre-tiling space.

Proof. Since $(X, d, \mathcal{P})$ and $(Y, e, \mathcal{Q})$ satisfy the conditions (S1), (S2) and (S3), so does $(X \times Y, d \times_p e, \mathcal{R})$. By the definition of ℓ^p-product metrics, we see that the conditions (T1) and (T2) are satisfied. Hence the triple $(X \times Y, d \times_p e, \mathcal{R})$ is a (N^2, s)-pre-tiling space. □

Remark 5.1.2. The author does not know whether it is true that if X and Y satisfy (U), then so does $X \times_p Y$ for any $p \in [1, \infty]$.

5.2 Tiling spaces and the Assouad dimension

In this section, we prove Theorem 1.2.7.

5.2.1 Proof of Theorem 1.2.7

Proposition 5.2.1 ([56]). *Let $(X, d, \mathcal{P})$ be a doubling pre-tiling space. Then for every $W \in (0, \infty)$, there exists $M_W \in \mathbb{N}_{\geq 1}$ such that for each $m \in \mathrm{dom}(\mathcal{P})$ and for each subset S of X with $\delta(S) \leq W s^m$, we have $\mathrm{card}(\{ A \in \mathcal{P}_m \mid A \cap S \neq \emptyset \}) \leq M_W$.*

Proof. Let D_2 and E be constants appearing in the conditions (T1) and (T2) for $(X, d, \mathcal{P})$. Let $W \in (0, \infty)$, and take $m \in \mathrm{dom}(\mathcal{P})$ and a subset S of X satisfying $\delta(S) \leq W s^m$. For each $A \in \mathcal{P}_m$ satisfying $A \cap S \neq \emptyset$, let $p_A \in A$ be a point appearing in the condition (T2) for $(X, d, \mathcal{P})$. Set $Z = \{ p_A \in X \mid A \in \mathcal{P}_m, \ A \cap S \neq \emptyset \}$. By Lemma 5.1.4 and the condition (T2), we have $\mathrm{card}(Z) = \mathrm{card}(\{ A \in \mathcal{P}_m \mid A \cap S \neq \emptyset \})$ and $\alpha(Z) \geq E s^m$. The condition (T2) implies that $\delta(A) \leq D_2 s^m$ for every $A \in \mathcal{P}_m$. For any point $x \in S$, we have $Z \subset B(x, (D_2 + W)s^m)$. Thus we have $\delta(Z) \leq 2(D_2 + W)s^m$. By Proposition 2.3.1, we can take $\gamma \in \mathbf{C}(X, d)$. Then for some $C \in (0, \infty)$ we obtain

$$\mathrm{card}(\{ A \in \mathcal{P}_m \mid A \cap S \neq \emptyset \}) \leq C \cdot \left(\frac{\delta(Z)}{\alpha(Z)} \right)^{\gamma} \leq C \cdot \left(\frac{2(D_2 + W)}{E} \right)^{\gamma}.$$

If we put $M_W = C \cdot (2(D_2 + W)/E)^{\gamma}$, then the proposition is proven. □

Definition 5.2.1 ([56]). Let $(X, d, \mathcal{P})$ be a pre-tiling space, and let F be a subset of X. For each pair $n, m \in \mathrm{dom}(\mathcal{P})$ with $n < m$ and for each $B \in \mathcal{P}_n$, we put

$$\mathbf{Q}^F_{n,m}(B) = \{ A \in \mathcal{P}_m \mid A \cap F \neq \emptyset, \ A \subset B \}. \tag{5.2.1}$$

Lemma 5.2.2 ([56]). *Let $(X, d, \mathcal{P})$ be a doubling pre-tiling space. Let F be a subset of X. Let Λ be the infimum of all $\beta \in (0, \infty)$ for which there exists $C \in (0, \infty)$ such that for every pair $n, m \in \dim(\mathcal{P})$ with $n < m$ and for every $B \in \mathcal{P}_n$ we have*

$$\mathbf{Q}^F_{n,m}(B) \leq C(s^{n-m})^{\beta}. \tag{5.2.2}$$

Then we have

$$\dim_A F = \Lambda.$$

Proof. Let $(X, d, \mathcal{P})$ be a doubling (N, s)-pre-tiling space, and let D_1 and D_2 be constants appearing in the condition (T1) for $(X, d, \mathcal{P})$. Take $\beta \in (0, \infty)$ satisfying (5.2.2) with $\Lambda < \beta$. Let $S \subset F$ be a bounded set. Take $n \in \operatorname{dom}(\mathcal{P})$ with $D_1 s^{n-1} \leq \delta(S) < D_1 s^n$. Let $r \in (0, \infty)$, and take $m \in \operatorname{dom}(\mathcal{P})$ with $s^{m-1} \leq r < s^m$. Take a constant M_{D_1} stated in Proposition 5.2.1 for $W = D_1$. Then S can be covered at most M_{D_1} many members in $\mathcal{P}_n$, and hence by (5.2.2), the set S also can be covered by at most $M_{D_1} C(s^{n-m})^{\beta}$ many members in $\mathcal{P}_m$. In particular, we have

$$\mathbf{Z}_{(X,d)}(S, r) \leq M_{D_1} C(s^{n-m})^{\beta} \leq M_{D_1} D_1^{-\beta} C s^{\beta} \cdot (\delta(S)/r)^{\beta}.$$

This implies $\beta \in \mathbf{A}(X, d)$. Hence, $\dim_A F \leq \Lambda$.

We next prove the opposite inequality. Take $\beta \in \mathbf{A}(X, d)$ and $B \in \mathcal{P}_n$. The set $B \cap F$ is a bounded set of X with $\delta(B \cap F) \leq D_2 s^n$. Thus $B \cap F$ can be covered by at most $C(s^{n-m})^{\beta}$ many bounded sets with diameters at most $D_2 s^m$. Write these bounded sets as $A_1, A_2, \dots, A_N$, where $N \leq C(s^{n-m})^{\beta}$. Take a constant M_{D_2} stated in Proposition 5.2.1 for $W = D_2$. Then each A_i can be covered by at most M_{D_2} many members in $\mathcal{P}_m$. Hence we have $\mathbf{Q}^F_{n,m}(B) \leq M_{D_2} \cdot C \cdot (s^{n-m})^{\beta}$. This implies $\Lambda \leq \dim_A(F)$. □

Applying Lemma 5.2.2 to a pre-tiling space or to a tile of it, we obtain:

Corollary 5.2.3 ([56]). *Let $(X, d, \mathcal{P})$ be a doubling (N, s)-pre-tiling space. Then for every tile T of $(X, d, \mathcal{P})$ we have*

$$\dim_A T = \dim_A X = \log(N)/\log(s^{-1}).$$

By using the condition (U), we obtain the following lemma:

Lemma 5.2.4 ([56]). *Let $(X, d, \mathcal{P})$ be a doubling tiling space. Let F be a subset of X. Let D_2 be a constant appearing in the condition (T1) for $(X, d, \mathcal{P})$. If $\mathcal{TPC}(F)$ contains no tiles of $(X, d, \mathcal{P})$, then there exists $k \in \mathbb{N}$ such that for each $n \in \operatorname{dom}(\mathcal{P})$ and for each $B \in \mathcal{P}_n$ we have $\mathcal{GH}(B, B \cap F) > D_2 s^{n+k}$.*

Proof. Suppose that for each $k \in \mathbb{N}$ there exist $n_k \in \operatorname{dom}(\mathcal{P})$ and $B_k \in P_{n_k}$ such that the inequality $\mathcal{GH}(B_k, B_k \cap F) \leq D_2 s^{n_k+k}$ holds. By the condition (T1), we have

$$\mathcal{GH}(B_k, B_k \cap F) \leq (D_2/D_1) s^k \cdot \delta(B_k), \tag{5.2.3}$$

where D_1 is a constant appearing in the condition (T1) for $(X, d, \mathcal{P})$. By the condition (U), there exists a subsequence $\{B_{\phi(k)}\}_{k \in \mathbb{N}}$ of the sequence $\{B_k\}_{k \in \mathbb{N}}$ such that the sequence $\{(B_{\phi(k)}, \delta(B_{\phi(k)})^{-1} d)\}_{k \in \mathbb{N}}$ converges to $(T, (\delta(T))^{-1} d)$ for some tile T of (X, P). From the inequality (5.2.3), it follows that

$$\mathcal{GH}((B_{\phi(k)}, \delta(B_{\phi(k)}))^{-1} d), ((B_{\phi(k)} \cap F), \delta(B_{\phi(k)}))^{-1} d) \leq (D_2/D_1) s^{\phi(k)}.$$

By $s^{\phi(k)} \to 0$ as $k \to \infty$, we conclude that $T \in \mathcal{TPC}(X, d)$. This is a contradiction. □

We next prove the following:

Lemma 5.2.5 ([56]). *Let $(X, d, \mathcal{P})$ be a doubling (N, s)-tiling space. Let F be a subset of X. If $\mathcal{TPC}(F, d)$ contains no tiles of $(X, d, \mathcal{P})$, then we have $\dim_A F < \log(N)/\log(s^{-1})$.*

Proof. Let D_1 and D_2 be constants appearing in the condition (T1) for $(X, d, \mathcal{P})$. Set

$$d = \log(N)/\log(s^{-1}).$$

Take $k \in \mathbb{N}$ stated in Lemma 5.2.4. Put $L = s^k$. By Lemma 5.2.4, for each $n \in \mathrm{dom}(\mathcal{P})$ and for each $B \in \mathcal{P}_n$, we have $d_H(B, B \cap F; X) > D_2 s^{n+k}$. Thus we can take a point $x \in B$ such that for every $y \in F$ we have $d_X(x, y) > D_2 s^{n+k}$. Take $C \in [B]_k$ with $x \in C$, then by the condition (T1) we have $C \cap F = \emptyset$. Therefore we obtain the following claim:

Sublemma 5.2.6. *For each $n \in \mathrm{dom}(\mathcal{P})$ and for each $B \in \mathcal{P}_n$, there exists $C \in \mathcal{P}_{n+k}$ with $C \subset B$ and $C \cap F = \emptyset$.*

Fix $a, b \in \mathrm{dom}(\mathcal{P})$ with $a > b$ and $B \in P_b$. Take $v \in \mathbb{N}$ such that

$$D_1 s^{b+k(v+1)} \leq D_2 s^a < D_1 s^{b+kv}. \tag{5.2.4}$$

Since $D_2 s^a < D_1 s^{b+kv}$, we have $b + kv < a$. Hence for each $A \in \mathcal{P}_{b+kv}$ the set $[A]_{a-(b+kv)}$ is non-empty. Let W be the set of all words generated by $\{0, \dots, N^k - 1\}$ whose length is at most v. Note that W contains the empty word. For $w \in W$, we denote by $|w|$ the length of the word w. For $u, v \in W$, we denote by uv the word product of u and v.

Let the set $\bigcup_{i=0}^{v} [B]_{ki}$ be indexed by W, say $\{T_w\}_{w \in W}$ such that for each $w \in W$ we have $T_w \in [B]_{k|w|}$, and such that if $|w| < v - 1$, then $T_{w0} \cap F = \emptyset$. This is possible by Sublemma 5.2.6. For each $i \in \{1, \dots, v\}$, define a set H_i by

$$H_i = \{\, w0 \mid w \in W,\ |w| = i - 1 \text{ and all entries of } w \text{ are not } 0 \,\}.$$

Put $R_w = [T_w]_{a-(b+k|w|)}$. Remark that $R_w = \{\, A \in \mathcal{P}_a \mid A \subset T_w \,\}$. Put $H = \bigcup_{i=1}^{v} H_i$. Note that for all distinct $v, w \in H$, the sets R_v and R_w are disjoint.

Let $G = \bigcup_{w \in H} R_w$. We find that $G = \coprod_{i=1}^{v} \coprod_{w \in H_i} R_w$. Let $\mathbf{Q}^F_{b,a}(B)$ be the quantity defined in Definition 5.2.1. Then we have $\mathbf{Q}^F_{b,a}(B) \leq \mathrm{card}([B]_{a-b}) - \mathrm{card}(G)$. Since $d = \log(N)/\log(s^{-1})$, we obtain the equalities $\mathrm{card}(R_w) = N^{a-b-k|w|} = s^{-d(a-b-k|w|)}$ and $\mathrm{card}(H_i) = (N^k - 1)^{i-1} = (L^{-d} - 1)^{i-1}$. By these equalities, we obtain

$$\begin{aligned}
\mathrm{card}(G) &= \mathrm{card}\left(\coprod_{i=1}^{v} \coprod_{w \in H_i} R_w \right) = \sum_{i=1}^{v} \mathrm{card}\left(\coprod_{w \in H_i} R_w \right) = \sum_{i=1}^{v} \sum_{w \in H_i} s^{-d(a-b-k|w|)} \\
&= \sum_{i=1}^{v} \sum_{w \in H_i} s^{-d(a-b-ki)} = \sum_{i=1}^{v} s^{d(b-a)} s^{kdi} (L^{-d} - 1)^{i-1} = s^{d(b-a)} \sum_{i=1}^{v} L^{di} (L^{-d} - 1)^{i-1}.
\end{aligned}$$

Since for each $w \in H$ we have $T_w \cap F = \emptyset$, by the definition (5.2.1) of $\mathbf{Q}^F_{b,a}(B)$ we obtain

$$\mathbf{Q}^F_{b,a}(B) \leq N^{a-b} - \mathrm{card}(G) = (s^{b-a})^d \left(1 - \sum_{i=1}^{v} L^{id} (L^{-d} - 1)^{i-1} \right).$$

Note that we have

$$\begin{aligned}
\sum_{i=1}^{v} L^{di} (L^{-d} - 1)^{i-1} &= (L^{-d} - 1)^{-1} \sum_{i=1}^{v} (1 - L^d)^i \\
&= (L^d - 1)^{-1} (1 - L^d)(1 - (1 - L^d)^v) L^{-d} = 1 - (1 - L^d)^v.
\end{aligned}$$

By (5.2.4), we have $L^{v+1} \leq (D_2/D_1)s^{a-b}$, then $\log((D_2/D_1)s^{a-b})/\log L - 1 \leq v$, and hence

$$\begin{aligned}
\mathbf{Q}^F_{b,a}(B) &\leq (s^{b-a})^d(1 - \sum_{i=1}^{v} L^{di}(L^{-d} - 1)^{i-1}) = (s^{b-a})^d(1 - L^d)^v \\
&\leq (s^{b-a})^d(1 - L^d)^{\log((D_2/D_1)s^{a-b})/\log L - 1} \\
&= (s^{b-a})^d \frac{1}{1 - L^d}((D_1/D_2)s^{b-a})^{-\log(1-L^d)/\log L} \\
&= \left(\frac{1}{1 - L^d} \left(\frac{D_1}{D_2} \right)^{-\log(1-L^d)/\log L} \right) \cdot (s^{b-a})^{d - \log(1-L^d)/\log L}.
\end{aligned}$$

By Lemma 5.2.2, we obtain $\dim_A(F) \leq d - \log(1 - L^d)/\log L < d$. Thus we conclude that $\dim_A(F) < d$. This finishes the proof. $\square$

Lemma 5.2.7 ([56])**.** *Let $(X, d, \mathcal{P})$ be a pre-tiling space. Then the following are equivalent:*

(1) *there exists a tile A of $(X, d, \mathcal{P})$ such that $A \in \mathcal{PC}(X, d)$;*

(2) *there exists a tile of A of $(X, d, \mathcal{P})$ such that F satisfies the asymptotic Steinhaus property for A.*

Proof. We first show that (2) $\Longrightarrow$ (1). Take a tile A of $(X, d, \mathcal{P})$ stated in the condition (2). For each $n \in \mathbb{N}_{\geq 1}$, take a $(1/n)$-net S_n of A. By the condition (2), we can take a finite subset T_n of F and $\delta_n \in (0, \infty)$ satisfying that $\mathcal{GH}((T_n, d), (S_n, \delta_n d)) < \delta_n/n$. Set $u_n = \delta_n^{-1}$, then we obtain the estimation $\mathcal{GH}((T_n, u_n d), (S_n, d)) \leq 1/n$. Hence $\mathcal{GH}((T_n, u_n d), (A, d)) \to 0$ as $n \to \infty$. This implies $(A, d) \in \mathcal{PC}(X, d)$.

We next show (1) $\Longrightarrow$ (2). Take a tile A of $(X, d, \mathcal{P})$ stated in the condition (1). Take a finite subset S of A. Since $(A, d) \in \mathcal{PC}(F, d)$, there exist a sequence $\{T_n\}_{n\in\mathbb{N}}$ of subsets of F and a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $(0, \infty)$ such that $\lim_{n\to\infty} \mathcal{GH}((T_n, u_n d), (A, d)) = 0$. By Lemma 5.1.5, the tile A is totally bounded. Then, by Proposition 2.4.5, for each $\epsilon \in (0, \infty)$, we can take a finite subset Y_N of T_N such that $\mathcal{GH}((Y_n, u_n d), (A, d)) < \epsilon$. Since S is finite, we can take a subset U_N of Y_N such that $\mathcal{GH}((U_N, d), (S, u_n^{-1} d)) < u_n^{-1} \cdot \epsilon$. Thus F satisfies the asymptotic Steinhaus property for A. $\square$

We now prove Theorem 1.2.7.

Proof of Theorem 1.2.7. By the definitions, the implications (3)$\Longrightarrow$ (2) and (4) $\Longrightarrow$ (2) are true. Theorem 1.2.1 implies that (2) $\Longrightarrow$ (1). Lemma 5.2.5 is equivalent to the implication (1) $\Longrightarrow$ (3). Lemma 5.2.7 states that the equivalence (2) $\Longleftrightarrow$ (5) is true. Therefore it suffices to show the implication (2) $\Longrightarrow$ (4).

Let $(X, d, \mathcal{P})$ be a doubling tiling space. By Lemma 5.1.5, every tile of a tiling space is totally bounded. From this property and Proposition 2.4.5, it follows that a tile of $(X, d, \mathcal{P})$ in $\mathcal{PC}(F, d)$ is approximated by a sequence of scalings of finite sets of F in the sense of Gromov–Hausdorff. This completes the proof of Theorem 1.2.7. $\square$

5.3 Tiling spaces induced from iterated function systems

In this section, we prove Theorem 1.2.8, stating that attractors of iterated function systems are tiling spaces, and we introduce the notion of an extended attractor.

5.3.1 Attractors

In this subsection, we prove Theorem 1.2.8. Before proving it, we prepare or recall the notation which we use in this subsection. Let (X, d) be a complete metric space. Let $N \in \mathbb{N}_{\geq 2}$ and $s \in (0, 1)$. Let $\mathcal{S} = \{S_i\}_{i=0}^{N-1}$ be an (N, s)-similar iterated function system on X. Assume that the attractor $A_{\mathcal{S}}$ of $\mathcal{S}$ satisfies the strong open set condition. Let V be an open set appearing in the strong open set condition. Let W be the set of all words generated by $\{0, \dots, N-1\}$. For every $q \in A_{\mathcal{S}}$, and for each $w \in W$, put $q_w = S_w(q)$. We also put $d_{\mathcal{S}} = d|_{A_{\mathcal{S}}}$. Let $\mathcal{P}_{\mathcal{S}}$ be the map defined in Definition 1.2.4.

We first prove that $(A_{\mathcal{S}}, d_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}})$ is an N-tiling set.

Lemma 5.3.1. *The covering pair $(A_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}})$ is an N-tiling set.*

Proof. By the definition of $\mathcal{P}_{\mathcal{S}}$, we see that the condition (S3) is satisfied. We next verify that the condition (S1) is satisfied. Take $q \in A_{\mathcal{S}} \cap V$. For each pair $n, m \in \mathbb{N}$ with $n < m$, and for each $B \in \mathcal{P}_n$, by the definitions of attractors and $\mathcal{P}$, we have $B = \bigcup [B]_{m-n}$.

We now prove $\operatorname{card}([B]_{m-n}) = N^{m-n}$. By the definition of $\mathcal{P}_{\mathcal{S}}$, we have

$$[B]_{m-n} = \{\, S_w(B) \mid w \in W \text{ and } |w| = m-n \,\}.$$

Write $B = S_v(A_{\mathcal{S}})$, where $v \in W$. By the condition (O2) in the strong open set condition, the family $\{\, S_{vw}(V) \mid w \in W \text{ and } |w| = m-n \,\}$ is disjoint. This implies that if $w \neq w'$ with $|w| = |w'| = m-n$, then we have $q_{vw} \neq q_{vw'}$. Therefore we see that the set $\{\, q_{vw} \in A_{\mathcal{S}} \mid w \in W \text{ and } |w| = m-n \,\}$ has cardinality N^{m-n}. Since for each $w \in W$ with $|w| = m-n$ we have $q_{vw} \in S_{vw}(V) \cap S_{vw}(A_{\mathcal{S}})$, we obtain $\operatorname{card}([B]_{m-n}) = N^{m-n}$.

By the boundedness of $A_{\mathcal{S}}$, the pair $(A_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}})$ satisfies the condition (S2). Therefore the pair $(A_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}})$ is an N-tiling set. $\square$

We next prove that $(A_{\mathcal{S}}, d_{\mathcal{S}}, P_{\mathcal{S}})$ is an (N, s)-tiling space.

Lemma 5.3.2. *The attractor $A_{\mathcal{S}}$ of $\mathcal{S}$ is contained in $\mathrm{CL}(V)$. Moreover, for every word $w \in W$, the inclusion $S_w(A_{\mathcal{S}}) \subset \mathrm{CL}(S_w(V))$ is satisfied.*

Proof. Take $q \in A_{\mathcal{S}} \cap V$. Put $M_0 = \{q\}$, and for each $n \in \mathbb{N}_{\geq 1}$ put $M_n = \bigcup_{i=0}^{N-1} S_i(M_{n-1})$. Then M_n converges to $A_{\mathcal{S}}$ in the Hausdorff topology, in particular, $\mathrm{CL}(\bigcup_{n \in \mathbb{N}} M_n) = A_{\mathcal{S}}$. By the definition of M_n, for each $n \in \mathbb{N}$ we have $M_n \subset V$. Thus $A_{\mathcal{S}} \subset \mathrm{CL}(V)$. Since S_w is a topological embedding for any $w \in W$, the latter part follows from the former one. $\square$

We now prove Theorem 1.2.8.

Proof of Theorem 1.2.8. Since $(A_{\mathcal{S}}, d_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}})$ is self-similar, by Lemma 5.1.15 the triple $(A_{\mathcal{S}}, d_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}})$ satisfies the conditions (T1) and (U). It suffices to show that $(A_{\mathcal{S}}, d_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}})$ satisfies the condition (T2). By $A_{\mathcal{S}} \cap V \neq \emptyset$, we can take $q \in A_{\mathcal{S}} \cap V$. Then there exists $E \in (0, \infty)$ such that $U(q, E) \subset V$. Since $\mathcal{S}$ consists of s-similar transformations, for each $w \in W$ we have $B(q_w, Es^{|w|}) \subset S_w(V)$. Lemma 5.3.2 implies that for each $w \in W$ we have $S_w(A_{\mathcal{S}}) \subset \mathrm{CL}(S_w(V))$, thus the ball $B(q_w, Es^{|w|})$ in X meets only $S_w(A_{\mathcal{S}})$. Hence the ball $B(q_w, Es^{|w|})$ in $A_{\mathcal{S}}$ is a subset of $S_w(A_{\mathcal{S}})$. Therefore we conclude that $(A_{\mathcal{S}}, P_{\mathcal{S}})$ satisfies the condition (T2). This completes the proof of Theorem 1.2.8. $\square$

Remark 5.3.1. An iterated function system $\mathcal{S}$ on a metric space is said to satisfy the *open set condition* if $\mathcal{S}$ satisfies the conditions (O1) and (O2) in the strong open set condition. Schief [100] proved that the open set condition implies the strong open set condition in the Euclidean setting. Schief [101] also proved that the implication mentioned above does not always hold in a general setting (see [101, Example 3.1]).

5.3.2 Extended attractors

We define unbounded tiling spaces induced from similar iterated function systems.

Definition 5.3.1 ([56]). Let $N \in \mathbb{N}_{\geq 2}$ and $s \in (0,1)$. Let $\mathcal{S}$ be an (N,s)-similar iterated function system on a complete metric space with the strong open set condition, say $\mathcal{S} = \{S_i\}_{i=0}^{N-1}$. Define a sequence $\{(F_k, e_k)\}_{k\in\mathbb{N}}$ of metric spaces by $(F_k, e_k) = (A_{\mathcal{S}}, s^{-k}d_{\mathcal{S}})$, where $A_{\mathcal{S}}$ is the attractor of $\mathcal{S}$. Note that for each $k \in \mathbb{N}$, each S_i is an s-similar transformation on F_k. By the definition of $A_{\mathcal{S}}$, we find that $(S_0(F_{k+1}), e_{k+1})$ is isometric to (F_k, e_k). Thus we can identify F_k with $S_0(F_{k+1})$, and we may consider that $F_k \subset F_{k+1}$ for each $k \in \mathbb{N}$. Put $E_{\mathcal{S}} = \bigcup_{k\in\mathbb{N}} F_k$. The metric $d_{\mathcal{S}}$ on $E_{\mathcal{S}}$ is naturally obtained by identifying F_k with $S_0(F_{k+1})$. Note that $E_{\mathcal{S}}$ is unbounded. Let W be the set of all words generated by $\{0,\dots,N-1\}$. Define a map $\mathcal{Q}_{\mathcal{S}} : \mathbb{Z} \to \mathrm{cov}(E_{\mathcal{S}})$ by

$$(\mathcal{Q}_{\mathcal{S}})_n = \{\, S_w(F_k) \mid \ w \in W \text{ and } |w| - k = n \,\}. \tag{5.3.1}$$

We call $E_{\mathcal{S}}$ an *extended attractor of* $\mathcal{S}$.

Similarly to Theorem 1.2.8, we obtain the following:

Theorem 5.3.3 ([56]). *For $N \in \mathbb{N}_{\geq 2}$ and $s \in (0,1)$, let $\mathcal{S}$ be an (N,s)-similar iterated function system on a complete metric space satisfying the strong open set condition. Let $E_{\mathcal{S}}$ be the extended attractor of $\mathcal{S}$, and $\mathcal{Q}_{\mathcal{S}}$ the map defined by* (5.3.1)*. Then $(E_{\mathcal{S}}, d_{\mathcal{S}}, \mathcal{Q}_{\mathcal{S}})$ is an unbounded (N,s)-tiling space.*

Proof. Since all the tiles of $(E_{\mathcal{S}}, d_{\mathcal{S}}, \mathcal{Q}_{\mathcal{S}})$ are similar to $A_{\mathcal{S}}$, by a similar argument to Lemma 5.3.1, we see that the condition (S1) is satisfied. Lemma 5.1.15 implies that $(E_{\mathcal{S}}, d_{\mathcal{S}}, \mathcal{Q}_{\mathcal{S}})$ satisfies the conditions (T1) and (U). By $E_{\mathcal{S}} = \bigcup_{k\in\mathbb{N}} F_k$, and by the definition of $\mathcal{Q}_{\mathcal{S}}$, the conditions (S2) and (S3) are satisfied. Thus $(E_{\mathcal{S}}, \mathcal{Q}_{\mathcal{S}})$ is an N-tiling set. Similarly to the proof of Theorem 1.2.8, we see that the condition (T2) is satisfied. Therefore $(E_{\mathcal{S}}, d_{\mathcal{S}}, \mathcal{Q}_{\mathcal{S}})$ is an (N,s)-tiling space. □

5.3.3 Examples of attractors

In this subsection, we see that classical examples of fractals are tiling spaces.

Example 5.3.1 (The middle-third Cantor set). Let C be the middle-third Cantor set (this is also just called the Cantor set). For each $i \in \{0,1\}$, define a map $f_i : \mathbb{R} \to \mathbb{R}$ by

$$f_i(x) = \frac{1}{3}x + \frac{2}{3}i.$$

Put $\mathcal{S} = \{f_0, f_1\}$. Hence $\mathcal{S}$ is a $(2, 3^{-1})$-similar iterated function system on $\mathbb{R}$, and the middle-third Cantor se C is the attractor of $\mathcal{S}$. Since the open set $(0,1)$ of $\mathbb{R}$ satisfies the conditions (O1), (O2) and (O3), the iterated function system $\mathcal{S}$ satisfies the strong open set condition. Let $\mathcal{P}_{\mathcal{S}} : \mathbb{N} \to C$ be the map defined in Definition 1.2.4. Theorem 1.2.8 implies that $(C, d_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}})$ is a $(2, 3^{-1})$-tiling space.

Example 5.3.2 (The Sierpiński gasket). Referring to the cubic roots of unity in the complex plane, put $w_0 = (1,0)$, $w_1 = 2^{-1}(-1, \sqrt{3})$ and $w_2 = 2^{-1}(-1, -\sqrt{3})$. For each $i \in \mathbb{N}$, we define a map $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f_i(x) = \frac{1}{2}x + \frac{1}{2}w_i.$$

Put $\mathcal{S} = \{f_0, f_1, f_2\}$. Hence $\mathcal{S}$ is a $(3, 2^{-1})$-similar iterated function system on $\mathbb{R}^2$. The attractor $A_{\mathcal{S}}$ of $\mathcal{S}$ is the so-called Sierpiński gasket. Let V be the interior of the triangle with vertices $\{w_0, w_1, w_2\}$, then V satisfies the conditions (O1), (O2) and (O3). Thus $\mathcal{S}$ satisfies the strong open set condition. Let $\mathcal{P}_{\mathcal{S}} : \mathbb{N} \to \mathrm{cov}(A_{\mathcal{S}})$ be the map defined in Definition 1.2.4. Then Theorem 1.2.8 implies that $(A_{\mathcal{S}}, d_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}})$ is a $(3, 2^{-1})$-tiling space.

Example 5.3.3 (Euclidean spaces)**.** Consider the N-dimensional normed vector space $\mathbb{R}^N$ with the ℓ^p-Euclidean metric, where $p \in [1, \infty]$.

Let $A = \{\, v \in \mathbb{R}^N \mid \text{the entries of } v \text{ are } 0,\ 1 \text{ or } -1 \,\}$. Since A has cardinality 3^N, it can be indexed by $\{1, \dots, 3^N\}$, say $A = \{v(i)\}_{i=1}^{3^N}$. For each $i \in \{1, \dots, 3^N\}$, define a $(1/3)$-similar transformation $f_i : \mathbb{R}^N \to \mathbb{R}^N$ by

$$f_i(x) = \frac{1}{3}x + \frac{1}{3}v(i).$$

Put $\mathcal{S} = \{f_i\}_{i=1}^{3^N}$. Then $\mathcal{S}$ is a $(3^N, 3^{-1})$-similar iterated function system on $\mathbb{R}^N$, and $[-2^{-1}, 2^{-1}]^N$ is the attractor of $\mathcal{S}$. Since the open set $(-2^{-1}, 2^{-1})^N$ of $\mathbb{R}^N$ satisfies the conditions (O1), (O2) and (O3), the iterated function system $\mathcal{S}$ satisfies the strong open set condition. Let $\mathcal{P}_{\mathcal{S}} : \mathbb{N} \to \mathrm{cov}([-2^{-1}, 2^{-1}]^N)$ be the map defined in Definition 1.2.4, then this map is described as $(\mathcal{P}_{\mathcal{S}})_n = \{\, 3^{-n}v + 3^{-n}[-2^{-1}, 2^{-1}]^N \mid v \in \mathbb{Z}^N \,\}$. Theorem 1.2.8 implies that $([-2^{-1}, 2^{-1}]^N, d_{\mathcal{R}^N}, \mathcal{P}_{\mathcal{S}})$ is a $(3^N, 3^{-1})$-tiling space.

We next consider the extended attractor $E_{\mathcal{S}}$ of the iterated function system $\mathcal{S}$. Since

$$\mathbb{R}^N = \bigcup_{i \in \mathbb{N}} [-2^{-1} \cdot 3^i, 2^{-1} \cdot 3^i]^N,$$

the space $E_{\mathcal{S}}$ is isomeric to $\mathbb{R}^N$ with the ℓ^p-Euclidean metric. Under a natural identification between $E_{\mathcal{S}}$ and $\mathbb{R}^N$, the map $\mathcal{Q}_{\mathcal{S}} : \mathbb{Z} \to \mathrm{cov}(\mathbb{R}^N)$ defined in Definition 5.3.1 can be described as $(\mathcal{Q}_{\mathcal{S}})_n = \{\, 3^{-n}v + 3^{-n}[-2^{-1}, 2^{-1}]^N \mid v \in \mathbb{Z}^N \,\}$. Theorem 5.3.3 implies that the extended attractor $(\mathbb{R}^N, d_{\mathbb{R}^N}, \mathcal{Q}_{\mathcal{S}})$ is a $(3^N, 3^{-1})$-tiling space.

Applying Theorem 1.2.7 to the tiling space $(\mathbb{R}^N, d_{\mathbb{R}^N}, \mathcal{Q}_{\mathcal{S}})$ discussed in Example 5.3.3, we obtain the Fraser–Yu characterization in [35] in a slightly different formulation:

Corollary 5.3.4. *For every subset F of $\mathbb{R}^N$, the following are equivalent:*

(1) $\dim_A(F, d_{\mathbb{R}^N}) = N$;

(2) $[0, 1]^N \in \mathcal{PC}(F, d_{\mathbb{R}^N})$;

(3) $[0, 1]^N \in \mathcal{TPC}(F, d_{\mathbb{R}^N})$;

(4) $[0, 1]^N \in \mathcal{KPC}(F, d_{\mathbb{R}^N})$;

(5) *F satisfies the asymptotic Steinhaus property for $[0, 1]^N$.*

Proof. Let $\mathcal{Q}_{\mathcal{S}}$ be the map described in Example 5.3.3. Since all the tiles of $(\mathbb{R}^N, d_{\mathbb{R}^N}, \mathcal{Q}_{\mathcal{S}})$ are similar to $[0, 1]^N$, Theorem 1.2.7 leads to the claim. □

Example 5.3.4 (p-adic numbers)**.** Let p be a prime number and let $\mathbb{Q}_p$ be the set of all p-adic numbers. Let $v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ be the p-adic valuation. Let $s \in (0, 1)$, and define $d_{\mathbb{Q}_p}(x, y) = s^{v_p(x-y)}$, then $d_{\mathbb{Q}_p}$ is an ultrametric on $\mathbb{Q}_p$ generating the same topology as $\mathbb{Q}_p$. For each $k \in \{0, \dots, p-1\}$, define an s-similar transformation $f_k : \mathbb{Q}_p \to \mathbb{Q}_p$ by

$$f_k(x) = xp + k.$$

Put $\mathcal{S} = \{f_k\}_{k=0}^{p-1}$. Then $\mathcal{S}$ is a (p, s)-similar iterated function system on $\mathbb{Q}_p$. We see that the ball $B(0, 1)$ centered at 0 with radius 1 in $\mathbb{Q}_p$ is the attractor of $\mathcal{S}$. Since for each $k \in \{0, \dots, p-1\}$ we have $f_k(B(0, 1)) = B(k, s)$, and since $d_{\mathbb{Q}_p}$ is an ultrametric, the open set $B(0, 1)$ satisfies the conditions (O1), (O2) and (O3). Thus the iterated function system $\mathcal{S}$ satisfies the strong open set condition. Let $\mathcal{P}_{\mathcal{S}} : \mathbb{N} \to \mathrm{cov}(B(0, 1))$ be the map defined in Definition 1.2.4, then this map can be described as $(\mathcal{P}_{\mathcal{S}})_n = \{\, B(a, s^{-n}) \mid a \in \mathbb{Q}_p \,\}$. By Theorem 1.2.8, we conclude that $(B(0, 1), d_{\mathbb{Q}_p}, \mathcal{P}_{\mathcal{S}})$ is a (p, s)-tiling space.

We next consider the extended attractor of $\mathcal{S}$. Since

$$\mathbb{Q}_p = \bigcup_{i \in \mathbb{N}} B(0, s^{-i}),$$

the space $E_{\mathcal{S}}$ is isometric to $\mathbb{Q}_p$. Under a natural identification between $E_{\mathcal{S}}$ and $\mathbb{Q}_p$, we can describe the map $\mathcal{Q}_{\mathcal{S}} : \mathbb{Z} \to \mathrm{cov}(\mathbb{Q}_p)$ in Definition 5.3.1 as $(\mathcal{Q}_{\mathcal{S}})_n = \{\, B(a, s^{-n}) \mid a \in \mathbb{Q}_p \,\}$. Theorem 5.3.3 implies that $(\mathbb{Q}_p, d_{\mathbb{Q}_p}, \mathcal{Q}_{\mathcal{S}})$ is a (p, s)-tiling space. By Corollary 5.2.3, we obtain the equality $\dim_A \mathbb{Q}_p = \log(p)/\log(s^{-1})$.

5.4 Counterexamples

In this section, we construct examples of (pre-)tiling space related to the doubling property, bi-Lipschitz maps, and similarity classes of tiles.

5.4.1 A non-doubling tiling space

We first provide a tiling space that is not doubling.

Example 5.4.1 ([56])**.** Let $N \in \mathbb{N}_{\geq 2}$, and let $s \in (0, 1)$. We denote by T the set of all sequences $x : \mathbb{N} \to \{0, \dots, N-1\}$ satisfying that $x_0 \in \{0, \dots, N-2\}$. The set T can be described as

$$T = \{0, \dots, N-2\} \times \prod_{n=1}^{\infty} \{0, \dots, N-1\}.$$

For $x, y \in T$, define a valuation $v : T \times T \to \mathbb{N} \cup \{\infty\}$ by

$$v(x, y) = \begin{cases} \min\{\, n \in \mathbb{N} \mid x_n \neq y_n \,\} & \text{if } x \neq y, \\ \infty & \text{if } x = y. \end{cases}$$

For each $i \in \mathbb{N}$, let T_i be a copy of T. Define a metric d_i on T_i by $d_i(x, y) = s^{v(x,y)+i}$.

Recall that, in this thesis, whenever we consider the disjoint union $\coprod_{i \in I} A_i$ of a family $\{A_i\}_{i \in I}$ of sets, we identify the family $\{A_i\}_{i \in I}$ with its disjoint copy. Under this identification, we may assume that $T_i \cap T_j = \emptyset$ for all distinct $i, j \in \mathbb{N}$.

For each $i \in \mathbb{N}$, the symbol o_i stands for the sequence whose all entries are 0 in T_i. For each $k \in \mathbb{N}$, put $O_k = \coprod_{i=k} \{o_i\}$, and put

$$X(k) = \left(\coprod_{i \in \mathbb{N}_{\geq k}} T_i \right) / O(k).$$

Namely, $X(k)$ is the set constructed by identifying the zero sequences in the direct sum set $\coprod_{i \in \mathbb{N}_{\geq k}} T_i$. Set $X = X(0)$. We may consider that $X(k+1) \subset X(k)$ and $T_k \subset X$ for all $k \in \mathbb{N}$. The symbol o stands for the zero sequence in X, which is also the identified point in X.

We define a function $d_X : X^2 \to [0, \infty)$ by

$$d_X(x, y) = \begin{cases} d_i(x, y) & \text{if } x, y \in T_i \text{ for some } i, \\ d_i(x, o) + d_j(o, y) & \text{if } x \in T_i \text{ and } y \in T_j \text{ for some } i \neq j. \end{cases}$$

The function d_X is a metric on X (see Proposition 3.2.1).

We next define a tiling structure on X. Let W be the set of all words whose 0-th entry is in $\{0, \dots, N-2\}$ and other entries are in $\{0, \dots, N-1\}$. Remark that the set W does not contain the empty word. For each word $w = w_0 \cdots w_l \in W$, we define

$$(T_i)_w = \{\, x \in T_i \mid x_0 = w_0, \dots, x_l = w_l \,\},$$

where $w_0 \in \{0, \dots, N-2\}$ and $w_i \in \{0, \dots, N-1\}$ $(i \geq 1)$. For each $k \in \mathbb{N}$ and for each $l \in \mathbb{N}_{\geq 1}$, we define a family $\mathcal{S}_{k,l}$ by

$$\mathcal{S}_{k,l} = \{\, (T_k)_w \mid w \in W \text{ and } |w| = l \,\}.$$

We define a map $\mathcal{P} : \mathbb{N} \to \mathrm{cov}(X)$ by

$$\mathcal{P}_n = \{X(n)\} \cup \bigcup_{k+l=n} \mathcal{S}_{k,l}.$$

We now show that $(X, d_X, \mathcal{P})$ is an N-tiling set. By the definition of $\mathcal{P}$, the condition (S3) is satisfied. For each $w \in W$, we have $(T_i)_w = \bigcup_{v=0}^{N-1} (T_i)_{wv}$. For each $n \in \mathbb{N}$, we have

$$X(n) = X(n+1) \cup \bigcup_{v=0}^{N-2} (T_n)_v.$$

Thus, the conditions (S1) is satisfied. By the boundedness of X, the condition (S2) is satisfied. Thus, the pair $(X, \mathcal{P})$ is an N-tiling set.

We next show that $(X, d_X, \mathcal{P})$ is an (N, s)-tiling space. By the definition of the metric d_X, for each $n \in \mathbb{N}$, and for all $k \in \mathbb{N}$ and $l \in \mathbb{N}_{\geq 1}$ with $k + l = n$, for each $(T_i)_w \in \mathcal{S}_{k,l}$, we have $\delta((T_i)_w) = s^{-n}$ and $\delta(X(n)) = s^{-n} + s^{-n-1}$. Then by $s^{-n} \leq s^{-n} + s^{-n-1} \leq 2s^{-n}$, the condition (T1) is satisfied. By the definition of $\{T_i\}_{i \in \mathbb{N}}$, for every $a \in T_i$ we have $(T_i)_w = B(a, s^{-n})$. For every $a \in (T_n)_1$ we also have

$$B(a, s^{-n-1}) \subset T_n \subset X(n).$$

Then the condition (T2) is satisfied. For each $n \in \mathbb{N}$, the spaces $sX(n)$ and $X(n+1)$ are isometric to each other. For all $i, j \in \mathbb{N}$ and for all $u, v \in W$, the spaces $(T_i)_u$ and $(T_j)_v$ are similar. Thus the tiles of $(X, d_X, \mathcal{P})$ have only two similarity classes, and hence the condition (U) is satisfied. Therefore $(X, d_X, \mathcal{P})$ is an (N, s)-tiling space.

For each $n \in \mathbb{N}$, we have the equality $\mathrm{card}(\{\, A \in \mathcal{P}_n \mid o \in A \,\}) = (n-1)(N-1) + N$. By Proposition 5.2.1, and by $(n-1)(N-1) + N \to \infty$ as $n \to \infty$, we conclude that (X, d_X) is not doubling.

Remark 5.4.1. Due to Brouwer's topological characterization of the Cantor set (see [13], or [66, Theorem 7.4]), the space X constructed in Example 5.4.1 is homeomorphic to the middle-third Cantor set. Indeed, the space X is topologically 0-dimensional and compact, and it has no isolated points.

Remark 5.4.2. In Example 5.4.1, by replacing the role of $\mathbb{N}$ with that of $\mathbb{Z}$, we also obtain an unbounded non-doubling (N, s)-tiling space that is not locally compact. Therefore being a tiling space does not imply the local compactness.

Remark 5.4.3. The construction of Example 5.4.1 is inspired by constructions of hedgehog spaces (see, for example, [117] or [110]).

5.4.2 A bi-Lipschitz image of a tiling space

We next construct a pre-tiling space that is not a tiling space. The space constructed below is also a bi-Lipschitz image of a tiling space.

Example 5.4.2 ([56]). Let $2^{\mathbb{N}}$ be the set of all sequences valued in $\{0, 1\}$. For $x, y \in 2^{\mathbb{N}}$, define a valuation $v : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{N} \cup \{\infty\}$ by

$$v(x, y) = \begin{cases} \min\{\, n \in \mathbb{N} \mid x_n \neq y_n \,\} & \text{if } x \neq y, \\ \infty & \text{if } x = y. \end{cases}$$

For $n \in \mathbb{N}$, set $a(n) = (1 - 1/(n + 3))2^{-(n+3)}$. Let d_X be a metric on $2^{\mathbb{N}}$ defined by $d_X(x, y) = a(v(x, y))$. Let d_Y be a metric on $2^{\mathbb{N}}$ defined by $d_Y(x, y) = 2^{-v(x,y)}$. Then the two metrics d_X and d_Y are ultrametrics. Define two maps $\mathcal{P}, \mathcal{Q} : \mathbb{N} \to \mathrm{cov}(2^{\mathbb{N}})$ by $\mathcal{P}_n = \{B(x, a(n)) \mid x \in X\}$ and $\mathcal{Q}_n = \{B(x, 2^{-n}) \mid x \in Y\}$. Then $(2^{\mathbb{N}}, d_X, \mathcal{P})$ and $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$ are 2-tiling sets.

We now prove that $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$ is a $(2, 2^{-1})$-tiling space. For each $i \in \{0, 1\}$, we define a $(1/2)$-similar map $f_i : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$(f_i(x))_j = \begin{cases} i & \text{if } j = 0, \\ x_{j-1} & \text{if } j \geq 1, \end{cases}$$

where $(f_i(x))_j$ is the j-th entry of $f_i(x)$. Then $\{f_0, f_1\}$ is a $(2, 2^{-1})$-similar iterated function system on $(2^{\mathbb{N}}, d_Y)$, and $(2^{\mathbb{N}}, d_Y)$ is the attractor of $\{f_0, f_1\}$. Since $(2^{\mathbb{N}}, d_Y)$ is the whole space, the iterated function system $\{f_0, f_1\}$ satisfies the strong open set condition. The map $\mathcal{P}_{\{f_0, f_1\}} : \mathbb{N} \to \mathrm{cov}(2^{\mathbb{N}})$ coincides with $\mathcal{Q}$. Thus Theorem 1.2.8 implies that the triple $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$ is a $(2, 2^{-1})$-tiling space.

We now prove the following sublemma:

Lemma 5.4.1. *The tiles of* $(2^{\mathbb{N}}, d_X, \mathcal{P})$ *have infinitely many similarity classes.*

Proof. Since d_X is an ultrametric, we have $\delta(B(0, a(k))) = a(k)$ for all $k \in \mathbb{N}$. Thus it suffices to show that for all $n, m \in \mathbb{N}$, if the metric spaces $((B(0, a(n))), (a(n))^{-1}d_X)$ and $((B(0, a(m))), (a(m))^{-1}d_X)$ are isometric to each other, then $n = m$.

For each $k \in \mathbb{N}$, we put $S_k = \{\, (a(k))^{-1}d(x, y) \mid x, y \in B(0, a(k)) \,\}$. Assume that $((B(0, a(n))), (a(n))^{-1}d_X)$ and $((B(0, a(m))), (a(m))^{-1}d_X)$ are isometric. Then the sets S_n and S_m coincide with each other. Real numbers $a(n + 1)/a(n)$ and $a(m + 1)/a(m)$ are the maximals of the sets $S_n \setminus \{1\}$ and $S_m \setminus \{1\}$, respectively. Thus, we find that $a(n + 1)/a(n) = a(m + 1)/a(m)$. For $k \in \mathbb{N}$ we have

$$\frac{a(k + 1)}{a(k)} = \frac{1}{2} \frac{(k + 2)(k + 4)}{(k + 3)^2},$$

Then the equality $a(n + 1)/a(n) = a(m + 1)/a(m)$ implies

$$\frac{(n + 2)(n + 4)}{(n + 3)^2} = \frac{(m + 2)(m + 4)}{(m + 3)^2},$$

and hence $(m + 3)^3(n + 2)(n + 4) = (n + 3)^2(m + 2)(m + 4)$. Note that if $a, b \in \mathbb{Z}$ satisfy $a - b = 1$, then a and b are coprime. Thus we see that $(m + 3)^2$ divides $(n + 3)^2$, and $(n + 3)^2$ divides $(m + 3)^2$, which implies $(n + 3)^2 = (m + 3)^2$. Therefore $n = m$. This finishes the proof of the lemma. □

Comparing the sets of values of the metrics, we see that the similarity classes of the tiles of $(2^{\mathbb{N}}, d_X, \mathcal{P})$ do not contain that of $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$.

The identity map $1_{2^{\mathbb{N}}} : (2^{\mathbb{N}}, d_X) \to (2^{\mathbb{N}}, d_Y)$ is bi-Lipschitz, in particular, the metric space $(2^{\mathbb{N}}, d_X)$ is a bi-Lipschitz image of $(2^{\mathbb{N}}, d_Y)$. Since $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$ is a $(2, 2^{-1})$-pre-tiling space, by Corollary 5.1.12, so is $(2^{\mathbb{N}}, d_X, \mathcal{P})$.

Take a sequence $\{A_i\}_{i\in\mathbb{N}}$ of tiles of $(2^{\mathbb{N}}, d_X, \mathcal{P})$ with $A_i \in \mathcal{P}_i$. Since for all $N \in \mathbb{N}$ and for all $n \in \mathbb{N}$ we have $|a(n+N)/a(N) - 2^{-n}| < 1/(N+2)$, the sequence $(A_N, (\delta(A_N))^{-1}d_X)$ converges to the space $(2^{\mathbb{N}}, d_Y)$ in the of Gromov–Hausdorff topology. Thus $(2^{\mathbb{N}}, d_X, \mathcal{P})$ is a pre-tiling space which does not satisfy the condition (U).

In summary, the pre-tiling space $(2^{\mathbb{N}}, d_X, \mathcal{P})$ is a non-tiling space which is a bi-Lipschitz image of the tiling space $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$.

5.4.3 A tiling space with infinitely many similarity classes of tiles

Combining the metric spaces provided in Example 5.4.2, we can construct a tiling space whose tiles have infinitely many similarity classes.

Example 5.4.3 ([56])**.** Let $(2^{\mathbb{N}}, d_X, \mathcal{P})$ and $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$ be the pre-tiling space and the tiling space constructed in Example 5.4.2, respectively. Put $Z = 2^{\mathbb{N}} \sqcup 2^{\mathbb{N}}$ and define a metric d_Z on Z by

$$d_Z(x,y) = \begin{cases} d_X(x,y) & \text{if } x, y \in X, \\ d_Y(x,y) & \text{if } x, y \in Y, \\ 2 & \text{if } x \text{ and } y \text{ lie in distinct components.} \end{cases}$$

Define a map $\mathcal{R} : \mathbb{N} \to \mathrm{cov}(X \sqcup Y)$ by $\mathcal{R}_0 = \{Z\}$ for $n = 0$, and by $\mathcal{R}_n = \mathcal{P}_{n-1} \cup \mathcal{Q}_{n-1}$ for $n \in \mathbb{N}_{\geq 1}$. Since $(2^{\mathbb{N}}, d_X, \mathcal{P})$ and $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$ are $(2, 2^{-1})$-pre-tiling spaces, $(Z, d_Z, \mathcal{R})$ is a $(2, 2^{-1})$-pre-tiling space.

We now prove that $(Z, d_Z, \mathcal{R})$ satisfies the condition (U). Take an arbitrary sequence $\{A_n\}_{n\in\mathbb{N}}$ of tiles of $(Z, d_Z, \mathcal{R})$. Then there exists a subsequence $\{A_{\phi(i)}\}_{i\in\mathbb{N}}$ of $\{A_n\}_{n\in\mathbb{N}}$ consisting of either tiles of $(2^{\mathbb{N}}, d_X, \mathcal{P})$ or that of $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$. If $\{A_{\phi(i)}\}_{i\in\mathbb{N}}$ consists of tiles of $(2^{\mathbb{N}}, d_X, \mathcal{Q})$, then by the argument in Example 5.4.2, there exists a subsequence $\{A_{\psi(i)}\}_{i\in\mathbb{N}}$ of $\{A_{\phi(i)}\}_{i\in\mathbb{N}}$ satisfying that $(A_{\psi(i)}, \delta(A_{\psi(i)})^{-1}d_X)$ converges to either $(2^{\mathbb{N}}, d_Y)$ or $(T, \delta(T)^{-1}d_X)$ for some tile T of $(2^{\mathbb{N}}, d_X, \mathcal{P})$. In the case where $\{A_{\phi(i)}\}_{i\in\mathbb{N}}$ consists of tiles of $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$, since $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$ satisfies the condition (U), there exists a subsequence $\{A_{\psi(i)}\}_{i\in\mathbb{N}}$ of $\{A_{\phi(i)}\}_{i\in\mathbb{N}}$ satisfying that $(A_{\psi(i)}, \delta(A_{\psi(i)})^{-1}d_Z)$ converges to $(T, \delta(T)^{-1}d_Y)$ for some tile T of $(2^{\mathbb{N}}, d_Y, \mathcal{Q})$. Therefore $(Z, d_Z, \mathcal{R})$ satisfies the condition (U).

As a summary, we obtain a $(2, 2^{-1})$-tiling space $(Z, d_Z, \mathcal{R})$ whose tiles have infinitely many similarity classes.

Chapter 6

Spaces of metrics

In this chapter, we first prove Theorem 1.2.9. In Section 6.2, we next discuss the basic properties of transmissible properties, and we prove Theorems 1.2.10 and 1.2.11 as applications of Theorem 1.2.9.

6.1 An interpolation theorem of metrics

In this section, we prove Theorem 1.2.9.

6.1.1 Proof of Theorem 1.2.9

Before proving Theorem 1.2.9, recall that a family of subsets of a topological space is said to be discrete if every point in the space has a neighborhood intersecting at most a single member of the family. By this definition, we obtain:

Proposition 6.1.1. *For a discrete family of closed subsets of a topological space, its union is closed.*

Proof of Theorem 1.2.9. Let X be a metrizable space, and let $\{A_i\}_{i\in I}$ be a discrete family of closed subsets of X. Take a metric $d \in \mathrm{M}(X)$, and a family $\{e_i\}_{i\in I}$ such that $e_i \in \mathrm{M}(A_i)$.

In the case of $\sup_{i\in I} \mathcal{D}_{A_i}(e_i, d|_{A_i^2}) = \infty$, by Lemma 3.2.4, we obtain a metric k in $\mathrm{M}(\coprod_{i\in I} A_i)$ such that for every $i \in I$ we have $k|_{A_i^2} = e_i$. Since k generates the same topology as $\coprod_{i\in I} A_i$, and $\coprod_{i\in I} A_i$ is closed in X (see Proposition 6.1.1), we can apply Theorem 2.2.3 to the metric k, and hence there exists a metric $r \in \mathrm{M}(X)$ such that we have $r|_{A_i^2} = e_i$ for all $i \in I$. Then, Theorem 1.2.9 is proven in this case.

We next deal with the case of $\sup_{i\in I} \mathcal{D}_{A_i}(e_i, d|_{A_i^2}) < \infty$. Put $\eta = \sup_{i\in I} \mathcal{D}_{A_i}(e_i, d|_{A_i^2})$. Let $\{B_i\}_{i\in I}$ and $\tau : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ be the same family and map as in Lemma 3.2.5, respectively. Put $Z = X \sqcup \coprod_{i\in I} B_i$. By Lemma 3.2.5, we find a metric h on Z such that

(1) for every $i \in I$ we have $h|_{B_i^2} = (\tau_i^{-1})^* e_i$;

(2) $h|_{X^2} = d$;

(3) for every $x \in \coprod_{i\in I} A_i$ we have $h(x, \tau(x)) = \eta/2$.

We can take an isometric embedding $H : Z \to Y$ from (Z, h) into a Banach space $(Y, \| * \|_Y)$ (see Subsection 2.2.1). Define a map $\phi : Z \to \mathcal{CC}(Y)$ by $\phi(x) = B(H(x), \eta/2)$. Corollary 2.7.7 implies that the map ϕ is lower semi-continuous.

Define a map $f : \bigcup_{i\in I} A_i :\to Y$ by $f(x) = H(\tau(x))$. Then the map f is continuous. By the property (3) of h, for every $x \in \bigcup_{i\in I} A_i$ we have $f(x) \in \phi(x)$. Due to the Stone theorem 2.7.1 and Proposition 6.1.1, the space X is paracompact and the set $\bigcup_{i l I} A_i$ is closed in X. Thus we can apply the Michael selection theorem 2.7.3 to the map f, and hence we obtain a continuous map $F : X \to Y$ such that $F|_{\bigcup_{i\in I} A_i} = f$ and for every $x \in \bigcup_{i\in I} A$ we have $F(x) \in \phi(x)$. Note that $F(x) \in \phi(x)$ means that

$$\|F(x) - H(x)\|_Y \leq \eta/2.$$

Let $r \in \mathrm{M}(X)$ be the metric constructed in the case of $\eta = \infty$. Put $l = \min\{r, \eta/2\}$. Note that $l \in \mathrm{M}(X)$. We consider that the product metric space $Y \times X$ is equipped with the metric $d_Y \times_\infty l$, where d_Y is the metric induced by $\| * \|_Y$.

Define a map $E : X \to Y \times X$ by

$$E(x) = (F(x), x).$$

Since the second component of E is the identity map, the map E is a topological embedding. Take a fixed base point $o \in X$. We also define a map $K : X \to Y \times X$ by

$$K(x) = (H(x), o).$$

Then, by the definition of $d_Y \times_\infty l$, the map K from (X, d) to $(Y \times X, d_Y \times_\infty l)$ is isometric. Since for every $x \in X$ we have $\|F(x) - H(x)\|_Y \leq \eta/2$ and $\delta_l(X) \leq \eta/2$, we obtain

$$\|E(x) - K(x)\| = \|F(x) - H(x)\|_Y \vee l(x, o) \leq \eta/2.$$

Define a function $m : X^2 \to [0, \infty)$ by $m(x, y) = \|E(x) - E(y)\|$, then m is a metric on X. Since E is a topological embedding, we see that $m \in \mathrm{M}(X)$. For every $i \in I$, and for all $x, y \in A_i$, we have $\|F(x) - F(y)\|_Y = e_i(x, y)$ and $l(x, y) \leq r(x, y) = e_i(x, y)$. Thus,

$$\|E(x) - E(y)\| = \|F(x) - F(y)\|_Y \vee l(x, y) = e_i(x, y),$$

and hence $m|_{A_i^2} = e_i$. By the definition of $\mathcal{D}_X$, we have $\eta \leq \mathcal{D}_X(m, d)$. We also obtain the opposite inequality $\mathcal{D}_X(m, d) \leq \eta$; indeed, for all $x, y \in X$,

$$\begin{aligned} |m(x, y) - d(x, y)| &= \Big| \|E(x) - E(y)\| - \|K(x) - K(y)\| \Big| \\ &\leq \|E(y) - K(y)\| + \|E(x) - K(x)\| \leq \eta/2 + \eta/2 = \eta. \end{aligned}$$

Therefore we conclude that $\mathcal{D}_X(m, d) = \eta$. This proves the former part of Theorem 1.2.9.

By the latter part of Theorem 2.2.3, the metric l can be chosen as a complete one. Then m becomes a complete metric. This leads to the latter part of Theorem 1.2.9. □

In Theorem 1.2.9, by letting I be a singleton, we obtain the following:

Corollary 6.1.2. *Let X be a metrizable space, and let A be a closed subset of X. Then for every $d \in \mathrm{M}(X)$, and for every $e \in \mathrm{M}(A)$, there exists a metric $m \in \mathrm{M}(X)$ satisfying the following:*

(1) $m|_{A^2} = e$;

(2) $\mathcal{D}_X(m, d) = \mathcal{D}_A(e, d|_{A^2})$.

Moreover, if X is completely metrizable, and if $e \in \mathrm{M}(A)$ is a complete metric, then the metric $m \in \mathrm{M}(X)$ can be chosen as a complete one.

6.2 Transmissible properties

In this section we discuss transmissible properties, and prove Theorem 1.2.10. We also show that various properties in metric geometry are transmissible properties.

6.2.1 Proof of Theorem 1.2.10

By the condition (TP2) in Definition 1.2.5, we obtain the following:

Lemma 6.2.1. *Let $\mathfrak{G}$ be a transmissible parameter. If a metric space (X, d) satisfies the $\mathfrak{G}$-transmissible property, then so does every metric subspace of (X, d).*

By the virtue of Lemma 6.2.1, we use the word "transmissible".

Corollary 6.2.2. *Let $\mathfrak{G}$ be a transmissible parameter. Let (X, d) be a metric space. If there exists a subspace of (X, d) with the anti-$\mathfrak{G}$-transmissible property, then so does (X, d).*

Let X be a metrizable space, and let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$ be a transmissible parameter. For $q \in Q$, for $a \in \operatorname{Seq}(G(q), X)$ and for $z \in Z$, we denote by $S(X, \mathfrak{G}, q, a, z)$ the set of all $d \in \mathrm{M}(X)$ such that $\phi^{q,X}(a, z, d) \in P \setminus F(q)$. We also denote by $S(X, \mathfrak{G})$ the set of all $d \in \mathrm{M}(X)$ such that (X, d) satisfies the anti-$\mathfrak{G}$-transmissible property.

Proposition 6.2.3. *Let X be a metrizable space. Let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$ be a transmissible parameter. Then for all $q \in Q$, $a \in \operatorname{Seq}(G(q), X)$ and $z \in Z$, the set $S(X, \mathfrak{G}, q, a, z)$ is open in $\mathrm{M}(X)$.*

Proof. Fix $q \in Q$, $a \in \operatorname{Seq}(G(q), X)$ and $z \in Z$. We see that

$$S(X, \mathfrak{G}, q, a, z) = \left(\phi^{q,X}(a, z)\right)^{-1}(P \setminus F(q)).$$

Since $P \setminus F(q)$ is open in P, and since the map $\phi^{q,X}(a, z) : \mathrm{M}(X) \to P$ is continuous, the set $S(X, \mathfrak{G}, q, a, z)$ is open in $\mathrm{M}(X)$. □

Corollary 6.2.4 ([59])**.** *Let X be a metrizable space, and let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$ be a transmissible parameter. Then the set $S(X, \mathfrak{G})$ is G_δ in $\mathrm{M}(X)$. Moreover, if the set Q is finite, then $S(X, \mathfrak{G})$ is open in $\mathrm{M}(X)$.*

Proof. By the definitions of $S(X, \mathfrak{G})$ and $S(X, \mathfrak{G}, q, a, z)$, we have

$$S(X, \mathfrak{G}) = \bigcap_{q \in Q} \bigcup_{a \in \operatorname{Seq}(G(q), X)} \bigcup_{z \in Z} S(X, \mathfrak{G}, q, a, z).$$

This equality together with Proposition 6.2.3 proves the lemma. □

We say that a topological space is an $(\omega_0 + 1)$*-space* if it is homeomorphic to the one-point compactification of the countable discrete topological space.

Lemma 6.2.5 ([59])**.** *Let $\mathfrak{G}$ be a transmissible parameter. Then $\mathfrak{G}$ is singular if and only if there exists a metric $(\omega_0 + 1)$-space with arbitrary small diameter satisfying the anti-$\mathfrak{G}$-transmissible property.*

Proof. Let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$. First assume that there exists a metric $(\omega_0 + 1)$-space with arbitrary small diameter satisfying the anti-$\mathfrak{G}$-transmissible property. By the definition of the anti-$\mathfrak{G}$-transmissible property, we see that $\mathfrak{G}$ is singular.

We next assume that $\mathfrak{G}$ is singular. Fix $\epsilon \in (0, \infty)$. Take a surjective map $\theta : \mathbb{N} \to Q$. By the singularity of $\mathfrak{G}$, there exists a sequence $\{(R_i, d_i)\}_{i \in \mathbb{N}}$ of finite metric spaces such that for each $i \in \mathbb{N}$ there exists $z_i \in Z$ satisfying

(1) $\delta_{d_i}(R_i) \le \epsilon \cdot 2^{-i}$;

(2) $\mathrm{card}(R_i) \in G(\theta(i))$;

(3) $\phi^{\theta(i),R_i}\left(\{r_{i,j}\}_{j=1}^{\mathrm{card}(R_i)}, z_i, d_i\right) \in P \setminus F(\theta(i))$.

In order to construct a desired space, we use the telescope construction discussed in Section 3.1. Let $\mathbf{A} = \left(\mathbb{N} \cup \{\infty\}, D, 1_{\mathbb{N}\cup\{\infty\}}\right)$ be a telescope base defined in Definition 3.1.2. Put $\mathbf{R} = \{(R_i, d_i)\}_{i\in\mathbb{N}}$, and put $\mathbf{B} = \left(\mathbb{N} \cup \{\infty\}, \epsilon \cdot D, 1_{\mathbb{N}\cup\{\infty\}}\right)$. Then $\mathcal{L} = (\mathbf{R}, \mathbf{B})$ is a compatible pair. By the properties (2) and (3) of $\{(R_i, d_i)\}_{i\in\mathbb{N}}$, the telescope metric space $(T(\mathcal{L}), d_{T(\mathcal{L})})$ of $\mathcal{L}$ satisfies the anti-$\mathfrak{G}$-transmissible property. By Lemma 3.1.3, $(T(\mathcal{L}), d_{T(\mathcal{L})})$ is an $(\omega_0 + 1)$-space. □

Let $\mathfrak{G}$ be a transmissible parameter. For a non-discrete metrizable space X, and for an $(\omega_0 + 1)$-subspace R of X, we denote by $T(X, R, \mathfrak{G})$ the set of all $d \in \mathrm{M}(X)$ for which $(R, d|_{R^2})$ satisfies the anti-$\mathfrak{G}$-transmissible property.

As a consequence of Corollary 6.1.2, we obtain the following:

Proposition 6.2.6. *Let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$ be a singular transmissible parameter. Then for every non-discrete metrizable space X, and for every $(\omega_0 + 1)$-subspace R of X, the set $T(X, R, \mathfrak{G})$ is dense in $\mathrm{M}(X)$.*

Proof. Fix $d \in \mathrm{M}(X)$ and $\epsilon \in (0, \infty)$. From the singularity of $\mathfrak{G}$, by Lemma 6.2.5, it follows that there exists an $(\omega_0 + 1)$-metric space (L, e) satisfying the anti-$\mathfrak{G}$-transmissible property and $\delta_e(L) < \epsilon/2$. Since R is an $(\omega_0 + 1)$-space, there exists an $(\omega_0 + 1)$ subspace S of R with $\delta_d(S) < \epsilon/2$. Let $\tau : S \to L$ be a homeomorphism. By the definitions of S and e, we have

$$\mathcal{D}_S(d|_{S^2}, \tau^*e) = \sup_{x,y\in S} |d(x, y) - \tau^*e(x, y)| \le \delta_d(S) + \delta_{\tau^*e}(S) \le \epsilon.$$

By Corollary 6.1.2, we obtain a metric $m \in \mathrm{M}(X)$ such that $m|_{S^2} = \tau^*e$ and $\mathcal{D}_X(m, d) \le \epsilon$. By Corollary 6.2.2, the metric space (X, m) satisfies the anti-$\mathfrak{G}$-transmissible property. Since d and ϵ are arbitrary, we conclude that $T(X, R, \mathfrak{G})$ is dense in $\mathrm{M}(X)$. □

Proof of Theorem 1.2.10. Let X be a non-discrete metrizable space, and let $\mathfrak{G}$ be a singular transmissible parameter. Since X is non-discrete, there exists an $(\omega_0 + 1)$-subspace R of X. By the definitions, we have $T(X, R, \mathfrak{G}) \subset S(X, \mathfrak{G})$. By Proposition 6.2.6 and Corollary 6.2.4, the set $S(X, \mathfrak{G})$ is dense G_δ in $\mathrm{M}(X)$. This finishes the proof. □

For a complete metrizable space X, we denote by $\mathrm{CM}(X)$ the set of all complete metrics in $\mathrm{M}(X)$. From the latter part of Corollary 6.1.2, we deduce the following:

Theorem 6.2.7 ([59])**.** *Let $\mathfrak{G}$ be a singular transmissible parameter. For every non-discrete completely metrizable space X, the set of all $d \in \mathrm{CM}(X)$ for which (X, d) satisfies the anti-$\mathfrak{G}$-transmissible property is dense G_δ in $\mathrm{CM}(X)$.*

6.2.2 Proof of Theorem 1.2.11

In this subsection, we prove Theorem 1.2.11.

Proof of Theorem 1.2.11. Let X be a second countable, locally compact locally non-discrete space, and let $\mathfrak{G}$ be a singular transmissible parameter. Let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$. Let S be the set of all metrics $d \in \mathrm{M}(X)$ for which (X, d) satisfies the local anti-$\mathfrak{G}$-transmissible property. Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable open base of X, and let $\{R_i\}_{i\in\mathbb{N}}$ be a family of $(\omega_0 + 1)$-subspaces of X with $R_i \subset U_i$. Since $\{U_i\}_{i\in\mathbb{N}}$ is an open base of X, Lemma 6.2.1 implies that

$$S = \bigcap_{i\in\mathbb{N}} \bigcap_{q\in Q} \bigcup_{z\in Z} \bigcup_{a\in\mathrm{Seq}(G(q),U_i)} S(X, \mathfrak{G}, q, a, z).$$

Corollary 6.2.4 implies that S is G_δ in $\mathrm{M}(X)$. For each $i \in \mathbb{N}$, the set

$$\bigcap_{q\in Q} \bigcup_{z\in Z} \bigcup_{a\in\mathrm{Seq}(G(q),U_i)} S(X, \mathfrak{G}, q, a, z)$$

contains $T(X, R_i, \mathfrak{G})$. Proposition 6.2.6 implies that each $T(X, R_i, \mathfrak{G})$ is dense in $\mathrm{M}(X)$. By the former part of Lemma 3.4.7, the space $\mathrm{M}(X)$ is a Baire space, and hence S is dense G_δ in $\mathrm{M}(X)$ (see Lemma 2.2.9). This completes the proof. □

6.3 Examples of transmissible properties

6.3.1 The doubling property

By the definitions of the topology of $\mathrm{M}(X)$, α_d and δ_d, we obtain:

Lemma 6.3.1. *Let X be a metrizable space. Fix a finite subset A of X. Then maps $B, D : \mathrm{M}(X) \to \mathbb{R}$ defined by $B(d) = \alpha_d(A)$ and $D(d) = \delta_d(A)$ are continuous.*

Proposition 6.3.2 ([59])**.** *The doubling property on metric spaces is a transmissible property with a singular transmissible parameter.*

Proof. Define a map $D : (\mathbb{Q}_{>0})^2 \to \mathcal{F}((\mathbb{R}_{>0})^2)$ by

$$D((q_1, q_2)) = \{\, (x, y) \in (\mathbb{R}_{>0})^2 \mid x \leq q_1 y^{q_2} \,\},$$

and define a constant map $G_D : (\mathbb{Q}_{>0})^2 \to \mathcal{P}(\mathbb{N})^*$ by $G_D(q) = [2, \infty)$. Put $Z_D = \{1\}$. For each metrizable space X, and for each $q \in (\mathbb{Q}_{>0})^2$, define a map

$$\phi_D^{q,X} : \mathrm{Seq}(G(q), X) \times Z_D \times \mathrm{M}(X) \to \mathbb{R}$$

by

$$\phi_D^{q,X}(\{a_i\}_{i=1}^N, 1, d) = \left(N, \frac{\delta_d(\{\, a_i \mid i \in \{1, \dots, N\} \,\})}{\alpha_d(\{\, a_i \mid i \in \{1, \dots, N\} \,\})} \right).$$

Let $\mathfrak{DB} = ((\mathbb{Q}_{>0})^2, (\mathbb{R}_{>0})^2, D, G_D, \{1\}, \phi_D)$. Then $\mathfrak{DB}$ satisfies the condition(TP2) in Definition 1.2.5. By Lemma 6.3.1, we see that $\mathfrak{DB}$ satisfies the condition (TP1). Hence $\mathfrak{DB}$ is a transmissible parameter. The $\mathfrak{DB}$-transmissible property is equivalent to the doubling property. We next prove that $\mathfrak{DB}$ is singular. For $q = (q_1, q_2) \in (\mathbb{Q}_{>0})^2$ and for $\epsilon \in (0, \infty)$, we denote by (R_q, d_q) a finite metric space with $\mathrm{card}(R_q) > q_1 + 1$ on which all distances of distinct two points are equal to ϵ. Then $\delta_{d_q}(R_q) = \epsilon$, and

$$\phi^{q,R_q}(R_q, 1, d_q) = (\mathrm{card}(R_q), 1) \notin D(q).$$

We next show that the $\mathfrak{DB}$-transmissible property is equivalent to the doubling property. This equivalence follows from the fact that a metric space (X, d) satisfies the $\mathfrak{DB}$-transmissible property if and only if there exists $q = (q_1, q_2) \in (\mathbb{Q}_{>0})^2$ such that for every $\{a_i\}_{i=1}^N \in \mathrm{Seq}([2, \infty), X)$ we have

$$N \leq q_1 \left(\frac{\delta_d(\{\, a_i \mid i \in \{1, \ldots, N\} \,\})}{\alpha_d(\{\, a_i \mid i \in \{1, \ldots, N\} \,\})} \right)^{q_2}.$$

This finishes the proof. □

6.3.2 Uniform disconnectedness

A metric space (X, d) is said to be *uniformly disconnected* if there exists $\delta \in (0, 1)$ such that if a finite sequence $\{z_i\}_{i=1}^N$ in X satisfies $\max_{1 \leq i \leq N-1} d(z_i, z_{i+1}) < \delta d(z_1, z_N)$, then we have $z_1 = z_2 = \cdots = z_N$. Note that a metric space is uniformly disconnected if and only if it is bi-Lipschitz to an ultrametric space (see e.g., [76, Lemma 5.1.10]).

By the definition of the topology of $\mathrm{M}(X)$, we obtain:

Lemma 6.3.3. *Let X be a metrizable space. Fix two points a, b in X. Then a map $f : \mathrm{M}(X) \to \mathbb{R}$ defined by $f(d) = d(a, b)$ is continuous.*

Proposition 6.3.4 ([59])**.** *The uniform disconnectedness on metic spaces is a transmissible property with a singular parameter.*

Proof. Define a map $U : \mathbb{Q} \cap (0, 1) \to \mathcal{F}((\mathbb{R}_{\geq 0})^2)$ by

$$U(q) = \{\, (x, y) \in (\mathbb{R}_{\geq 0})^2 \mid x \geq qy \,\},$$

and define a constant map $G_U : \mathbb{Q} \cap (0, 1) \to \mathcal{P}^*(\mathbb{N})$ by $G_U(q) = [2, \infty)$. Put $Z_U = \{1\}$. For each metrizable space X, and for each $q \in \mathbb{Q} \cap (0, 1)$, define a map

$$\phi_U^{q,X} : \mathrm{Seq}(G_U(q), X) \times Z_U \times \mathrm{M}(X) \to (\mathbb{R}_{\geq 0})^2$$

by

$$\phi_{UD}^{q,X}(\{a_i\}_{i=1}^N, 1, d) = \left(\max_{1 \leq i \leq N-1} d(a_i, a_{i+1}), d(a_1, a_N) \right).$$

Let $\mathfrak{UD} = (\mathbb{Q} \cap (0, 1), (\mathbb{R}_{>0})^2, U, G_U, \{1\}, \phi_{UD})$. Then $\mathfrak{UD}$ satisfies the condition (TP2) in Definition 1.2.5. By Lemma 6.3.3, we see that $\mathfrak{UD}$ satisfies the condition (TP1). Hence $\mathfrak{UD}$ is a transmissible parameter, and the $\mathfrak{UD}$-transmissible property is equivalent to the uniform disconnectedness on metric spaces. We next prove that $\mathfrak{UD}$ is singular. For every $q \in \mathbb{Q} \cap (0, 1)$, take $n \in \mathbb{N}$ with $1/n < q$. Put

$$R_q = \{\, \epsilon \cdot i/n \mid i \in \mathbb{Z} \cap [0, n] \,\},$$

and let d_q be the relative metric on R_q induced from the Euclidean metric. Then we obtain $\delta_{d_q}(R_q) = \epsilon$, and

$$\phi_{UD}^{q,R_q}(\{a_i\}_{i=1}^N, 1, d_q) = (\epsilon/n, \epsilon) \notin U(q).$$

We next show that the $\mathfrak{UD}$-transmissible property is equivalent to uniform disconnectedness. This equivalence follows from the fact that a metric space (X, d) satisfies the $\mathfrak{DB}$-transmissible property if and only if there exists $q \in \mathbb{Q} \cap (0, 1)$ such that for every $\{a_i\}_{i=1}^N \in \mathrm{Seq}([2, \infty), X)$ we have

$$qd(a_1, a_N) \leq \max_{1 \leq i \leq N-1} d(a_i, a_{i+1}).$$

This finishes the proof. □

6.3.3 Necessity of an assumption of Theorem 1.2.10

Let D be a metric on $\mathbb{Z}$ on which all distances of two distinct points in $\mathbb{Z}$ are equal to 1. Let R be the relative Euclidean metric on $\mathbb{Z}$. Note that $E, D \in \mathrm{M}(\mathbb{Z})$. The following proposition tells us that the assumption on discreteness in Theorem 1.2.10 is necessary.

Proposition 6.3.5. *For the metric D and R on $\mathbb{Z}$ defined above, the following are satisfied:*

(1) *there exists a neighborhood U of R in $\mathrm{M}(\mathbb{Z})$ such that for every $d \in U$ the space $(\mathbb{Z}, d)$ is doubling.*

(2) *there exists a neighborhood U of D in $\mathrm{M}(\mathbb{Z})$ such that for every $d \in U$ the space $(\mathbb{Z}, d)$ is uniformly disconnected.*

Proof. We first prove the statement (1). Put $U = U(R, 1/2; \mathrm{M}(\mathbb{Z}), \mathcal{D}_{\mathbb{Z}})$. Take $d \in U$. Then for every finite subset A of $\mathbb{Z}$ with cardinality at least 2, we have $\delta_d(A) \le \delta_R(A) + 1/2$ and $\alpha_d(A) \ge \alpha_R(A) - 1/2$. Since $1 \le \alpha_R(A)$ and $1 \le \delta_R(A)$, we have $\delta_d(A) \le (3/2)\delta_R(A)$ and $\alpha_d(A) \ge (1/2)\alpha_R(A)$. Since R is the relative 1-dimensional Euclidean metric, we have

$$\mathrm{card}(A) \le 2 \cdot \left(\frac{\delta_R(A)}{\alpha_R(A)}\right),$$

and hence we obtain

$$\mathrm{card}(A) \le 6 \cdot \left(\frac{\delta_d(A)}{\alpha_d(A)}\right).$$

This implies that d is doubling.

We next prove the statement (2). Put $U = U(D, 1/2; \mathrm{M}(\mathbb{Z}), \mathcal{D}_{\mathbb{Z}})$. Take $d \in U$. Let $\{z_i\}_{i=1}^N$ be a sequence in $\mathbb{Z}$ not consisting of a single point. Since D is an ultrametric, it satisfies the uniform disconnectedness for $\delta = 1/2$. Then we have

$$\frac{1}{2} D(z_1, z_N) < \max_{1 \le i \le N-1} D(z_i, z_{i+1}).$$

Since $\mathcal{D}_{\mathbb{Z}}(d, D) \le 1/2$, we have

$$\frac{1}{2} d(z_1, z_N) - \frac{1}{4} < \max_{1 \le i \le N-1} d(z_i, z_{i+1}) + \frac{1}{2}.$$

Since $1/2 < \max_{1 \le i \le N-1} d(z_i, z_{i+1})$, we also have

$$\frac{2}{7} d(z_1, z_N) < \max_{1 \le i \le N-1} d(z_i, z_{i+1}).$$

Therefore we conclude that d is uniformly disconnected for $\delta = 2/7$. □

6.3.4 Rich pseudo-cones

Let $\mathcal{F}$ be the class of all finite metric spaces on which all distances are in rational numbers. We denote by $\mathcal{G}$ the quotient class of $\mathcal{F}$ divided by the isometric equivalence. Note that $\mathcal{G}$ is countable.

We say that a metric space (X, d) *has rich pseudo-cones* if $\mathcal{F}$ is contained in $\mathcal{PC}(X, d)$.

Proposition 6.3.6 ([59]). *The rich pseudo-cones property on metric spaces is an anti-transmissible property with a singular transmissible parameter.*

Proof. Let $\{(F_n, d_n)\}_{n\in\mathbb{N}}$ be a complete representation system of $\mathcal{G}$. Let $F_n = \{f_{n,l}\}_{l=1}^{\operatorname{card}(F_i)}$. Define a function $R : \mathbb{N}^2 \to \mathbb{R}$ by

$$R(n, m) = \{\, y \in \mathbb{R} \mid y \geq 2^{-m} \,\},$$

and define a map $G_R : \mathbb{N}^2 \to \mathcal{P}^*(\mathbb{N})$ by $G_R(n, m) = \{\operatorname{card}(F_n)\}$.

For each $k = (n, m) \in \mathbb{N}^2$, for each metrizable space X, for each finite sequence $\{a_i\}_{i=1}^M \in \operatorname{Seq}(G_R(k), X)$, and for all $i, j \in \{1, \dots, M\}$ we define a function

$$r_{i,j}^k(\{a_i\}_{i=1}^M) : (0, \infty) \times \mathrm{M}(X) \to \mathbb{R}$$

by

$$r_{i,j}^k(\{a_i\}_{i=1}^M)(z, d) = |z^{-1}d(a_i, a_j) - d_n(f_{n,i}, f_{n,j})|$$

if $i, j \in \{1, \dots, M\}$; otherwise, we define $r_{i,j}^k(\{a_i\}_{i=1}^M)(z, d) = 0$. By Lemma 6.3.3, the map $r_{i,j}^k(\{a_i\}_{i=1}^M)$ is continuous.

Define a map $\phi_R^{k,X} : \operatorname{Seq}(G(k), X) \times (0, \infty) \times \mathrm{M}(X) \to \mathbb{R}$ by

$$\phi_R^q(\{a_i\}_{i=1}^M, z, d) = \max_{i,j \in \{1,\dots,M\}} r_{i,j}^k(\{a_i\}_{i=1}^M)(z, d).$$

Let $\mathfrak{R} = (\mathbb{N}^2, \mathbb{R}, R, G_R, (0, \infty), \phi_r)$. Then $\mathfrak{R}$ satisfies the conditions (TP1) and (TP2) in Definition 1.2.5, and hence it is a transmissible parameter.

For a metric space (X, d), the anti-$\mathfrak{R}$-transmissible property means that for every $n \in \mathbb{N}$, and for every $m \in \mathbb{N}$, there exist a finite subspace $A = \{a_i\}_{i=1}^{\operatorname{card}(F_n)}$ of X and a positive number $z \in (0, \infty)$ such that for all $i, j \in \{1, \dots, \operatorname{card}(F_n)\}$ we have

$$|z^{-1}d(a_i, a_j) - d_n(f_{n,i}, f_{n,j})| < 2^{-m};$$

in particular, $\mathcal{GH}((A, z^{-1}d|_{A^2}), (F_n, d_n)) < 2^{-(m+1)}$. Thus $\mathcal{F}$ is contained in $\mathcal{PC}(X, d)$. This implies that (X, d) has rich pseudo-cones. We next prove the opposite. If (X, d) has rich pseudo-cones, then for every $(F, d_F) \in \mathcal{F}$, and for every $\epsilon \in (0, \infty)$, there exist a positive number $z \in (0, \infty)$ and a subset A of X with $\operatorname{card}(A) = \operatorname{card}(F)$ such that $\mathcal{GH}((A, z^{-1}d|_{A^2}), (F, d_F)) < \epsilon$. Thus (X, d) satisfies the anti-$\mathfrak{R}$-transmissible property. We next prove that $\mathfrak{R}$ is singular. For each $(n, m) \in \mathbb{N}^2$ and for each $\epsilon \in (0, \infty)$, we put

$$(R, d_R) = (F_n, (\epsilon/\delta_{d_n}(F_n)) \cdot d_n).$$

Then we have $\delta_{d_R}(R) = \epsilon$, and

$$\phi_R^{(n,m),R}\left(\{f_{n,l}\}_{l=1}^{\operatorname{card}(F_n)}, \delta_{d_n}(F_n)/\epsilon, d_R\right) = 0 \notin R(n, m).$$

Therefore $\mathfrak{R}$ is singular. This completes the proof. □

Since every compact metric space is arbitrarily approximated by members of $\mathcal{F}$ in the sense of Gromov–Hausdorff, we obtain:

Proposition 6.3.7. *A metric space (X, d) has rich pseudo-cones if and only if $\mathcal{PC}(X, d)$ contains all compact metric spaces.*

From Theorem 1.2.10, we deduce the following:

Theorem 6.3.8. *For every metrizable space X, the set of all metrics $d \in \mathrm{M}(X)$ for which (X, d) has rich pseudo-cones is dense G_δ in $\mathrm{M}(X)$.*

Remark 6.3.1. Chen and Rossi [18] introduced the notion of a locally rich compact metric space. They investigated distributions of locally rich metric spaces in a space of compact metric spaces with respect to the Gromov–Hausdorff distance, and they also studied this subject in the Euclidean setting in a space of compact subspaces. Let S be the set of all compact subsets K of $[0,1]^N$ whose tangent cones contain all similarity classes of compact subsets of $[0,1]^N$ for each point in K. They proved that the complement of S is of first category in the space of all compact subsets of $[0,1]^N$ (see [18, Theorem 3.6]). Theorems 1.2.10, 1.2.11, 1.2.16 and 1.2.17 are inspired by this result of Chen and Rossi.

6.3.5 Metric inequality

Let $n \in \mathbb{N}$. We denote by $P(n)$ the set of all point in $\mathbb{R}^{\binom{n}{2}}$ whose all coordinates are positive. Let $f : P(n) \to \mathbb{R}$ be a continuous function. We say that a metric space (X, d) satisfies the (n, f)-*metric inequality* if for all n points $\{a_i\}_{i=1}^n$ in X we have $f(\{d(a_i, a_j)\}_{i\neq j}) \geq 0$. We say that a function $f : P(n) \to \mathbb{R}$ is *positively sub-homogeneous* if there exists $c \in [0, \infty)$ such that for every $x \in P(n)$ and for every $r \in (0, \infty)$ we have $f(r \cdot x) \leq r^c f(x)$.

Proposition 6.3.9 ([59])**.** *For $n \in \mathbb{N}$, let $f : P(n) \to \mathbb{R}$ be a continuous function. Then satisfying the (n, f)-metric inequality on metric spaces is a transmissible property. Moreover, if f is positively sub-homogeneous, and if there exists a metric space not satisfying the (n, f)-metric inequality, then satisfying the (n, f)-metric inequality on metric spaces is a transmissible property with a singular transmissible parameter.*

Proof. Let $Q = \{1\}$ and define a map $F : Q \to \mathcal{F}(\mathbb{R})$ by $F(1) = [0, \infty)$. Define a map $G : Q \to \mathcal{P}^*(\mathbb{N})$ by $G(1) = \{n\}$. For each metrizable space X, we define a map $\phi^{1,X} : \mathrm{Seq}(n, X) \times \{1\} \times \mathrm{M}(X) \to \mathbb{R}$ by

$$\phi^{1,X}(\{a_i\}_{i=1}^n, 1, d) = f(\{d(a_i, a_j)\}_{i\neq j}).$$

Let $\mathfrak{G} = (\{1\}, \mathbb{R}, F, G, \{1\}, \phi)$. Then $\mathfrak{G}$ is a transmissible parameter.

We next show the latter part. Since there exists a metric space not satisfying the (n, f)-metric inequality, there exists a metric space (S, d_S) with $\mathrm{card}(S) = n$ not satisfying the (n, f)-metric inequality. Let $c \in (0, \infty)$ be a positive number such that for every $x \in P(n)$, and for every $r \in (0, \infty)$ we have $f(r \cdot x) \leq r^c f(x)$. Let $S = \{s_i\}_{i=1}^n$ and assume that $f(\{d_S(s_i, s_j)\}_{i\neq j}) < 0$. For every $\epsilon \in (0, \infty)$, put $(R, d_R) = (S, \epsilon \cdot d_S)$. Thus we have $\delta_{d_R}(R) = \epsilon$, and

$$\phi^{1,R}(\{s_i\}_{i=1}^n, 1, d_R) = f(\{\epsilon \cdot d_S(s_i, s_j)\}) = \epsilon^c f(\{d_S(s_i, s_j)\}) < 0.$$

This implies that $\mathfrak{G}$ is singular. This finishes the proof. □

Combining Theorem 1.2.10, Corollary 6.2.4 and Proposition 6.3.9, we obtain the following corollary:

Corollary 6.3.10 ([59])**.** *Let X be a non-discrete metrizable space. For a number $n \in \mathbb{N}$, let $f : P(n) \to \mathbb{R}$ be a continuous function. If f is positively sub-homogeneous, and if there exists a metric space not satisfying the (n, f)-metric inequality, then the set of all metrics d in $\mathrm{M}(X)$ for which the space (X, d) does not satisfy the (n, f)-metric inequality is dense open in $\mathrm{M}(X)$.*

Proposition 6.3.11. *Define a function $f : P(3) \to \mathbb{R}$ by*

$$f(x) = \max\{x_{1,2}, x_{2,3}\} - x_{1,3}.$$

Then the strong triangle inequality on metric spaces is equivalent to the $(3, f)$-metric inequality, and f is positively sub-homogeneous.

We say that a metric space (X, d) satisfies the *Ptolemy inequality* if for all four points a_1, a_2, a_3, a_4 in X we have

$$d(a_1, a_3)d(a_2, a_4) \leq d(a_1, a_2)d(a_3, a_4) + d(a_1, a_4)d(a_2, a_3).$$

Proposition 6.3.12. *Define a function $f : P(4) \to \mathbb{R}$ by*

$$f(x) = x_{2,3}x_{1,4} + x_{1,2}x_{3,4} - x_{1,3}x_{2,4}.$$

Then the Ptolemy inequality on metric spaces is equivalent to the $(4, f)$-metric inequality, and f is positively sub-homogeneous.

Gromov [42] introduced the cycle condition for metric spaces as follows: Let $m \in \mathbb{N}$ and $\kappa \in \mathbb{R}$. Let $(M(\kappa), d_{M(\kappa)})$ be the two-dimensional space form of constant curvature κ. We say that a metric space (X, d) satisfies the $\mathrm{Cycl}_m(\kappa)$ *condition* if for every map $f : \mathbb{Z}/m\mathbb{Z} \to X$ there exists a map $g : \mathbb{Z}/m\mathbb{Z} \to M(\kappa)$ such that

(1) for all $i \in \mathbb{Z}/m\mathbb{Z}$, we have $d_{M(\kappa)}(g(i), g(i + 1)) \leq d(f(i), f(i + 1))$;

(2) for all $i, j \in \mathbb{Z}/m\mathbb{Z}$ with $i - j \neq \pm 1$, we have $d_{M(\kappa)}(g(i), g(j)) \geq d(f(i), f(j))$, where the symbol $+$ stands for the addition of $\mathbb{Z}/m\mathbb{Z}$.

Proposition 6.3.13. *For every $m \in \mathbb{N}$, the* $\mathrm{Cycl}_m(0)$ *condition can be represented by an (m, C)-metric inequality for some positively sub-homogeneous function C.*

Proof. For a map $g : \mathbb{Z}/m\mathbb{Z} \to \mathbb{R}^2$, we define two functions $C_{1,g}, C_{2,g} : P(m) \to \mathbb{R}$ by

$$C_{1,g}(x) = \min_{i \in \mathbb{Z}/m\mathbb{Z}} \{x_{i,i+1} - d_{M(0)}(g(i), g(i + 1))\}, \tag{6.3.1}$$

$$C_{2,g}(x) = \min_{i,j \in \mathbb{Z}/m\mathbb{Z},\ i-j \neq \pm 1} \{d_{M(0)}(g(i), g(j)) - x_{i,j}\}. \tag{6.3.2}$$

We define a function $C : P(m) \to \mathbb{R}$ by

$$C(x) = \sup_{g:\mathbb{Z}/m\mathbb{Z} \to M(0)} \{C_{1,g}(x),\ C_{2,g}(x)\}.$$

Then C is continuous. For every $r \in (0, \infty)$ we have

$$\begin{aligned} C(r \cdot x) &= \sup_{g:\mathbb{Z}/m\mathbb{Z} \to M(0)} \{C_{1,g}(r \cdot x),\ C_{2,g}(r \cdot x)\} \\ &= r \cdot \sup_{g:\mathbb{Z}/m\mathbb{Z} \to M(0)} \{C_{1,g/r}(r \cdot x),\ C_{2,g/r}(r \cdot x)\} \\ &= r \cdot \sup_{g:\mathbb{Z}/m\mathbb{Z} \to M(0)} \{C_{1,g}(x),\ C_{2g}(x)\}. \end{aligned}$$

Thus the function C is positively sub-homogeneous.

If m many points $a_1, \dots, a_m$ in X satisfy the inequality $C(\{d(a_i, a_j)\}_{i \neq j}) \geq 0$, then there exists a map $g : \mathbb{Z}/m\mathbb{Z} \to M(0)$ satisfying that $C_{1,g}(\{d(a_i, a_j)\}_{i \neq j}) \geq 0$ and $C_{2,g}(\{d(a_i, a_j)\}_{i \neq j}) \geq 0$. These two inequalities are equivalent to the conditions (1) and (2) in the $\mathrm{Cycl}_m(0)$ condition, respectively. Therefore the $\mathrm{Cycl}_m(0)$ condition is equivalent to the (m, C)-metric inequality. □

Gromov [40] introduced the notion which today we call the Gromov hyperbolicity. We say that a metric space (X, d) is *Gromov hyperbolic* if there exists $\delta \in [0, \infty)$ such that for all four points a_1, a_2, a_3, a_4 in X we have

$$d(a_1, a_3) + d(a_2, a_4) \leq \max\{d(a_1, a_2) + d(a_3, a_4), d(a_1, a_4) + d(a_2, a_3)\} + 2\delta,$$

Proposition 6.3.14. *Define a function* $f : P(4) \to \mathbb{R}$ *by*

$$f(x) = \sup_{\delta \in [0,\infty)} \{\max\{x_{1,2} + x_{3,4}, x_{1,4} + x_{2,3}\} + 2\delta - (x_{1,3} + x_{2,4})\}.$$

Then the Gromov hyperbolicity is equivalent to satisfying the $(4, f)$*-metric inequality.*

Since for every metrizable space X the set of all bounded metrics in $\mathrm{M}(X)$ is open in $\mathrm{M}(X)$, and since every bounded metric space is Gromov hyperbolic, we obtain:

Proposition 6.3.15 ([59]). *The Gromov hyperbolicity on metric spaces is not equivalent to any transmissible property with a singular transmissible parameter.*

As consequences of Theorems 1.2.10 and 1.2.11, we obtain:

Corollary 6.3.16. *Let* X *be a non-discrete metrizable space. Then the following sets are dense* G_δ *in the space* $\mathrm{M}(X)$ *of metrics:*

(1) *the set of all metric* $d \in \mathrm{M}(X)$ *for which* (X, d) *has infinite Assouad dimension;*

(2) *the set of all* $d \in \mathrm{M}(X)$ *for which* (X, d) *is not bi-Lipschitz embeddable into any ultrametric space;*

(3) *the set of all* $d \in \mathrm{M}(X)$ *for which* $\mathcal{PC}(X, d)$ *contains all compact metric spaces;*

(4) *the set of all* $d \in \mathrm{M}(X)$ *for which* (X, d) *is not an ultrametric;*

(5) *the set of all* $d \in \mathrm{M}(X)$ *for which* (X, d) *is not a Ptolemaic metric;*

(6) *for each* $m \in \mathbb{N}$ *the set of all* $d \in \mathrm{M}(X)$ *for which* (X, d) *does not satisfy the* $\mathrm{Cycl}_m(0)$ *condition.*

Corollary 6.3.17. *Let* X *be a second countable locally compact, locally non-discrete metrizable space. Then the following sets are dense* G_δ *in the space* $\mathrm{M}(X)$ *of metrics:*

(1) *the set of all metric* $d \in \mathrm{M}(X)$ *for which all non-empty open metric subspaces of* (X, d) *have infinite Assouad dimension;*

(2) *the set of all* $d \in \mathrm{M}(X)$ *for which all non-empty open metric subspaces of* (X, d) *are not bi-Lipschitz embeddable into any ultrametric space;*

(3) *the set of all* $d \in \mathrm{M}(X)$ *for which all non-empty open metric subspace of* (X, d) *contain all compact metric spaces as its pseudo-cones;*

(4) *the set of all* $d \in \mathrm{M}(X)$ *for which all non-empty open metric subspaces of* (X, d) *are not ultrametrics;*

(5) *the set of all* $d \in \mathrm{M}(X)$ *for which all non-empty open metric subspaces of* (X, d) *are not Ptolemaic metrics;*

(6) *for each* $m \in \mathbb{N}$ *the set of all* $d \in \mathrm{M}(X)$ *for which all non-empty open metric subspaces of* (X, d) *do not satisfy the* $\mathrm{Cycl}_m(0)$ *condition.*

Chapter 7

Spaces of ultrametrics

In this chapter, we prove Theorem 1.2.12, which is an ultrametric version of the Arens–Eells isometric embedding theorem. We also prove an extension theorem of ultrametrics (Theorem 1.2.13) while referring to Toruńczyk's proof of the Hausdorff extension theorem with the Arens–Eells isometric embedding theorem. Due to Theorem 1.2.13, we can prove an interpolation theorem of ultrametrics (Theorem 1.2.9), and theorems on topological distributions of ultrametrics (Theorems 1.2.10 and 1.2.11).

7.1 An embedding theorem of ultrametric spaces

In this section, we prove Theorem 1.2.12.

7.1.1 Proof of Theorem 1.2.12

We first discuss general algebraic facts.

Lemma 7.1.1. *Let R be a commutative ring, and let V be an R-module. Let P be an R-independent set of V, and let Q be a subset of P. Let H be an R-submodule of V generated by Q. Then $H \cap P = Q$.*

Proof. By the definition of H, first we have $Q \subset H \cap P$. Since P is R-independent, we see that $(P \setminus Q) \cap H = \emptyset$. Thus every $x \in H \cap P$ must belong to Q, and hence we conclude that $H \cap P \subset Q$. □

Let R be a commutative ring. Let X be a set, and let $o \notin X$. We denote by $\mathrm{F}(R, X, o)$ the free R-module M satisfying that

(1) $X \sqcup \{o\} \subset M$;

(2) o is the zero element of M;

(3) X is an R-independent generator of M.

Note that by the construction of free modules generated by given sets, the module $\mathrm{F}(R, X, o)$ uniquely exists up to isomorphism.

For two sets A, B, we denote by $\mathrm{Map}(A, B)$ the set of all maps from A into B. If R is a commutative ring and V is an R-module, and if E is a set, then the set $\mathrm{Map}(E, V)$ becomes an R-module with respect the coordinate-wise addition and scalar multiplication. Note that the zero element of $\mathrm{Map}(E, V)$ is the zero function of $\mathrm{Map}(E, V)$; namely, the constant function valued at the zero element of V. In what follows, for a set E, and for an R-module V, the set $\mathrm{Map}(E, V)$ will be always equipped with this module structure.

We next discuss a construction of universal ultrametric spaces of Lemin–Lemin type [74]. Let S be a range set. Let M be a set, and let $o \in M$ be a fixed base point. A map $f : S_+ \to M$ is said to be *eventually o-valued* if there exists $C \in S_+$ such that for every $q > C$ we have $f(q) = o$. We denote by $\mathrm{L}(S, M, o)$ the set of all eventually o-valued maps from S_+ to M. Define a metric Δ on $\mathrm{L}(S, M, o)$ by

$$\Delta(f, g) = \sup\{\, q \in S_+ \mid f(q) \neq g(q) \,\}.$$

Note that Δ takes values in the closure $\mathrm{CL}(S)$ of S in $[0, \infty)$.

The next lemma follows from the definitions of $\mathrm{L}(S, M, o)$ and Δ.

Lemma 7.1.2. *For every range set S, for every set M and for every point $o \in M$, the space $(\mathrm{L}(S, M, o), \Delta)$ is a complete $\mathrm{CL}(S)$-valued ultrametric space.*

Proof. For all $f, g, h \in (\mathrm{L}(S, M, o), \Delta)$, take $q \in S_+$ with $\Delta(f, h) \vee \Delta(h, g) < q$. Then we have $f(q) = h(q)$ and $h(q) = g(q)$, a hence $f(q) = g(q)$. Thus we have $\Delta(f, g) < q$. This implies that $\Delta(f, g) \leq \Delta(f, h) \vee \Delta(h, g)$, and hence Δ is an S-valued ultrametric.

We next prove that $(\mathrm{L}(S, M, o), \Delta)$ is complete. Let $\{f_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{L}(S, M, o)$. Since $\{f_i\}_{i\in\mathbb{N}}$ is Cauchy, for every $s \in S_+$, there exists $\alpha(s) \in \mathbb{N}$ such that for all $m \geq \alpha(s)$, we have $\Delta(f_{\alpha(s)}, f_m) < s$. We define a map $F \in \mathrm{L}(S, M, o)$ by $F(s) = f_{\alpha(s)}(s)$. Then, F is a limit of $\{f_i\}_{i\in\mathbb{N}}$. This finishes the proof. □

In the next theorem, we review the Lemin–Lemin construction [74] of embeddings into their universal spaces in order to obtain more detailed information of their construction. The condition (3) in Theorem 7.1.3 was first observed in [58].

Theorem 7.1.3. *Let S be a range set. Let $(X \sqcup \{o\}, d)$ be an S-valued ultrametric space with $o \notin X$. Let K be a set with $X \sqcup \{o\} \subset K$. Then there exists an isometric embedding $L : X \sqcup \{o\} \to \mathrm{L}(S, K, o)$ satisfying that*

1. *for every $q \in S_+$ we have $L(o)(q) = o$;*
2. *for every $x \in X$ the function $L(x)$ is valued in $X \sqcup \{o\}$;*
3. *for all $x, y \in X$ we have $(0, d(x, y)] \cap S_+ = \{\, q \in S_+ \mid L(x)(q) \neq L(y)(q) \,\}$.*

Proof. Let $X \sqcup \{o\} = \{x(\alpha)\}_{\alpha<\kappa}$ be an injective index with $x(0) = o$, where κ is a cardinal. By following the Lemin–Lemin's way [74], we construct an isometric embedding $L : X \sqcup \{o\} \to \mathrm{L}(S, K, o)$ by transfinite induction. First put $L(x(0)) = o$. Let $\gamma < \kappa$. Assume that an isometric embedding $L : \{\, x(\alpha) \mid \alpha < \gamma \,\} \to \mathrm{L}(S, K, o)$ is already defined. Set $D_\gamma = \inf\{\, d(x(\alpha), x(\gamma)) \mid \alpha < \gamma \,\}$.

Case 1. (There exists an ordinal $\beta < \gamma$ with $D_\gamma = d(x(\beta), x(\gamma))$.) We define an eventually o-valued map $L(x(\gamma)) : S_+ \to K$ by

$$L(x(\gamma))(q) = \begin{cases} x(\gamma) & \text{if } q \in (0, D_\gamma]; \\ L(x(\beta))(q) & \text{if } q \in (D_\gamma, \infty). \end{cases}$$

Case 2. (No ordinal $\beta < \gamma$ satisfies $D_\gamma = d(x(\beta), x(\gamma))$.) Take a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ with $\alpha_n < \gamma$ and $d(x(\alpha_n), x(\gamma)) < D_\gamma + 1/n$ for all $n \in \mathbb{N}$. We define an eventually o-valued map $L(x(\gamma)) : S_+ \to K$ by

$$L(x(\gamma))(q) = \begin{cases} x(\gamma) & \text{if } q \in (0, D_\gamma]; \\ L(x(\alpha_n))(q) & \text{if } D_\gamma + 1/n < q. \end{cases}$$

In the same way as [74], we see that the map $L : X \sqcup \{o\} \to \mathrm{L}(S, K, o)$ is well-defined and isometric, and the conditions (1) and (2) are satisfied.

We now prove the condition (3). Note that for each $\alpha < \kappa$, the function $L(x(\alpha))$ is valued in $\{\, x(\beta) \mid \beta \leq \alpha \,\}$. Let $\gamma < \kappa$. Assume that for all $\alpha, \beta < \gamma$, the condition (3) is satisfied for $x = x(\alpha)$ and $y = x(\beta)$. We prove that for every $\alpha < \gamma$, the condition (3) is satisfied for $x = x(\alpha)$ and $y = x(\gamma)$.

In Case 1, by the definition of D_γ and $\beta < \gamma$, we have $d(x(\beta), x(\gamma)) \leq d(x(\alpha), x(\gamma))$. This inequality and the strong triangle inequality (or Lemma 3.3.7) yield the inequality $d(x(\alpha), x(\beta)) \leq d(x(\alpha), x(\gamma))$. Thus, by the hypothesis of transfinite induction and the definition of $L(x(\gamma))$, we conclude that the condition (3) is satisfied.

In Case 2, by the definition of D_γ, we have $D_\gamma < d(x(\alpha), x(\gamma))$, and for all sufficiently large $n \in \mathbb{N}$, we obtain $d(x(\alpha_n), x(\gamma)) < d(x(\alpha), x(\gamma))$. Lemma 3.3.7 implies that $d(x(\alpha_n), x(\alpha)) = d(x(\alpha), x(\gamma))$. Since on the set $(D_\gamma + 1/n, d(x(\alpha), x(\alpha_n))]$ the function $L(x(\gamma))$ coincides with $L(x(\alpha_n))$, by the hypothesis of transfinite induction we have

$$(D_\gamma + 1/n, d(x(\alpha), x(\gamma))] \cap S_+ \subset \{\, q \in S_+ \mid L(x(\alpha))(q) \neq L(x(\gamma))(q) \,\}.$$

By $L(x(\alpha))(S_+) \subset \{\, x(\beta) \mid \beta \leq \alpha \,\}$ and $L(x(\gamma))|_{(0,D_\gamma]} = x(\gamma)$, we also have

$$(0, D_\gamma] \cap S_+ \subset \{\, q \in S_+ \mid L(x(\alpha))(q) \neq L(x(\gamma))(q) \,\}.$$

These imply the condition (3) for $x = x(\alpha)$ and $y = x(\gamma)$. □

The following lemma plays a central role in the proof of our embedding theorem from ultrametric spaces into ultra-normed modules.

Lemma 7.1.4. *Let S be a range set. Let R be a commutative ring. If M is an R-module, then the following are satisfied:*

(1) *the space $\mathrm{L}(S, M, 0)$ becomes an R-submodule of $\mathrm{Map}(S_+, M)$;*

(2) *the ultrametric Δ on $\mathrm{L}(S, M, 0)$ is invariant under the addition; namely, the space $(\mathrm{L}(S, M, 0), \Delta)$ is ultra-normed;*

(3) *if M is torsion-free, and if R is an integral domain, then for every $r \in R \setminus \{0\}$, and for every $x \in \mathrm{L}(S, M, o)$, we have $\Delta(r \cdot x, 0) = \Delta(x, 0)$.*

Proof. By the definition of an eventually 0-valued map, the set $\mathrm{L}(S, M, 0)$ is closed under the coordinate-wise addition and scalar multiplication. Thus the statement (1) follows from $\mathrm{L}(S, M, 0) \subset \mathrm{Map}(S_+, M)$.

We prove the statement (2). For all $f, g, h \in \mathrm{L}(S, M, 0)$, and for every $q \in S_+$, we have $f(q) \neq g(q)$ if and only if $f(q) + h(q) \neq g(q) + h(q)$. Thus, the metric Δ is invariant under the addition. Similarly, since R is an integral domain, the statement (3) holds true. □

Lemma 7.1.5 ([58])**.** *Let R be a commutative ring. Let S be a range set. Let X be a set, and let $o \notin X$. Let $(X \sqcup \{o\}, D)$ be an S-valued ultrametric space. Put $M = \mathrm{F}(R, X, o)$. Let $L : X \sqcup \{o\} \to \mathrm{L}(S, M, o)$ be an isometric embedding constructed in Theorem 7.1.3. Then the image set $L(X)$ of X under L is R-independent in the R-module $\mathrm{L}(S, M, o)$.*

Proof. In this proof, we denote by 0_M and by 0_{L} the zero element o of R-modules M and $\mathrm{L}(S, M, o)$, respectively. Let $C = \{x_1, \dots, x_n\}$ be an arbitrary finite subset of X. Assume that $\sum_{i=1}^{n} N_i \cdot L(x_i) = 0_{\mathrm{L}}$, where $N_i \in R$ for all i. Put

$$c = \min\{\, \Delta(L(x), L(y)) \mid x, y \in C \sqcup \{o\} \text{ and } x \neq y \,\}.$$

Since L is isometric, $c \in S_+$. By the definition of Δ and the conditions (1) and (3) stated in Theorem 7.1.3, we see that for all $i, j = 1, \dots, n$, we have $L(x_i)(c) \neq L(x_j)(c)$, and for all i we see that $L(x_i)(c) \neq 0_M$. Since X is R-independent in $\mathrm{F}(R, X, o)$, the set $\{L(x_1)(c), \dots, L(x_n)(c)\}$ is R-independent in M. Since $\sum_{i=1}^{n+1} N_i \cdot L(x_i)(c) = 0_M$, we have $N_i = 0$ for all i. Thus $\{L(x_1), \dots, L(x_n)\}$ is R-independent in $\mathrm{L}(S, M, o)$. Since $S = \{x_1, \dots, x_n\}$ is arbitrary, we conclude that $L(X)$ is R-independent in $\mathrm{L}(S, M, o)$. $\square$

Lemma 7.1.6. *Let R be a commutative ring. Let S be a range set. Let X be a set, and let $o \notin X$. Let $(X \sqcup \{o\}, D)$ be an S-valued ultrametric space. Put $M = \mathrm{F}(R, X, o)$. Let $L : X \sqcup \{o\} \to \mathrm{L}(S, M, o)$ be an isometric embedding constructed in Theorem 7.1.3. Let Q be an R-submodule of $\mathrm{L}(S, M, o)$ generated by $L(X)$. Then the metric $\Delta|_{Q^2}$ takes values in the range set S.*

Proof. In this proof, we denote by 0_M the zero element o of M.

Since Δ is invariant under the addition, it suffices to show that every point x in Q satisfies $\Delta(x, 0_{\mathrm{L}}) \in S$, where 0_{L} is the zero function of $\mathrm{L}(S, N, o)$. Take $x \in Q$. Then there exist a finite subset $\{x_1, \dots, x_n\}$ of X and a finite subset $\{N_1, \dots, N_n\}$ of $R \setminus \{0\}$ such that $x = \sum_{i=1}^{n} N_i \cdot L(x_i)$. Let $p_0, p_1, \dots, p_k$ be a sequence in S such that

(1) $p_0 = 0$;

(2) $p_j < p_{j+1}$ for all j;

(3) $\{\, d(x_i, 0_M) \mid i = 1, \dots, n \,\} \cup \{\, d(x_i, x_j) \mid i \neq j \,\} = \{p_1, \dots, p_k\}$.

For $l \in \{0, \dots, k-1\}$, we put $I(j) = (p_j, p_{j+1}] \cap S$, and we put $I(k) = (p_k, \infty) \cap S$. By the definition of $\{p_j\}_{j=0}^{k}$, and by the properties (2) and (3) of the map L stated in Theorem 7.1.3, we obtain:

(A) for all $a \in \{1, \dots, n\}$ we have $L(x_a) = 0_M$ on $I(k)$;

(B) for every $a \in \{1, \dots, n\}$, and for every $j \in \{0, \dots, k\}$, if there exists $c \in I(j)$ satisfying that $L(x_a)(c) = 0_M$, then we have $L(x_a) = 0_M$ on $I(j)$;

(C) for all $a, b \in \{1, \dots, n\}$, and for every $j \in \{0, \dots, k\}$, if there exists $c \in I(j)$ satisfying that $L(x_a)(c) = L(x_b)(c)$, then we have $L(x_a) = L(x_b)$ on $I(j)$.

Suppose that $\Delta(x, 0) \notin S$. By using the property (A), take $j \in \{0, \dots, k-1\}$ such that $\Delta(x, 0) \in I(j)$. By the definition of Δ, there exists $p \in I(j)$ with $x(p) \neq 0_M$, and we see that $x(p_{j+1}) = 0_M$. Put $q = p_{j+1}$. Take a subset $\{y_1, \dots, y_m\}$ of $\{x_1, \dots, x_n\}$ such that

(a) $L(y_1)(q), \dots, L(y_m)(q)$ are not equal to the zero element 0_M of M, and they are different to each other;

(b) m is maximal in cardinals of all subsets of the set $\{x_1, \dots, x_n\}$ satisfying the property (a).

The properties (B) and (C) imply that the set $\{L(y_1)|_{I(j)}, \dots, L(y_m)|_{I(j)}\}$ is a maximal R-independent subset of $\{L(x_1)|_{I(j)}, \dots, L(x_n)|_{I(j)}\}$ in the R-module $\mathrm{Map}(I(j), M)$. Then there exists a subset $\{C_1, \dots, C_m\}$ of R such that

$$x|_{I(j)} = \sum_{l=1}^{m} C_l \cdot L(y_l)|_{I(j)}.$$

Since $x(q) = 0_M$, we have

$$\sum_{l=1}^{m} C_l \cdot L(y_l)(q) = 0_M.$$

Since $\{L(y_1)(q), \dots, L(y_m)(q)\}$ is a subset of X, it is R-independent in M. Thus we have $C_l = 0$ for all $l \in \{1, \dots, m\}$. This implies that $x = 0_M$ on $I(j)$. This contradicts the existence of $p \in I(j)$ with $x(p) \neq 0_M$. Therefore, we conclude that $\Delta(x, 0_{\mathrm{L}}) \in S$. □

We recall that every free module on an integral domain is torsion-free.

Proof of Theorem 1.2.12. Let S be a range set. Let R be a commutative ring, and let (X, d) be an ultrametric space.

We first deal with the case where (X, d) is complete. Take $o \notin X$. Put $M = \mathrm{F}(R, X, o)$. Applying Corollary 3.2.7 to (X, d) and o, then we obtain a one-point extension $(X \sqcup \{o\}, D)$ of (X, d). Let $L : (X \sqcup \{o\}, D) \to (\mathrm{L}(S, M, o), \Delta)$ be an isometric embedding stated in Theorem 7.1.3. Let Q be an R-submodule of $\mathrm{L}(S, M, o)$ generated by $L(X)$, and let (V, Ξ) be the completion of $(Q, \Delta|_{Q^2})$. By Lemmas 3.3.6, 7.1.6, and Proposition 3.3.8, the ultrametric space (V, Ξ) is an S-valued ultra-normed R-module. Since complete metric subspaces are closed in metric spaces, Lemma 7.1.5 implies that the space (V, Ξ) and the map $L : (X, d) \to (V, \Xi)$ satisfy the conditions (1) and (2) stated in Theorem 1.2.12. This also proves the latter part of the theorem.

In the case where (X, d) is not complete, let (Y, e) be the completion of (X, d). As in the above, we can take an ultra-normed R-module (W, D) and an isometric embedding $I : Y \to W$ satisfying the conditions (1) and (2) in Theorem 1.2.12. Let H be an R-submodule of W generated by $I(X)$. Since $I(Y)$ is R-independent, Lemma 7.1.1 yields $H \cap I(Y) = I(X)$. Thus $I(X)$ is closed in H, and hence $(H, D|_{H^2})$ and I are desired ones. This completes the proof of Theorem 1.2.12. □

Remark 7.1.1. If a range set S is closed under the supremum operator, then we can replace the assumption that R is an integral domain in the statement of Theorem 1.2.12 with the condition that R is a commutative ring. In this case, the space $(\mathrm{L}(S, M, o), \Delta)$ is an S-valued ultrametric space, and in the proof of Theorem 1.2.12, we can use the space $(\mathrm{L}(S, M, o), \Delta)$ instead of the space (V, Ξ).

7.1.2 Ultrametrics taking values in general totally ordered sets

We say that an ordered set is *bottomed* if it has a minimal element. Let $(T, \leq_T)$ be a bottomed totally ordered set. Let X be a set. A function $d : X \times X \to T$ is said to be a *$(T, \leq_T)$-valued ultrametric on* X if the following are satisfied:

(1) for all $x, y \in X$ we have $d(x, y) = 0_T$ if and only if $x = y$, where 0_T stands for the minimal element of $(T, \leq_T)$;

(2) for all $x, y \in X$ we have $d(x, y) = d(y, x)$;

(3) for all $x, y, z \in X$ we have $d(x, y) \leq_T d(x, z) \vee_T d(z, y)$, where $\vee_T$ is the maximal operator of $(T, \leq_T)$.

Such general ultrametric spaces, or general metric spaces on which distances are valued in a totally ordered Abelian group are studied for a long time as a natural extension of metric spaces (see e.g., [104], [19], [106], [88] and [20]).

Amalgamation arguments and a one-point extension of ultrametrics (Subsection 3.2.2), the construction of the universal ultrametric space of Lemin–Lemin-type mentioned in Section 7.1, and the proof of Theorem 1.2.12 are still valid for $(T, \leq_T)$-valued ultrametric spaces for all bottomed totally ordered set $(T, \leq_T)$. In this case, note that the ultrametric Δ on the space $\mathrm{L}(T, M, o)$ takes values in the Dedekind completion of $(T, \leq_T)$. For simplicity, and for necessity of our study, we omit the details of the following:

Theorem 7.1.7 ([58]). *Let $(T, \leq_T)$ be a bottomed totally ordered set. Let (X, d) be a $(T, \leq_T)$-valued ultrametric space. Let R be an integral domain. Then there exist a $(T, \leq_T)$-valued ultra-normed R-module $(V, \| * \|)$ and an isometric embedding $I : X \to V$ such that*

(1) *$I(X)$ is closed in V;*

(2) *$I(X)$ is R-independent in V.*

Moreover, if (X, d) is complete, then $(V, \| * \|)$ can be chosen as a complete $(T, \leq_T)$-valued ultrametric space.

For a bottomed totally ordered set $(T, \leq_T)$, we define the *coinitiality* $\mathrm{coi}(T, \leq_T)$ of T as the minimal cardinal $\kappa > 0$ such that there exists a strictly decreasing map $f : \kappa + 1 \to T$ with $f(\kappa) = 0_T$ such that for every $t \in T$, there exists $\alpha < \kappa$ with $f(\alpha) \leq t$. The coinitiality for ordered sets is the dual concept of the cofinality for ordered sets. Note that a range set S has countable coinitiality if and only if $\mathrm{coi}(\mathrm{CL}(S), \leq) = \omega_0$. Some readers may think our results such as Corollary 1.2.14 and Theorems 1.2.13–1.2.17 in this thesis can be generalized for $(T, \leq_T)$-valued ultrametrics for a bottomed totally ordered set $(T, \leq_T)$. If $\mathrm{coi}(T, \leq_T) = \omega_0$, then it is possible to obtain analogues of our results for general ultrametrics. However, in this case, the $(T, \leq_T)$-ultrametrizability is equivalent to the ordinal ultrametrizability. Thus, it seems not to be a vast generalization. Unfortunately, in the case of $\mathrm{coi}(T, \leq_T) > \omega_0$, it seems to be quite difficult to generalize our theory of ultrametrics. Our proofs of Theorems 1.2.13–1.2.17 require extensions of continuous maps on ultrametric spaces (Corollary 2.7.12). An analogue for $(T, \leq_T)$-valued ultrametric spaces of Corollary 2.7.12 seems not to hold true.

7.2 An extension theorem of ultrametrics

In this section, as an application of Theorem 1.2.12, we prove Theorem 1.2.13, which is an extension theorem of ultrametrics. By using this extension theorem, we can develop an ultrametric analogue of the theory of spaces of metrics in Chapter 6, and we prove Theorems 1.2.15 and 1.2.16, and 1.2.17.

In this section, by following the methods of Toruńczyk [113] and Hausdorff [45], we prove Theorem 1.2.13 and Corollary 1.2.14, respectively. Since Toruńczyk's proof of Lemma in [113] on real linear spaces does not depend on the coefficient ring $\mathbb{R}$, we can apply that argument to all ultra-normed modules over all commutative rings. Toruńczyk's lemma in [113] relies upon the Dugundji extension theorem in the proof. Instead of that extension theorem, we use Corollary 2.7.12, which is an extension theorem for continuous functions on ultrametrizable spaces.

Lemma 7.2.1. *Let R be a commutative ring. Let (E, D_E) and (F, D_F) be two ultra-normed R-modules. Let K and L be closed subsets of E and F, respectively. Let $f : K \to L$ be a homeomorphism. Let $g : K \times \{0\} \to \{0\} \times L$ be a homeomorphism defined by $g(x, 0) = (0, f(x))$, where we regard $K \times \{0\}$ and $\{0\} \times L$ as subsets of $E \times F$ and $E \times F$, respectively. Then there exists a homeomorphism $h : E \times F \to E \times F$ with $h|_{K \times \{0\}} = g$.*

Proof. By Corollary 2.7.12, we obtain a continuous map $\beta : F \to E$ which is an extension of the map $f^{-1} : L \to K$. Define a map $J : E \times F \to E \times F$ by $J(x, y) = (x + \beta(y), y)$. Lemma 3.3.5 implies that the addition and the inversion on E is continuous, and hence J is continuous. The map $Q : E \times F \to E \times F$ defined by $Q(x, y) = (x - \beta(y), y)$ is also continuous, and it is the inverse map of J. Hence J is a homeomorphism. Similarly, by Corollary 2.7.12, we obtain a continuous map $\alpha : E \to F$ which is an extension of $f : K \to L$. Define a map $I : E \times F \to E \times F$ by $I(x, y) = (x, y + \alpha(x))$. Then I is a homeomorphism. Define a homeomorphism $h : E \times F \to E \times F$ by $h = J^{-1} \circ I$. Since for every $x \in K$ we have $I(x, 0) = (x, \alpha(x)) = (x, f(x))$, we obtain

$$\begin{aligned} h(x, 0) &= J^{-1}(x, f(x)) = Q(x, f(x)) = (x - \beta(f(x)), f(x)) \\ &= (x - f^{-1}(f(x)), f(x)) = (0, f(x)) = g(x, 0), \end{aligned}$$

and hence h is an extension of g. This completes the proof. □

We now prove Theorem 1.2.13.

Proof of Theorem 1.2.13. Let S be a range set. Let X be an S-valued ultrametrizable space, and let A be a closed subset of X. Let $e \in \mathrm{UM}(A, S)$. Take $d \in \mathrm{UM}(X, S)$. Theorem 1.2.12 implies that there exist an S-valued ultra-normed $\mathbb{Z}$-module (E, D_E) and a closed isometric embedding $i : (X, d) \to (E, D_E)$. Similarly, there exist an S-valued ultra-normed $\mathbb{Z}$-module (F, D_F) and a closed isometric embedding $j : (A, e) \to (F, D_F)$.

Since A is closed in X, the set $i(A)$ is closed in E. Since i and j are topological embeddings, $i(A)$ and $j(A)$ are homeomorphic. Based on this observation, define a map $f : i(A) \to j(A)$ by $f = j \circ (i|_A)^{-1}$, and by applying Lemma 7.2.1 to f, we obtain a homeomorphism $h : E \times F \to E \times F$ which is an extension of $g : i(A) \times \{0\} \to \{0\} \times j(A)$ defined by $g(i(a), 0) = (0, j(a))$.

Let $k : E \to E \times F$ be a natural embedding defined by $k(x) = (x, 0)$. The map $H : X \to E \times F$ defined by $H = h \circ k \circ i$ is a topological embedding. Note that H is a closed map. Define a metric D on X by $D(x, y) = (D_E \times_\infty D_F)(H(x), H(y))$. Since H is a topological embedding, we have $D \in \mathrm{UM}(X, S)$. Since for every $a \in A$ we have $H(a) = (0, j(a))$, and since the map $j : (A, \rho) \to (F, D_F)$ is an isometric embedding, we have $D|_{A^2} = e$. This completes the proof of the former part.

We next show the latter part. Assume that X is completely metrizable, and the metric $e \in \mathrm{UM}(A, S)$ is complete. Then by Proposition 3.3.14, we can choose $d \in \mathrm{UM}(X, S)$ as a complete S-valued ultrametric. Thus, we can choose (E, D_E) and (F, D_F) as complete ultrametric spaces, and hence the metric space (X, D) can be regarded as a closed metric subspace of the complete metric space $(E \times F, D_F \times_\infty D_E)$. Therefore D is complete. □

Remark 7.2.1. In the proof of Theorem 1.2.13, for simplicity, we use $\mathbb{Z}$-modules. The proof described above is still valid even if we use any integral domain as a coefficient ring.

We next prove Corollary 1.2.14, which characterizes the compactness on metric spaces in terms of the completeness of ultrametrics.

Lemma 7.2.2. *Let S be a range set with the countable coinitiality. Let M be a countable discrete space. Then there exists a non-complete S-valued ultrametric $d \in \mathrm{UM}(M, S)$.*

Proof. Take a non-zero strictly decreasing sequence $\{a(i)\}_{i \in \mathbb{N}}$ in S convergent to 0 as $i \to \infty$. We may assume that $M = \mathbb{N}$. Define a metric d on M by

$$d(n, m) = \begin{cases} a(n) \vee a(m) & \text{if } n \neq m; \\ 0 & \text{if } n = m. \end{cases}$$

Then d is in $\mathrm{UM}(M,S)$, and it is non-complete. In particular, the sequence $\{n\}_{n\in\mathbb{N}}$ is Cauchy in $(\mathbb{N},d)$, and it does not have any limit point in (M,d). □

Proof of Corollary 1.2.14. Assume that X is not compact. Then there exists a closed countable discrete subset M of X. By Theorem 1.2.13 and Lemma 7.2.2, we obtain an S-valued ultrametric $D \in \mathrm{UM}(X,S)$ such that $D|_{M^2}$ is not complete. Since every closed metric subspace of a complete metric space is complete, the metric D is not complete. This leads to the corollary. □

7.3 An interpolation theorem of ultrametrics

In this section, we prove Theorem 1.2.15.

7.3.1 Proof of Theorem 1.2.15

In the proof of Theorem 1.2.9, the author used the Michael continuous selection theorem for paracompact spaces (Theorem 2.7.3). Instead of it, in order to prove Theorem 1.2.15, we now use the 0-dimensional Michael continuous selection theorem (Theorem 2.7.4).

Proof of Theorem 1.2.15. Let $C \in [1,\infty)$, and let S be a C-quasi-complete range set. Let X be an ultrametrizable space. Let $\{A_i\}_{i\in I}$ be a discrete family of closed subsets of X. Let $d \in \mathrm{UM}(X,S)$, and let $\{e_i\}_{i\in I}$ be a family of S-valued ultrametrics with $e_i \in \mathrm{UM}(A_i,S)$.

In the case of $\sup_{i\in I}\mathcal{UD}^S_{A_i}(e_i, d|_{A_i^2}) = \infty$, by Lemma 3.2.10, we obtain an S-valued ultrametric $k \in \mathrm{UM}(\coprod_{i\in I} A_i, S)$ such that $k|_{A_i^2} = e_i$ for all $i \in I$. Since the metric k generates the same topology as the direct sum space $\coprod_{i\in I} A_i$, and since $\coprod_{i\in I} A_i$ is closed in X (see Proposition 6.1.1), we can apply Theorem 1.2.13 to the S-valued ultrametric k, and hence there exists an S-valued ultrametric $r \in \mathrm{UM}(X,S)$ such that for every $i \in I$ we have $r|_{A_i^2} = e_i$. Then in this case, Theorem 1.2.15 is proven.

We next deal with the case of $\sup_{i\in I}\mathcal{UD}^S_{A_i}(e_i, d|_{A_i^2}) < \infty$. Let $\eta \in S$ satisfy

$$\sup_{i\in I}\mathcal{UD}^S_{A_i}(e_i, d|_{A_i^2}) \le \eta \le C \cdot \sup_{i\in I}\mathcal{UD}^S_{A_i}(e_i, d|_{A_i^2}).$$

Let $\{B_i\}_{i\in I}$ and $\tau : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ be the same family and map as in Lemma 3.2.11, respectively. Put $Z = X \sqcup \coprod_{i\in I} B_i$. By Lemma 3.2.11, we find an S-valued ultrametric h on Z such that

(1) for every $i \in I$ we have $h|_{B_i^2} = (\tau_i^{-1})^* e_i$;

(2) $h|_{X^2} = d$;

(3) for every $x \in \coprod_{i\in I} A_i$ we have $h(x,\tau(x)) = \eta$.

By Theorem 1.2.12, we can take an isometric embedding H from (Z,h) into a complete S-valued ultra-normed $\mathbb{Z}$-module (Y, D_Y). Define a set-valued map $\phi : Z \to \mathcal{C}(Y)$ by $\phi(x) = B(H(x),\eta)$. By Corollary 2.7.8, the map ϕ is lower semi-continuous. We define a map $f : \bigcup_{i\in I} A_i \to Y$ by $f_i(x) = H(\tau(x))$. Then f is continuous. By the property (3) of h, for every $x \in \bigcup_{i\in I} A_i$ we have $f(x) \in \phi(x)$.

Since (Y, D_Y) is complete, we can apply the 0-dimensional Michael continuous selection theorem (Theorem 2.7.4) to the maps f and ϕ, and hence we obtain a continuous map $F : X \to Y$ such that $F|_{\bigcup_{i\in I} A_i} = f$ and for every $x \in X$ we have $F(x) \in \phi(x)$. Note that $F(x) \in \phi(x)$ means that $D_Y(F(x), H(x)) \le \eta$.

Let $r \in \mathrm{UM}(X, S)$ be the metric constructed in the case of $\sup_{i \in I} \mathcal{UD}_{A_i}^S(e_i, d|_{A_i^2}) = \infty$. Put $l = \min\{r, \eta\}$. Note that by Lemma 3.3.3, we have $l \in \mathrm{UM}(X, S)$.

Put $D = D_Y \times_\infty l$. Then D is an S-valued ultrametric on $Y \times X$. Take a base point $o \in X$. Define a map $E : X \to Y \times X$ by

$$E(x) = (F(x), x).$$

Since the second component of E is a topological embedding, so is E.

We also define a map $K : X \to Y \times X$ by

$$K(x) = (H(x), o).$$

Then, by the definition of the ultrametric D on $Y \times X$, the map K from (X, d) to $(Y \times X, D)$ is an isometric embedding. Since for every $x \in X$ we have $D_Y(F(x), H(x)) \leq \eta$ and $l(x, o) \leq \eta$, we obtain

$$D(E(x), K(x)) = D_Y(F(x), H(x)) \vee l(x, o) \leq \eta.$$

Define a function $m : X^2 \to [0, \infty)$ by $m(x, y) = D(E(x), E(y))$, then m is an S-valued ultrametric on X. Since E is a topological embedding, we see that $m \in \mathrm{UM}(X, S)$. We have $D_Y(F(x), F(y)) = e_i(x, y)$ and $l(x, y) \leq r(x, y) = e_i(x, y)$ for all $i \in I$, and for all $x, y \in A_i$. Thus we obtain

$$D(E(x), E(y)) = D_Y(F(x), F(y)) \vee l(x, y) = e_i(x, y),$$

and hence $m|_{A_i^2} = e_i$. Moreover, we have $\sup_{i \in I} \mathcal{UD}_{A_i}^S(e_i, d|_{A_i^2}) \leq \mathcal{UD}_X^S(m, d)$. We also obtain the inequality $\mathcal{UD}_X^S(m, d) \leq \eta$; indeed, for all $x, y \in X$,

$$\begin{aligned} m(x, y) &= D(E(x), E(y)) \\ &\leq D(E(x), K(x)) \vee D(K(x), K(y)) \vee D(K(y), E(y)) \\ &\leq D(K(x), K(y)) \vee \eta = d(x, y) \vee \eta, \end{aligned}$$

and

$$\begin{aligned} d(x, y) &= D(K(x), K(y)) \\ &\leq D(K(x), E(x)) \vee D(E(x), E(y)) \vee D(E(y), K(y)) \\ &\leq D(E(x), E(y)) \vee \eta = m(x, y) \vee \eta. \end{aligned}$$

Therefore $\mathcal{UD}_X^S(m, d) \leq \eta$, and hence we conclude that

$$\sup_{i \in I} \mathcal{UD}_{A_i}^S(e_i, d|_{A_i^2}) \leq \mathcal{UD}_X^S(m, d) \leq C \cdot \sup_{i \in I} \mathcal{UD}_{A_i}^S(e_i, d|_{A_i^2}).$$

This completes the proof of the former part of Theorem 1.2.15.

By the latter part of Theorem 1.2.13, we can choose l as a complete S-valued ultrametric. Then m becomes a complete S-valued ultrametric. This leads to the proof of the latter part of Theorem 1.2.15. □

In Theorem 1.2.15, by letting I be a singleton, we obtain the following:

Corollary 7.3.1. *Let $C \in [1, \infty)$, and let S be a C-quasi-complete range set. Let X be an ultrametrizable space, and let A be a closed subset of X. Then for every $d \in \mathrm{UM}(X, S)$, and for every $e \in \mathrm{UM}(A, S)$, there exists an ultrametric $m \in \mathrm{UM}(X, S)$ such that*

(1) $m|_{A^2} = e$;

(2) $\mathcal{UD}_A^S(e, d|_{A^2}) \leq \mathcal{UD}_X^S(m, d) \leq C \cdot \mathcal{UD}_A^S(e, d|_{A^2})$.

Moreover, if X is completely ultrametrizable, and if $e \in \mathrm{UM}(A, S)$ is complete, then the metric $m \in \mathrm{UM}(X, S)$ can be chosen as a complete one.

7.4 Transmissible properties and ultrametrics

In this section, we discuss the S-ultra-singularity of transmissible parameters, and we prove Theorem 1.2.16.

7.4.1 Transmissible properties on ultrametric spaces

By the definitions of $\mathcal{D}_X$ and $\mathcal{UD}_X^S$, we obtain:

Lemma 7.4.1. *Let S be a range set. For every ultrametrizable space X, and for all $d, e \in \mathrm{UM}(X, S)$ we have*

$$\mathcal{D}_X(d, e) \leq \mathcal{UD}_X^S(d, e).$$

In particular, the identity map $1_{\mathrm{UM}(X,S)} : (\mathrm{UM}(X, S), \mathcal{UD}_X^S) \to \big(\mathrm{UM}(X, S), \mathcal{D}_X|_{\mathrm{UM}(X,S)^2}\big)$ is continuous.

Let S be a range set. Let X be an ultrametrizable space, and let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$ be a transmissible parameter. For $q \in Q$, for $a \in \mathrm{Seq}(G(q), X)$ and for $z \in Z$, we denote by $US(X, S, \mathfrak{G}, q, a, z)$ the set of all $d \in \mathrm{UM}(X, S)$ satisfying $\phi^{q,X}(a, z, d) \in P \setminus F(q)$. We also denote by $US(X, S, \mathfrak{G})$ the set of all $d \in \mathrm{UM}(X, S)$ such that (X, d) satisfies the anti-$\mathfrak{G}$-transmissible property.

Proposition 7.4.2. *Let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$ be a transmissible parameter. Let S be a range set. Let X be an ultrametrizable space. Then for all $q \in Q$, for all $a \in \mathrm{Seq}(G(q), X)$, and for all $z \in Z$, the set $US(X, S, \mathfrak{G}, q, a, z)$ is open in $(\mathrm{UM}(X, S), \mathcal{UD}_X^S)$.*

Proof. Fix $q \in Q$, $a \in \mathrm{Seq}(G(q), X)$ and $z \in Z$. Since the map $\phi^{q,X}(a, z) : \mathrm{M}(X) \to P$ is continuous, Lemma 7.4.1 implies that the map $\phi^{q,X}(a, z)|_{\mathrm{UM}(X,S)} : \mathrm{UM}(X, S) \to P$ is also continuous, where $\mathrm{UM}(X, S)$ is equipped with the topology induced from the metric $\mathcal{UD}_X^S$. Since we have

$$US(X, S, \mathfrak{G}, q, a, z) = (\phi^{q,X}(a, z)|_{\mathrm{UM}(X,S)})^{-1}(P \setminus F(q)),$$

the set $US(X, S, \mathfrak{G}, q, a, z)$ is open in $(\mathrm{UM}(X, S), \mathcal{UD}_X^S)$. □

Corollary 7.4.3 ([58])**.** *Let S be a range set. Let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$ be a transmissible parameter. Let X be an S-valued ultrametrizable space. Then the set $US(X, S, \mathfrak{G})$ is G_δ in $\mathrm{UM}(X, S)$. Moreover, if the set Q is finite, then $US(X, S, \mathfrak{G})$ is open in $\mathrm{UM}(X, S)$.*

Proof. By the definitions of $US(X, S, \mathfrak{G})$ and $US(X, S, \mathfrak{G}, q, a, z)$, we have

$$US(X, S, \mathfrak{G}) = \bigcap_{q \in Q} \bigcup_{a \in \mathrm{Seq}(G(q), X)} \bigcup_{z \in Z} US(X, S, \mathfrak{G}, q, a, z).$$

This equality together with Proposition 7.4.2 proves the lemma. □

Lemma 7.4.4. *Let S be a range set with the countable coinitiality. Then a transmissible parameter $\mathfrak{G}$ is S-ultra-singular if and only if there exists an S-valued ultrametric (ω_0+1)-space with arbitrary small diameter satisfying the anti-$\mathfrak{G}$-transmissible property.*

Proof. Let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$. First assume that there exists an ultrametric $(\omega_0 + 1)$-space with arbitrary small diameter satisfying the anti-$\mathfrak{G}$-transmissible property. By the definition of the anti-$\mathfrak{G}$-transmissible property, we see that $\mathfrak{G}$ is S-ultra-singular.

Next assume that $\mathfrak{G}$ is S-ultra-singular. Take a non-zero strictly decreasing sequence $\{r(i)\}_{i \in \mathbb{N}}$ convergent to 0 as $i \to \infty$. Fix $\epsilon \in (0, \infty)$ and take a surjective map $\theta : \mathbb{N} \to Q$.

Take $N \in \mathbb{N}$ such that for every $n > N$ we have $r(n) < \epsilon$. Then there exists a sequence $\{(R_i, d_i)\}_{i\in\mathbb{N}}$ of finite ultrametric spaces such that for each $i \in \mathbb{N}$ there exist $z_i \in Z$ and an index $R_i = \{r_{i,j}\}_{j=1}^{\mathrm{card}(R_i)}$ satisfying

(1) $\delta_{d_i}(R_i) \le r(N + i)$;

(2) $\mathrm{card}(R_i) \in G(\theta(i))$;

(3) $\phi^{\theta(i),R_i}\left(\{r_{i,j}\}_{j=1}^{\mathrm{card}(R_i)}, z_i, d_i\right) \in P \setminus F(\theta(i))$.

In order to construct a desired space, we use the telescope construction (see Section 3.1). Put $r(\infty) = 0$. Define a metric on $\{r(n) \mid n \in \mathbb{N} \cup \{\infty\}\}$ by

$$D(r(n), r(m)) = \begin{cases} r(n) \vee r(m) & n \neq m; \\ 0 & \text{n=m}. \end{cases}$$

Let $C = \{r(n) \mid n \in \mathbb{N} \cup \{\infty\}\}$ and let $\mathbf{R} = \{(R_i, d_i)\}_{i\in\mathbb{N}}$. Then, the triple $\mathbf{C} = (C, D, 1_C)$ is a telescope base, and $\mathcal{L} = (\mathbf{R}, \mathbf{C})$ is a compatible pair. By Lemma 3.1.3, the telescope space $(T(\mathcal{L}), d_{\mathcal{L}})$ of $\mathcal{L}$ is a metric $(\omega_0 + 1)$-space with $\delta_{d_{T(\mathcal{L})}}(T(\mathcal{L})) \le \epsilon$. By Lemma 3.1.2, and by the definition of $d_{T(\mathcal{L})}$, the metric $d_{T(\mathcal{L})}$ is an S-valued ultrametric. By the properties (2) and (3) of $\{(R_i, d_i)\}_{i\in\mathbb{N}}$, the metric space $(T(\mathcal{L}), d_{\mathcal{L}})$ satisfies the anti-$\mathfrak{G}$-transmissible property. □

Let S be a range set. Let $\mathfrak{G}$ be a transmissible parameter. For a non-discrete ultrametrizable space X, and for an $(\omega_0 + 1)$-subspace R of X, we denote by $UT(X, S, R, \mathfrak{G})$ the set of all $d \in \mathrm{UM}(X, S)$ for which $(R, d|_{R^2})$ satisfies the anti-$\mathfrak{G}$-transmissible property.

Corollary 7.3.1 and Lemma 7.4.4 imply the following:

Proposition 7.4.5. *Let $C \in [1, \infty)$, and let S be a C-quasi-complete range set with the countable coinitiality. Let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$ be an S-ultra-singular transmissible parameter. Then for every non-discrete ultrametrizable space X, and for every $(\omega_0 + 1)$-subspace R of X, the set $UT(X, S, R, \mathfrak{G})$ is dense in the space* $(\mathrm{UM}(X, S), \mathcal{UD}_X^S)$.

Proof. Let $\epsilon \in (0, \infty)$ be an arbitrary number. Let $d \in \mathrm{UM}(X, S)$. Take an $(\omega_0 + 1)$-subspace R of X with $\delta_d(R) \le \epsilon$. By Lemma 7.4.4, there exists an S-valued ultrametric $e \in \mathrm{UM}(R, S)$ with $\delta_e(R) \le \epsilon$ such that (R, e) satisfies the anti-$\mathfrak{G}$-transmissible property. Since $\delta_d(R) \le \epsilon$ and $\delta_e(R) \le \epsilon$, by the definition of $\mathcal{UD}_R^S$ we have $\mathcal{UD}_R^S(d|_{R^2}, e) \le \epsilon$. Applying Corollary 7.3.1 to d and e, we obtain $m \in \mathrm{UM}(X, S)$ satisfying that

(1) $m|_{R^2} = e$;

(2) $\mathcal{UD}_X^S(d, m) \le C \cdot \mathcal{UD}_R^S(d|_{R^2}, e) \le C \cdot \epsilon$.

By Lemma 6.2.1, we see that (X, m) satisfies the anti-$\mathfrak{G}$-transmissible property. Since ϵ is arbitrary, the proposition follows. □

Proof of Theorem 1.2.16. Let $C \in [1, \infty)$, and let S be a C-quasi-complete range set with the countable coinitiality. Let X be a non-discrete metrizable space, and let $\mathfrak{G}$ be an S-ultra-singular transmissible parameter. Since X is non-discrete, there exists an $(\omega_0 + 1)$-subspace R of X. By the definitions, we have $UT(X, S, R, \mathfrak{G}) \subset US(X, S, \mathfrak{G})$. From Proposition 7.4.5 and Corollary 7.4.3, it follows that $US(X, S, \mathfrak{G})$ is dense G_δ in $(\mathrm{UM}(X, S), \mathcal{UD}_X^S)$. This finishes the proof. □

For a range set S, and for a complete metrizable space X, we denote by $\mathrm{CUM}(X,S)$ the set of all complete metrics in $\mathrm{UM}(X,S)$. From the latter part of Corollary 7.3.1, we deduce the following:

Theorem 7.4.6 ([58]). *Let S be a quasi-complete range set with the countable coinitiality. Let $\mathfrak{G}$ be an S-ultra-singular transmissible parameter. Then for every non-discrete completely ultrametrizable space X, the set of all $d \in \mathrm{CUM}(X,S)$ for which (X,d) satisfies the anti-$\mathfrak{G}$-transmissible property is dense G_δ in $\left(\mathrm{CUM}(X,S), \mathcal{UD}_X^S|_{\mathrm{CUM}(X,S)^2}\right)$.*

7.4.2 Proof of Theorem 1.2.17

Proof of Theorem 1.2.17. Let S be a quasi-complete range set with the countable coinitiality. Let X be a second countable, locally compact locally non-discrete ultrametrizable space, and let $\mathfrak{G}$ be an S-ultra-singular transmissible parameter. Put $\mathfrak{G} = (Q, P, F, G, Z, \phi)$. Let E be the set of all S-valued ultrametrics $d \in \mathrm{UM}(X,S)$ for which (X,d) satisfies the local anti-$\mathfrak{G}$-transmissible property. Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable open base of X, and let $\{R_i\}_{i\in\mathbb{N}}$ be a family of (ω_0+1)-subspaces of X with $R_i \subset U_i$. Since $\{U_i\}_{i\in\mathbb{N}}$ is an open base of X, by Lemma 6.2.1, we have

$$E = \bigcap_{i\in\mathbb{N}} \bigcap_{q\in Q} \bigcup_{z\in Z} \bigcup_{a\in \mathrm{Seq}(G(q),U_i)} US(X,S,\mathfrak{G},q,a,z).$$

Corollary 7.4.3 implies that E is G_δ in $\mathrm{UM}(X,S)$. By the definitions, for all $i \in \mathbb{N}$, the set

$$\bigcap_{q\in Q} \bigcup_{z\in Z} \bigcup_{a\in \mathrm{Seq}(G(q),U_i)} US(X,S,\mathfrak{G},q,a,z)$$

contains $UT(X,S,R_i,\mathfrak{G})$. From Proposition 7.4.5 it follows that each set $UT(X,S,R_i,\mathfrak{G})$ is dense in $\mathrm{UM}(X,S)$. By the latter part of Lemma 3.4.7, the space $\mathrm{UM}(X,S)$ is a Baire space. Since E is the intersection of countable dense G_δ sets in a Baire space $\mathrm{UM}(X,S)$, the set E is dense G_δ in $\mathrm{UM}(X,S)$. This completes the proof. □

7.5 Examples of S-singular transmissible properties

We show some examples of transmissible properties.

Similarly to Proposition 6.3.2, we obtain:

Proposition 7.5.1 ([58]). *Let S be a range subset with the countable coinitiality. The doubling property is a transmissible property with an S-ultra-singular parameter.*

Let S be a range set, and let T be a range subset of S which is countable dense subset of S. Let $\mathcal{U}_T$ be the class of all finite ultrametric spaces on which all distances are in T. We say that a metric space (X,d) *has rich S-ultra-pseudo-cones* if $\mathcal{U}_T$ is contained in the class $\mathcal{PC}(X,d)$ of all pseudo-cones of (X,d) for some countable dense range subset T of S.

Lemma 7.5.2. *Let S be a range set, and let T be a countable dense range subset of S. Let X be a finite discrete space, and let $d \in \mathrm{UM}(X,S)$. For every $\epsilon \in (0,\infty)$, there exists a T-valued ultrametric $e \in \mathrm{UM}(X,T)$ such that for all $x,y \in X$ we have $|d(x,y) - e(x,y)| < \epsilon$.*

Proof. Let $a_0, a_1, \dots, a_m$ be a sequence in S with $\{\, d(x,y) \mid x,y \in X \,\} = \{a_0, a_1, \dots, a_m\}$. We may assume that $a_0 = 0$ and $a_i < a_{i+1}$ for all i. Put $q_0 = a_0 (= 0)$. Since T is dense in S, we can take a sequence $q_1, \dots, q_m$ in T_+ such that $|a_i - q_i| < \epsilon$ and $q_i < q_{i+1}$ for all for all $i \in \{1, \dots, m\}$. Define a function $e : X \times X \to T$ by putting $e(x,y) = q_i$ if $d(x,y) = a_i$. Lemma 3.3.2 implies that e is an ultrametric. By the definition, the ultrametric e satisfies the conditions as required. □

Since every compact ultrametric space has an ϵ-net for all $\epsilon \in (0, \infty)$, Lemma 7.5.2 implies that for every range set S, every compact S-valued ultrametric space is arbitrarily approximated by members of $\mathcal{U}_T$ in the sense of Gromov–Hausdorff for every countable dense range subset T of S. Thus we have:

Corollary 7.5.3 ([58])**.** *Let S be a range set. Let (X, d) be an S-valued ultrametric space. Then the following are equivalent to each other:*

(1) *(X, d) has rich S-ultra-pseudo-cones;*

(2) *$\mathcal{PC}(X, d)$ contains all compact S-valued ultrametric spaces;*

(3) *$\mathcal{PC}(X, d)$ contains $\mathcal{U}_T$ for all countable dense range subset T of the range set S.*

In Proposition 6.3.6, it is proven that the rich pseudo-cones property is an anti-transmissible property with a singular parameter. Similarly, we obtain:

Proposition 7.5.4 ([58])**.** *Let S be a range set with the countable coinitiality. Then the rich S-ultra-pseudo-cones property is an anti-transmissible property with an S-ultra-singular transmissible parameter.*

Similarly to Corollary 6.3.10, we have:

Proposition 7.5.5. *Let S be a range set with the countable coinitiality. For $n \in \mathbb{N}$, let $f : P(n) \to \mathbb{R}$ be a continuous function. Then satisfying the (n, f)-metric inequality on metric spaces is a transmissible property. Moreover, if f is positively sub-homogeneous, and if there exists an ultrametric space not satisfying the (n, f)-metric inequality, then satisfying the (n, f)-metric inequality on metric spaces is a transmissible property with an S-ultra-singular transmissible parameter.*

We define continuous functions $s, t : P(3) \to \mathbb{R}$ by

$$s(x_{1,2}, x_{2,3}, x_{3,1}) = \max\{x_{1,2}, x_{2,3}, x_{3,1}\},$$

and

$$t(x_{1,2}, x_{2,3}, x_{3,1}) = \min\{x_{1,2}, x_{2,3}, x_{3,1}\}.$$

We also define a function $A : P(3) \to \mathbb{R}$ by

$$A(x_{1,2}, x_{2,3}, x_{3,1}) = \frac{2(s(x_{1,2}, x_{2,3}, x_{3,1}))^2 - (t(x_{1,2}, x_{2,3}, x_{3,1}))^2}{2(s(x_{1,2}, x_{2,3}, x_{3,1}))^2}.$$

Note that A is continuous. For every $\alpha \in (0, \pi/3)$, we define functions $f_\alpha, g_\alpha : P(3) \to \mathbb{R}$ by

$$f_\alpha(x_{1,2}, x_{2,3}, x_{3,1}) = \cos\alpha - A(x_{1,2}, x_{2,3}, x_{3,1}),$$

and by

$$g_\alpha(x_{1,2}, x_{2,3}, x_{3,1}) = A(x_{1,2}, x_{2,3}, x_{3,1}) - \cos\alpha.$$

Note that for every $\alpha \in (0, \pi/3)$, the functions f_α, g_α are continuous and positively sub-homogeneous.

Let (X, d) be an ultrametric space. For distinct three points $x, y, z \in X$, by the cosine formula, the value $A(d(x, y), d(y, z), d(z, x))$ is equal to the cosine of the apex angle of the triangle determined by $\{x, y, z\}$ (the angle between the sides with equal length of the triangle determined by $\{x, y, z\}$).

By Lemma 3.3.7, the value $A(d(x,y), d(y,z), d(z,x))$ is equal to or greater than $\cos(\pi/3)$. An ultrametric space (X, d) satisfies the $(3, f_\alpha)$-metric inequality (resp. $(3, g_\alpha)$-metric inequality) if and only if an angle between the legs of every triangle in (X, d) is equal to or greater than α (resp. less than α).

By Proposition 7.5.5, we have:

Proposition 7.5.6. *Let S be a range set with the countable coinitiality. Let $\alpha \in (0, \pi/3)$. Then the following two properties on ultrametric space are transmissible properties with S-ultra-singuler parameters:*

(1) *an angle between the legs (apex angle) of every triangle in a space is equal to or greater than α;*

(2) *an angle between the legs (apex angle) of every triangle in a space is equal to or less than α.*

As consequences of Theorems 1.2.16 and 1.2.17, we obtain:

Corollary 7.5.7. *Let S be a range set with the countable coinitiality. Let X be a non-discrete ultrametrizable space. Then the following sets are dense G_δ in the space $\mathrm{UM}(X, S)$ of metrics:*

(1) *the set of all metrics $d \in \mathrm{UM}(X, S)$ for which (X, d) has infinite Assouad dimension;*

(2) *the set of all metrics $d \in \mathrm{UM}(X, S)$ for which $\mathcal{PC}(X, d)$ contains all compact S-valued ultrametric spaces;*

(3) *for each $\alpha \in (0, \pi/3)$, the set of all metrics $d \in \mathrm{UM}(X, S)$ for which (X, d) contains a triangle whose apex angle is less than α;*

(4) *for each $\alpha \in (0, \pi/3)$, the set of all metrics $d \in \mathrm{UM}(X, S)$ for which (X, d) contains a triangle whose apex angle is greater than α.*

Corollary 7.5.8. *Let S be a range set with the countable coinitiality. Let X be a second countable locally compact locally non-discrete ultrametrizable space. Then the following sets are dense G_δ in the space $\mathrm{UM}(X, S)$ of metrics:*

(1) *the set of all metrics $d \in \mathrm{UM}(X, S)$ for which every non-empty open subspace of (X, d) has infinite Assouad dimension;*

(2) *the set of all metrics $d \in \mathrm{UM}(X, S)$ for which every non-empty open subspace contains all compact S-valued ultrametrics as its pseudo-cones;*

(3) *for each $\alpha \in (0, \pi/3)$, the set of all metrics $d \in \mathrm{UM}(X, S)$ for which every non-empty open metric subspace f (X, d) contains a triangle whose apex angle is less than α;*

(4) *for each $\alpha \in (0, \pi/3)$, the set of all metrics $d \in \mathrm{UM}(X, S)$ for which every non-empty open metric subspace of (X, d) contains a triangle whose apex angle is greater than α.*

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