Twisted forms of differential Lie algebras over $\mathbb{C}(t)$ associated with complex simple Lie algebras

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Abstract

Descent theory discusses twisted forms of a fixed object A defined over a base ring R, where the twisted forms are R-objects which turn to be isomorphic to A after some faithfully flat base extension of R. Here discussed is descent theory in the differential context in which everything is equipped with a differential operator. We determine all twisted forms of those differential Lie algebras over the differential field $\mathbb{C}(t)$ which are associated with complex simple Lie algebras; this solves the problem raised by A. Pianzola. Our crucial technical ingredient is Hopf-Galois Theory, a ring-theoretic counterpart of theory of torsors for group schemes; it plays an essential role when we grasp the above-mentioned twisted forms from torsors. As an important result we use Steinberg's cohomology-vanishing theorem. But we prove that its differential analogue does not hold.

Contents

1 Introduction—problem and answer

Rings and algebras are supposed to be associative and containing 1, and their morphisms are supposed to send 1 to 1. Moreover, rings, algebras and Hopf algebras are assumed to be commutative, unless otherwise stated.

We let δ mean "differential" and use the symbol δ to indicate differential operators in general. A *δ-ring* is thus a (commutative) ring R equipped with an additive operator $\delta : R \to R$ satisfying the Leibniz rule

$$\delta(xy) = (\delta x)y + x(\delta y)$$

for all $x, y \in R$. It is called a *δ-field* if the ring is a field. The rational function field $\mathbb{C}(t)$ in one variable is regarded as a δ-field with respect to the standard operator such that $\delta t = 1$ and $\delta c = 0$ for every $c \in \mathbb{C}$. The field

$$C_{\mathbb{C}(t)} = \{\, a \in \mathbb{C}(t) \mid \delta a = 0 \,\}$$

of constants is $\mathbb{C}$. A *δ-$\mathbb{C}(t)$-Lie algebra* is a Lie algebra $\mathfrak{g}$ over $\mathbb{C}(t)$ which is equipped with an additive operator $\delta : \mathfrak{g} \to \mathfrak{g}$ such that

$$\delta(aX) = (\delta a)X + a(\delta X), \quad \delta[X, Y] = [\delta X, Y] + [X, \delta Y]$$

for all $a \in \mathbb{C}(t)$ and $X, Y \in \mathfrak{g}$. In the same way a *δ-R-Lie algebra* is defined for any δ-ring R. Let $\mathfrak{g}$ be a δ-$\mathbb{C}(t)$-Lie algebra. Given a δ-ring map $\mathbb{C}(t) \to R$ (that is, a ring map preserving the δ-operator), the base extension $\mathfrak{g} \otimes_{\mathbb{C}(t)} R$ of $\mathfrak{g}$ is naturally a δ-R-Lie algebra. A *twisted form* of $\mathfrak{g}$ is a δ-$\mathbb{C}(t)$-Lie algebra $\mathfrak{f}$ such that

$$\mathfrak{g} \otimes_{\mathbb{C}(t)} R \simeq \mathfrak{f} \otimes_{\mathbb{C}(t)} R \quad \text{as } \delta\text{-}R\text{-Lie algebras}$$

for some δ-ring map $\mathbb{C}(t) \to R$ with $R \neq 0$.

Let $n \geq 2$. We can and do regard the $\mathbb{C}(t)$-Lie algebra $\mathfrak{sl}_n(\mathbb{C}(t))$ which consists of all traceless matrices $X = \big(x_{ij}\big)$ with $x_{ij} \in \mathbb{C}(t)$, as a δ-$\mathbb{C}(t)$-Lie algebra with respect to the entry-wise δ-operator $\delta\big(x_{ij}\big) := \big(\delta x_{ij}\big)$.

As an obvious generalization, one can replace $\mathfrak{sl}_n(\mathbb{C})$ with a complex simple Lie algebra $\mathfrak{g}_0$ (of finite dimension), and regard $\mathbb{C}(t)$-Lie algebra

$$\mathfrak{g}_0(\mathbb{C}(t)) = \mathfrak{g}_0 \otimes_{\mathbb{C}} \mathbb{C}(t) \tag{1.1}$$

as a δ-$\mathbb{C}(t)$-Lie algebra with the δ operating on the tensor factor $\mathbb{C}(t)$. The notation (1.1) is used since $\mathfrak{g}_0$ is seen to give the functor $R \mapsto \mathfrak{g}_0 \otimes_{\mathbb{C}} R$, and $\mathfrak{g}_0 \otimes_{\mathbb{C}} \mathbb{C}(t)$ is then its value. A. Pianzola raised the following problem, which we are going to solve.

Problem 1.1. *Given a complex simple Lie algebra $\mathfrak{g}_0$, describe all twisted forms of $\mathfrak{g}_0(\mathbb{C}(t))$.*

Our interest in this problem or in studying twisted forms of Lie algebras in the differential context comes from differential Galois Theory, in which examples of such twisted forms naturally arise. In fact, given a homogeneous linear differential equation with coefficients in a differential field, say $\mathbb{C}(t)$, we have the *Galois group* of the equation, G_0, which is an affine algebraic $\mathbb{C}$-group. In addition we naturally have an affine algebraic "differential" $\mathbb{C}(t)$-group G, called the *intrinsic Galois group* (or *Katz group*); see [6]. One sees that the Lie algebra $\mathrm{Lie}(\mathsf{G})$ of G is a δ-$\mathbb{C}(t)$-Lie algebra which is a twisted form of the δ-$\mathbb{C}(t)$-Lie algebra $\mathrm{Lie}(\mathsf{G}_0) \otimes_{\mathbb{C}} \mathbb{C}(t)$, where $\mathrm{Lie}(\mathsf{G}_0)$ is the (complex) Lie algebra of G_0.

Returning to the situation before, recall that the complex simple Lie algebras are classified, labeled by their root systems

$$\mathrm{A}_\ell\ (\ell \geq 1),\ \mathrm{B}_\ell\ (\ell \geq 2),\ \mathrm{C}_\ell\ (\ell \geq 3),\ \mathrm{D}_\ell\ (\ell \geq 4),\ \mathrm{E}_6,\ \mathrm{E}_7,\ \mathrm{E}_8,\ \mathrm{F}_4,\ \mathrm{G}_2.$$

See [4, Chapter IV], for example. Let $\mathfrak{g}_0$ be a complex simple Lie algebra, and let Γ denote the automorphism group of the associated Dynkin diagram. Explicitly, the group is

$$\Gamma = \begin{cases} \{1\} & \text{type } \mathrm{A}_1,\ \mathrm{B}_\ell\ (\ell \geq 2),\ \mathrm{C}_\ell\ (\ell \geq 3),\ \mathrm{E}_7,\ \mathrm{E}_8,\ \mathrm{F}_4 \text{ or } \mathrm{G}_2; \\ \mathbb{Z}_2 & \text{type } \mathrm{A}_\ell\ (\ell \geq 2),\ \mathrm{D}_\ell\ (\ell \geq 5)\ \text{ or }\ \mathrm{E}_6; \\ \mathfrak{S}_3 & \text{type } \mathrm{D}_4 \end{cases} \tag{1.2}$$

according to the type of $\mathfrak{g}_0$; see [11, Table 3 on Page 298]. Here and in what follows $\mathbb{Z}_n$ denotes the cyclic group of order n. In addition, $\mathfrak{S}_3$ denotes the symmetric group of degree 3. The action by Γ naturally (up to conjugation) gives rise to automorphisms of $\mathfrak{g}_0$, which forms a group naturally identified with the group $\mathsf{Out}(\mathfrak{g}_0)$ of outer-automorphisms of $\mathfrak{g}_0$.

Roughly speaking, our answer, Theorem 1.4, to the problem tells that all non-trivial twisted forms are obtained by the Galois descent (see [7, Section 18]) for which Γ (and its subgroups for type D_4) act as Galois groups. To make a precise statement we introduce below the notion of being *quasi-isomorphic*.

Lemma 1.2. *If $\mathfrak{g} = (\mathfrak{g}, \delta)$ is a δ-R-Lie algebra, then for any element $D \in \mathfrak{g}$,*

$$\delta + \mathsf{ad}(D) : \mathfrak{g} \to \mathfrak{g},\ X \mapsto \delta X + [D, X]$$

is a δ-operator with which $\mathfrak{g}$ is again a δ-R-Lie algebra.

Indeed, one sees, more generally, that for any R-linear derivation $\mathfrak{D} : \mathfrak{g} \to \mathfrak{g}$, $(\mathfrak{g}, \delta + \mathfrak{D})$ is a δ-R-Lie algebra. Note that the inner derivation $\mathsf{ad}(D)$ above is R-linear.

Definition 1.3. Let R be as above, We say that two δ-R-Lie algebras $\mathfrak{g} = (\mathfrak{g}, \delta)$ and $\mathfrak{g}' = (\mathfrak{g}', \delta')$ are *quasi-isomorphic*, if there is an element $D \in \mathfrak{g}$ such that

$$(\mathfrak{g}, \delta + \mathsf{ad}(D)) \simeq (\mathfrak{g}', \delta') \quad \text{as } \delta\text{-}R\text{-Lie algebras.}$$

The condition is equivalent to saying that there is an element $D' \in \mathfrak{g}'$ such that $(\mathfrak{g}, \delta) \simeq (\mathfrak{g}', \delta' + \mathsf{ad}(D'))$, as is easily seen. It follows that the quasi-isomorphism gives an equivalence relation among all δ-R-Lie algebras.

Theorem 1.4. *Suppose that $\mathfrak{g}_0$ is a complex simple Lie algebra, and let Γ be the automorphism group of the associated Dynkin diagram. Then a δ-$\mathbb{C}(t)$-Lie algebra is a twisted form of $\mathfrak{g}_0(\mathbb{C}(t))$ if and only if it is quasi-isomorphic to one of those listed below, according to the case $\Gamma = \{1\}$, $\mathbb{Z}_2$ or $\mathfrak{S}_3$; see* (1.2).

(1) *Case* $\Gamma = \{1\}$: $\mathfrak{g}_0(\mathbb{C}(t))$;

(2) *Case* $\Gamma = \mathbb{Z}_2$: (i) $\mathfrak{g}_0(\mathbb{C}(t))$;

(ii) $\mathfrak{g}_0(L)^\Gamma$, *where $L/\mathbb{C}(t)$ is a quadratic field extension.*

(3) *Case* $\Gamma = \mathfrak{S}_3$: (i) $\mathfrak{g}_0(\mathbb{C}(t))$;

(ii) $\mathfrak{g}_0(L)^{\mathbb{Z}_2}$, *where $L/\mathbb{C}(t)$ is a quadratic field extension*;

(iii) $\mathfrak{g}_0(L)^{\mathbb{Z}_3}$, *where $L/\mathbb{C}(t)$ is a cubic Galois extension*;

(iv) $\mathfrak{g}_0(L)^\Gamma$, *where $L/\mathbb{C}(t)$ is a Galois extension of fields with Galois group $\Gamma(= \mathfrak{S}_3)$.*

We should immediately add some explanations about the statement above. First, any finite field extension $L/\mathbb{C}(t)$ uniquely turns into an extension of δ-fields, whence $\mathfrak{g}_0(L)$ turns into a δ-L-Lie algebra. Second, in (ii) of (2) and (iv) of (3) above, the group Γ is supposed to act diagonally on $\mathfrak{g}_0(L) = \mathfrak{g}_0 \otimes_{\mathbb{C}} L$, as outer-automorphisms on $\mathfrak{g}_0$, and as the Galois group on L. In addition, $\mathfrak{g}_0(L)^\Gamma$ denotes the Γ-invariants in $\mathfrak{g}_0(L)$, which is in fact a δ-$\mathbb{C}(t)$-Lie algebra by Galois descent; see Section 4.3. Third, in (ii) of (3), we choose arbitrarily an order 2 subgroup $\mathbb{Z}_2$ of $\Gamma\,(= \mathfrak{S}_3)$, and let it act on $\mathfrak{g}_0$ by restriction. The δ-$\mathbb{C}(t)$-Lie algebra $\mathfrak{g}_0(L)^{\mathbb{Z}_2}$ which results in the same way as above does not depend (up to isomorphism) on the choice since the order 2 subgroups are conjugate to each other; on the other hand it may depend on L. Finally, in (iii) of (3), we suppose that $\mathbb{Z}_3$ is the unique order 3 subgroup of $\Gamma\,(= \mathfrak{S}_3)$, and let it act on $\mathfrak{g}_0$ by restriction, again. We add the following remark: there exist infinitely many quadratic and cubic Galois extensions over $\mathbb{C}(t)$, as is easily seen, while the existence of a Galois extension over $\mathbb{C}(t)$ with Galois group $\mathfrak{S}_3$ will be ensured by Example 4.9. The theorem will be proved in the final Section 4, which contains as well, explicit descriptions (see Section 4.6) of the non-trivial δ-$\mathbb{C}(t)$-Lie algebras listed in (ii) of (2) and (ii)-(iv) of (3). The preceding two sections provide preliminaries, some of which are beyond what will be needed, but are of interest by themselves. Section 2 presents descent theory in the differential context; Section 3 prepares technical tools mainly from Hopf-Galois Theory, which is a ring-theoretic counterpart of theory of torsors for group schemes. In particular, Schauenburg's bi-Galois Theory [13] will play a role in two stages (see Sections 3.5 and 4.3), when we grasp the twisted forms in question from δ-torsors.

In Case $\Gamma = \{1\}$, every twisted form of $\mathfrak{g}_0(\mathbb{C}(t))$ will be proved, as presented above, to be quasi-isomorphic to $\mathfrak{g}_0(\mathbb{C}(t))$. In the epilogue we prove that among those twisted forms, there is one which is not isomorphic to $\mathfrak{g}_0(\mathbb{C}(t))$. Thus "quasi-isomorphism" is not necessarily "isomorphism".

2 δ-R-Objects and Descent theory in differential context

Let R be a δ-ring. A *δ-R-module* is an R-module M equipped with an additive operator $\delta : M \to M$ satisfying

$$\delta(xm) = (\delta x)m + x(\delta m), \quad x \in R,\ m \in M.$$

All δ-R-modules form a symmetric tensor category δ-R-$\texttt{Modules}$ with respect to the tensor product $M_1 \otimes_R M_2$, the unit object R and the obvious symmetry

$$M_1 \otimes_R M_2 \to M_2 \otimes_R M_1, \quad m_1 \otimes m_2 \mapsto m_2 \otimes m_1.$$

The δ-operator on $M_1 \otimes_R M_2$ is given by

$$\delta(m_1 \otimes m_2) = \delta m_1 \otimes m_2 + m_1 \otimes \delta m_2.$$

The notion of δ-R-Lie algebra defined in the previous section is precisely a Lie algebra in the category δ-R-$\texttt{Modules}$. In general, any linear object, such as algebra or Hopf algebra, in δ-R-$\texttt{Modules}$ is called a *δ-R-object*, so as *δ-R-algebra* or *δ-R-Hopf algebra*; important is the fact that the structure is defined by morphisms of δ-R-$\texttt{Modules}$ between tensor powers of the object. Given a δ-R-algebra S, we have the base-extension functor

$$\otimes_R S : \delta\text{-}R\text{-}\texttt{Modules} \to \delta\text{-}S\text{-}\texttt{Modules},$$

which induces base-extension functors for linear objects such as above.

We are concerned with descent theory (see [15], for example) in differential context. To make this clearer, let us fix a δ-R-object A. A δ-R-object B is called an *S/R-form* of A, or a *twisted form* of A *split by* S, if S is a δ-R-algebra such that

(i) S is faithfully flat as an R-algebra, and

(ii) $A \otimes_R S \simeq B \otimes_R S$ as δ-S-objects.

A δ-R-object B is called a *twisted form* of A, if there exists a δ-R-algebra S which satisfies (i) and (ii) above.

The *δ-automorphism-group functor* of A is the functor

$$\mathsf{Aut}_\delta(A) : \delta\text{-}R\text{-}\texttt{Algebras} \to \texttt{Groups}, \quad T \mapsto \mathrm{Aut}_{\delta\text{-}T}(A \otimes_R T) \tag{2.1}$$

from the category δ-R-$\texttt{Algebras}$ of δ-R-algebras to the category $\texttt{Groups}$ of groups, which associates to each δ-R-algebra T, the automorphism-group $\mathrm{Aut}_{\delta\text{-}T}(A \otimes_R T)$ of the δ-T-object $A \otimes_R T$. When constructing the 1st Amitsur cohomology (pointed) set as in [15, Section 17.6], replace faithfully flat homomorphisms of rings and automorphism-group functors with our $R \to S$ (satisfying (i) above) and $\mathsf{Aut}_\delta(A)$, respectively. The resulting differential analogue is denoted by

$$H^1_\delta(S/R, \mathsf{Aut}_\delta(A)). \tag{2.2}$$

This is seen to classify the S/R-forms of A; to be more precise, there is a natural bijection from $H^1_\delta(S/R, \mathsf{Aut}_\delta(A))$ to the set of all δ-R-isomorphism classes of the S/R-forms. An important consequence is: if A' is another δ-R-object of some distinct kind, which has the δ-automorphism-group functor isomorphic to $\mathsf{Aut}_\delta(A)$, then there is a one-to-one correspondence (up to isomorphism) between the S/R-forms of A and S/R-forms of A'.

We remark that for any functor $\mathsf{G} : \delta\text{-}R\text{-}\mathtt{Algebras} \to \mathtt{Groups}$, the cohomology set $H^1_\delta(S/R, \mathsf{G})$ is defined just as the one in (2.2). The set will appear in what follows (see (3.6)) only when G is representable, and turns out, indeed, to be an automorphism-group functor.

Remark 2.1. We have used so far the *base-on-right* notation $A \otimes_R T$ which denotes the extended base on the right; it seemingly looks nicer than the *base-on-left* notation $T \otimes_R A$. But we may and do (when it is natural) use the latter notation.

Remark 2.2. We would like to clarify our use of the term "form". In this Remark the differential structure is ignored. Consider a faithfully flat homomorphism $R \to S$ of rings and an R-object A. If R and S are fields, it is more conventional for experts in representation theory or physicists to speak of R-form of the S-object $A \otimes S$. For example, $\mathfrak{so}_3(\mathbb{R})$ is a real form of $\mathfrak{sl}_2(\mathbb{C})$. Our terminology is that $\mathfrak{so}_3(\mathbb{R})$ is a twisted form, or a $\mathbb{C}/\mathbb{R}$-form of $\mathfrak{sl}_2(\mathbb{R})$. The terminology we have chosen, namely an S/R-form or twisted form of the R-object A, is familiar in number theory and algebraic geometry, and it is also the standard terminology in Grothendieck's descent theory. The base R is fixed, and the S can vary.

3 Affine δ-K-groups, their Lie algebras and torsors

In this section K denotes a δ-field. We assume that the characteristic $\operatorname{char} K$ of K is zero.

3.1 Affine δ-K-groups and their Lie algebras

An *affine δ-K-group scheme* is by definition a representable functor

$$\mathsf{G} : \delta\text{-}K\text{-}\mathtt{Algebras} \to \mathtt{Groups} \quad \text{(see (2.1))};$$

this will be called an *affine δ-K-group* for short. Such a functor G is uniquely represented by a δ-K-Hopf algebra, say H, and is presented so as $\mathsf{G} = \operatorname{Spec}_\delta(H)$ or $\operatorname{Spec}_{\delta\text{-}K}(H)$. We say that G is *algebraic*, or it is an *affine algebraic δ-K-group*, if H is finitely generated as a K-algebra. If one forgets δ, then $\mathsf{G} = \operatorname{Spec}(H)$ is an affine K-group, which has the Lie algebra

$$\operatorname{Lie}(\mathsf{G}) = \operatorname{Der}_\epsilon(H, K). \tag{3.1}$$

Recall that this consists of all K-linear maps $D : H \to K$ that satisfy

$$D(ab) = D(a)\epsilon(b) + \epsilon(a)D(b), \quad a, b \in H,$$

where $\epsilon : H \to K$ is the counit of H. This is in fact a δ-K-Lie algebra with respect to the operator defined by

$$(\delta D)(a) := \delta(Da) - D(\delta a), \quad D \in \mathrm{Lie}(\mathsf{G}),\ a \in H.$$

Note that the canonical pairing $H \otimes_K \mathrm{Lie}(\mathsf{G}) \to K$ is a morphism in δ-K-`Modules`. We have $\dim_K(\mathrm{Lie}(\mathsf{G})) < \infty$, if G is algebraic.

Remark 3.1. The notion of being "algebraic" defined above would be rather restricted for those who would like to work intensively in differential algebra. It should be distinguished from the more natural (for those above) notion of being "δ-algebraic", which will be discussed briefly in Section 3.3, being less crucial for our purpose though.

3.2 δ-K-Torsors and Galois δ-K-algebras

An *affine δ-K-scheme* is by definition a representable set-valued functor

$$\mathsf{X} : \delta\text{-}K\text{-}\mathtt{Algebras} \to \mathtt{Sets}.$$

It is uniquely represented by a δ-K-algebra, say A, being presented so as $\mathsf{X} = \mathrm{Spec}_\delta(A)$; it is said to be *algebraic* if A is finitely generated as a K-algebra. The category of affine δ-K-schemes, whose morphisms are natural transformations, has direct products. The direct product $\mathsf{X}_1 \times \mathsf{X}_2$ of two affine δ-K-schemes $\mathsf{X}_i = \mathrm{Spec}_\delta(A_i)$, $i = 1, 2$, is represented by $A_1 \otimes_K A_2$. The notion of *group object* of the category is naturally defined, and such an object is precisely an affine δ-K-group. Given an affine δ-K-group $\mathsf{G} = \mathrm{Spec}_\delta(H)$, the notion of right (or left) G-equivariant objects is defined, as well. Such an object is called a *right* (or *left*) *G-equivariant δ-K-scheme*. Giving such a δ-K-scheme $\mathsf{X} = \mathrm{Spec}_\delta(A)$ is the same as giving a *right* (or *left*) *H-comodule δ-K-algebra*; it is an object A in δ-K-`Algebras` equipped with a morphism $A \to A \otimes_K H$ (or $A \to H \otimes_K A$) in the category which satisfy the co-associativity and the counit property. Obviously, G itself is G-equivariant on both sides.

Let R be a δ-K-algebra. An affine δ-K-group or (equivariant or ordinary) δ-K-scheme $\mathsf{X} = \mathrm{Spec}_\delta(A)$ has the base change $\mathsf{X}_R = \mathrm{Spec}_{\delta\text{-}R}(A \otimes_K R)$; it is by definition the functor $T \mapsto \mathsf{X}(T)$ defined on δ-R-`Algebras`, where each $T \in \delta$-R-`Algebras` is regarded naturally as a δ-K-algebra. We can discuss twisted forms of X; it is the same as discussing twisted forms of A.

Let $\mathsf{G} = \mathrm{Spec}_\delta(H)$ be an affine δ-K-group. A twisted form of the right G-equivariant δ-K-scheme G is called a *right δ-K-torsor* for G. To be explicit it is a right G-equivariant δ-K-scheme X such that $\mathsf{X}_R \simeq \mathsf{G}_R$ as right G-equivariant δ-R-schemes for some non-zero δ-K-algebra R. Such an X is uniquely represented by a right H-comodule δ-K-algebra B which is a twisted form of H. Such a twisted form B is characterized as a *right H-Galois δ-K-algebra* [9, Section 8.1]; it is by definition a non-zero right H-comodule δ-K-algebra B such that the δ-K-algebra map

$$\tilde{\rho} : B \otimes_K B \to B \otimes_K H, \quad \tilde{\rho}(b \otimes c) = b\rho(c) \tag{3.2}$$

is an isomorphism. Here and in what follows, $\rho : B \to B \otimes_K H$ denotes the structure map. Note that $\tilde{\rho}$ is a δ-B-algebra isomorphism (with the base-on-left notation, see Remark 2.1), and B is split by B itself.

The analogous notions of *left δ-K-torsors for* G and of *left H-Galois δ-K-algebras* are defined in the obvious manner, and those two are in one-to-one correspondence.

3.3 Affine δ-algebraic δ-K-groups

An δ-K-algebra A is said to be *δ-finitely generated* if it is generated as a K-algebra by finitely many elements $a_1, \ldots, a_n$ together with their iterated differentials $\delta^r a_1, \ldots, \delta^r a_n$, $r > 0$. An extension L/K of δ-fields said to be *δ-finitely generated*, if L is the quotient field of some δ-K-finitely generated δ-K-subalgebra of L.

An affine δ-group $\mathsf{G} = \mathrm{Spec}_\delta(H)$ is said to be *δ-algebraic* if the δ-K-Hopf algebra H is δ-finitely generated as a δ-K-algebra; see Remark 3.1. Obviously, "algebraic" implies "δ-algebraic".

Lemma 3.2. *Every right (or left) δ-K-torsor for an affine δ-K-group G is split by some δ-K-field. It is split by a δ-finitely generated extension L/K of δ-fields, if G is δ-algebraic.*

Proof. Suppose that B is a right H-Galois δ-K-algebra, as above. Choose arbitrarily a maximal δ-stable ideal $\mathfrak{m}$ of B, and construct $R = B/\mathfrak{m}$, a simple δ-K-ring. Since $\mathrm{char}\, K = 0$, R is an integral domain by [12, Lemma 1.17]. The quotient field $L = Q(R)$ of R uniquely turns into a δ-K-field. By applying $L \otimes_B$ to $\tilde{\rho}$, it follows that B is split by L, proving the first assertion. If H is δ-finitely generated, then B and R are so. It follows that the L/K above is δ-finitely generated, proving the second. $\square$

Proposition 3.3. *Suppose that A is δ-K-object of finite K-dimension. Then $\mathsf{Aut}_\delta(A)$ is an affine δ-algebraic δ-K-group, and every twisted form of A is split by some δ-finitely generated extension L/K of δ-fields.*

Proof. We have only to prove that $\mathsf{Aut}_\delta(A)$ is an affine δ-algebraic δ-K-group, since the rest then follows from the preceding Lemma.

Choose a K-basis $v_1, \ldots, v_n$ of A. Let

$$F = K[x_{ij}, x'_{ij}, x''_{ij}, \ldots, x^{(r)}_{ij}, \ldots]$$

denote the free δ-K-algebra in indeterminates x_{ij}, where $1 \leq i, j \leq n$. Let

$$G = F_d \,(= F[1/d])$$

denote the localization by the determinant $d = \det X$ of the $n \times n$ matrix $X = \big(x_{ij}\big)_{i,j}$ which has the indeterminates above as entries. This G has the δ-operator uniquely extending the one $\delta x^{(r)}_{ij} = x^{(r+1)}_{ij}$, $r \geq 0$, on F. We have a G-linear bijection $\phi : A \otimes_K G \to A \otimes_K G$ determined by

$$\phi(v_j \otimes 1) = \sum_{i=1}^{n} v_i \otimes x_{ij}, \quad 1 \leq j \leq n.$$

This is alternatively expressed as

$$\phi(v_1 \otimes 1, \dots, v_n \otimes 1) = (v_1, \dots, v_n) \otimes X$$

by matrix presentation; such presentation will be used in (3.3), (3.4) and (3.5), as well.

Let

$$H = G/\mathfrak{a},$$

where $\mathfrak{a}$ is the smallest δ-stable ideal of G such that the base extension $\phi_H : A \otimes_K H \to A \otimes_K H$ of ϕ along $G \to G/\mathfrak{a} = H$ is an endomorphism of the δ-H-object $A \otimes_K H$; obviously, it is necessarily an automorphism. This $\mathfrak{a}$ is, in fact, given by the relations which ensure that ϕ_H commutes with the structure maps of A (see [15, Section 7.6]), and with the δ-operator. Explicitly, the latter relation for commuting with δ-operator is

$$XD = DX + \delta X, \tag{3.3}$$

where $D \in M_n(K)$ is the matrix determined by

$$\delta(v_1, \dots, v_n) = (v_1, \dots, v_n)D. \tag{3.4}$$

We see that H represents the functor $\mathsf{Aut}_\delta(A)$ regarded to be set-valued. In fact, for every $R \in \delta\text{-}K\text{-}\mathtt{Algebras}$, we have the natural bijection

$$\mathrm{Spec}_\delta(H)(R) \to \mathrm{Aut}_{\delta\text{-}R}(A \otimes_K R), \quad f \mapsto \text{ the base extension of } \phi_H \text{ along } f.$$

By Yoneda's Lemma, H uniquely turns into a δ-K-Hopf algebra with respect to the familiar Hopf-algebra structure

$$\Delta X = X \otimes X, \quad \epsilon X = I, \quad \mathcal{S}X = X^{-1}, \tag{3.5}$$

where Δ, ϵ and $\mathcal{S}$ denote the coproduct, the counit and the antipode, respectively, and it represents the group-valued functor $\mathsf{Aut}_\delta(A)$. Since H is obviously δ-finitely generated, the desired result follows. □

For K as above, we choose and fix an extension $\mathcal{U}/K$ of δ-fields into which every δ-finitely generated extension L/K of δ-fields can be embedded. There exists such an extension; a universal extension [8, Chapter III, Section 7] over K is an example.

For an affine δ-algebraic δ-K-group G, we define $H^1_\delta(K, \mathsf{G})$ by

$$H^1_\delta(K, \mathsf{G}) := H^1_\delta(\mathcal{U}/K, \mathsf{G}). \tag{3.6}$$

The δ-automorphism-group functor $\mathsf{Aut}_\delta(\mathsf{G}) : T \mapsto \mathrm{Aut}_{\delta\text{-}T}(\mathsf{G}_T)$ of the right G-equivariant δ-K-scheme G is naturally isomorphic to G itself; $\mathrm{Aut}_{\delta\text{-}T}(\mathsf{G}_T)$ consists of the natural automorphisms of the functor $\mathsf{G}_T : \delta\text{-}T\text{-}\mathtt{Algebras} \to \mathtt{Groups}$. This fact, combined with Lemma 3.2, shows that $H^1_\delta(K, \mathsf{G})$ classifies all right δ-K-torsors for G.

For a δ-K-object A of finite K-dimension, we define

$$H^1_\delta(K, \mathsf{Aut}_\delta(A)) := H^1_\delta(\mathcal{U}/K, \mathsf{Aut}_\delta(A)).$$

This classifies all twisted forms of A, as is seen from Proposition 3.3.

3.4 δ-K-Bi-torsors and bi-Galois δ-K-algebras

Let $\mathsf{G} = \mathrm{Spec}_\delta(H)$ be an affine δ-K-group. Suppose that $\mathsf{X} = \mathrm{Spec}_\delta(B)$ is a right δ-K-torsor for G, or in other words, $B = (B, \rho)$ is a right H-Galois δ-K-algebra. Tracing the argument of [13] modified into our differential situation, we see that there exists uniquely (up to isomorphism) a pair (H', λ) of a δ-K-Hopf algebra H' and a left H'-comodule δ-K-algebra structure $\lambda : B \to H' \otimes_K B$ such that (i) (B, λ) is a left H'-Galois δ-K-algebra, and (ii) λ and ρ commute in the sense that

$$(\lambda \otimes \mathrm{id}_H) \circ \rho = (\mathrm{id}_{H'} \otimes \rho) \circ \lambda. \tag{3.7}$$

We say that B is an (H', H)-*bi-Galois δ-K-algebra*. Accordingly, we have uniquely a pair of an affine δ-K-group G' and its action on X from the left, such that (i) X is a left δ-K-torsor for G', and (ii) the actions on X by G' and by G commute with each other. We say that X is a *δ-K-bi-torsor*. We write

$$H^B, \quad \mathsf{G}^{\mathsf{X}} \tag{3.8}$$

for H', G', respectively. If B (or equivalently, X) is trivial, or namely if $B = H$ (or $\mathsf{X} = \mathsf{G}$), then $H^B = H$ and $\mathsf{G}^{\mathsf{X}} = \mathsf{G}$. This, applied after base extension to B, shows the following; see the proof of Proposition 3.5 below for detailed argument.

Proposition 3.4. *H^B and G^{X} are B/K-forms of H and of G, respectively.*

With K replaced by a non-zero δ-ring R, the results above remain true if the relevant δ-R-Hopf algebra is flat over R. We remark that δ-R-torsors are then required, in addition to the $\tilde{\rho}$ being isomorphic, to be faithfully flat over R.

3.5 Interpretation of $H^1_\delta(K, \mathsf{G}) \to H^1_\delta(K, \mathsf{Aut}_\delta(\mathfrak{g}))$

Let $\mathsf{G} = \mathrm{Spec}_\delta(H)$ be an affine algebraic δ-K-group, and set $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$. Then $\mathfrak{g}$ is a δ-K-Lie algebra of finite K-dimension, whence the δ-automorphism-group functor $\mathsf{Aut}_\delta(\mathfrak{g})$ is an affine δ-algebraic δ-K-group by Proposition 3.3. We see that the left adjoint action by G on $\mathfrak{g}$ gives rise to a morphism of affine δ-algebraic δ-K-groups

$$\mathsf{Ad} : \mathsf{G} \to \mathsf{Aut}_\delta(\mathfrak{g}),$$

which induces naturally a map between the cohomology sets

$$\mathsf{Ad}_* : H^1_\delta(K, \mathsf{G}) \to H^1_\delta(K, \mathsf{Aut}_\delta(\mathfrak{g})). \tag{3.9}$$

Given a right δ-K-torsor X for G, we define

$$\mathfrak{g}^{\mathsf{X}} := \mathrm{Lie}(\mathsf{G}^{\mathsf{X}}). \tag{3.10}$$

This is a twisted form of $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$, since G^{X} is a twisted form of G; see Proposition 3.4.

Proposition 3.5. Ad_* *is interpreted in terms of twisted forms so as*

$$[\textit{a right } \delta\text{-}K\textit{-torsor } \mathsf{X} \textit{ for } \mathsf{G}] \ \mapsto \ [\mathfrak{g}^{\mathsf{X}}], \tag{3.11}$$

where [ ] *indicates isomorphism classes.*

Proof. In this proof we write $\otimes$ for $\otimes_K$, and use the base-on-left notation for base extensions; see Remark 2.1.

Suppose that $\mathsf{X} = \mathrm{Spec}_\delta(B)$ is a right δ-K-torsor for G, or in other words, $B = (B, \rho)$ is a right H-Galois δ-K-algebra.

Let $\gamma \in \mathsf{G}(B \otimes B)$. This gives the automorphism

$$\ell_\gamma : (B \otimes B) \otimes H \xrightarrow{\simeq} (B \otimes B) \otimes H$$

of the right $((B \otimes B) \otimes H)$-Galois δ-$(B \otimes B)$-algebra $(B \otimes B) \otimes H$ defined by

$$\ell_\gamma((b \otimes c) \otimes h) = (b \otimes c)\gamma(h_{(1)}) \otimes h_{(2)}, \quad b, c \in B, \ h \in H.$$

Here and in what follows, we let

$$\Delta(h) = h_{(1)} \otimes h_{(2)}, \quad (\Delta \otimes \mathrm{id}) \circ \Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$$

denote the coproduct on H. The *right co-adjoint action*

$$\mathsf{Coad}(\gamma) : (B \otimes B) \otimes H \xrightarrow{\simeq} (B \otimes B) \otimes H$$

by γ is defined by

$$\mathsf{Coad}(\gamma)((b \otimes c) \otimes h) = (b \otimes c)\gamma(h_{(1)})\gamma^{-1}(h_{(3)}) \otimes h_{(2)}.$$

This is an automorphism of the δ-$(B \otimes B)$-Hopf algebra $(B \otimes B) \otimes H$. Note that ℓ_γ turns into an isomorphism of left $((B \otimes B) \otimes H)$-Galois δ-$(B \otimes B)$-algebras, if one twists through $\mathsf{Coad}(\gamma)$ the obvious co-action by $(B \otimes B) \otimes H$ on the domain. Explicitly, this means that

$$(\mathsf{Coad}(\gamma) \ \otimes_{B \otimes B} \ \ell_\gamma) \circ \Delta_{(B \otimes B) \otimes H} = \Delta_{(B \otimes B) \otimes H} \circ \ell_\gamma \ \text{ on } \ (B \otimes B) \otimes H, \tag{3.12}$$

where $\Delta_{(B \otimes B) \otimes H}$ denotes the coproduct on $(B \otimes B) \otimes H$.

Suppose that the γ above is a cocycle for computing $H^1_\delta(B/K, \mathsf{G})$ which gives the B/K-form

B through $\tilde{\rho}$. This means that the commutative diagram

$$\begin{tikzcd} & (B \otimes B) \otimes B \arrow[dl, "d_1\tilde{\rho}"'] \arrow[dr, "d_2\tilde{\rho}"] & \\ (B \otimes B) \otimes H \arrow[rr, "\ell_\gamma"'] & & (B \otimes B) \otimes H \end{tikzcd}$$

of right $((B \otimes B) \otimes H)$-Galois δ-$(B \otimes B)$-algebras, where d_i, $i = 1, 2$, denote the base extensions along

$$B \to B \otimes B, \quad b \mapsto 1 \otimes b,\ b \otimes 1.$$

Recall that B is an (H^B, H)-bi-Galois δ-K-algebra. By [13, Theorem 3.5], the Hopf algebra H^B consists of the elements $\sum_i b_i \otimes c_i$ in $B \otimes B$ such that

$$\sum_i (b_i)_{(0)} \otimes (c_i)_{(0)} \otimes (b_i)_{(1)}(c_i)_{(1)} = \sum_i b_i \otimes c_i \otimes 1 \text{ in } (B \otimes B) \otimes H, \tag{3.13}$$

where $\rho(b) = b_{(0)} \otimes b_{(1)}$. Moreover,

$$\mu : B \otimes H^B \to B \otimes B, \quad \mu(b \otimes z) = bz \tag{3.14}$$

is an isomorphism of left $(B \otimes H^B)$-Galois δ-B-algebras. Define

$$\nu := \tilde{\rho} \circ \mu : B \otimes H^B \to B \otimes H.$$

Recall from Section 3.4 uniqueness of the pair (H', λ), and apply it first over B, and next over $B \otimes B$. Then one sees the following. First, there uniquely exists an isomorphism $\theta : B \otimes H^B \to B \otimes H$ of δ-B-Hopf algebras such that

$$(\theta \otimes \nu) \circ \Delta_{B \otimes H^B} = \Delta_{B \otimes H} \circ \nu,$$

where $\Delta_{B \otimes H^B}$ and $\Delta_{B \otimes H}$ denote the coproducts on the δ-B-Hopf algebras. In fact, this θ is the unique isomorphism between the two δ-B-Hopf-algebras that is compatible with their co-actions on $B \otimes B$. (Notice that this θ ensures Proposition 3.4.) Next, the last commutative diagram, with $((B \otimes B) \otimes H)^{(-)}$ applied (see (3.8)), induces the commutative diagram

$$\begin{tikzcd} & (B \otimes B) \otimes H^B \arrow[dl, "d_1\theta"'] \arrow[dr, "d_2\theta"] & \\ (B \otimes B) \otimes H \arrow[rr, "\mathsf{Coad}(\gamma)"'] & & (B \otimes B) \otimes H \end{tikzcd}$$

of δ-$(B \otimes B)$-Hopf algebras; notice from (3.12) that ℓ_γ induces $\mathsf{Coad}(\gamma)$.

Notice from (3.1) that $\mathfrak{g}^{\mathsf{X}} = \mathrm{Der}_\epsilon(H^B, K)$. Then one sees that θ induces an isomorphism

$$\theta^* : B \otimes \mathfrak{g} \xrightarrow{\simeq} B \otimes \mathfrak{g}^{\mathsf{X}}$$

of δ-B-Lie algebras. Moreover, the last commutative diagram induces by duality the commutative diagram

$$\begin{tikzcd}
 & (B \otimes B) \otimes \mathfrak{g}^{\mathsf{X}} & \\
(B \otimes B) \otimes \mathfrak{g} \arrow[ur, "d_1(\theta^*)"] & & (B \otimes B) \otimes \mathfrak{g} \arrow[ul, "d_2(\theta^*)"'] \arrow[ll, "\mathsf{Ad}(\gamma^{-1})"]
\end{tikzcd}$$

of δ-$(B \otimes B)$-Lie algebras, where the horizontal arrow indicates the left adjoint action by γ^{-1}. We may reverse the direction of the arrow, changing the label into the left adjoint action $\mathsf{Ad}(\gamma)$ by γ. The result shows that $\mathsf{Ad}(\gamma)$, regarded as a cocycle for computing $H^1_\delta(B/K, \mathsf{Aut}_\delta(\mathfrak{g}))$, gives the twisted form $\mathfrak{g}^{\mathsf{X}}$ of $\mathfrak{g}$ which is split by B, indeed.

Recall from (3.6) the definition $H^1(K, \mathsf{G}) := H^1(\mathcal{U}/K, \mathsf{G})$. Let ψ is an element of $H^1(K, \mathsf{G})$. Then this arises from a cocycle γ such as above, which gives a δ-K-torsor $\mathsf{X} = \mathrm{Spec}_\delta(B)$ for G, through a δ-K-algebra map $j : K \to \mathcal{U}$. Thus, ψ is represented by the cocycle given as the composite

$$H \xrightarrow{\gamma} B \otimes B \xrightarrow{j \otimes j} \mathcal{U} \otimes \mathcal{U}.$$

This cocycle is seen to give the $\mathcal{U}/K$-form B of the right H-comodule δ-K-algebra H. The argument in the preceding paragraphs shows that $\mathsf{Ad}_*(\psi)$ is represented by the base extension of the automorphism $\mathsf{Ad}(\gamma)$ along the δ-K-algebra map $j \otimes j$. This base extension is seen to be a cocycle which gives the $\mathcal{U}/K$-form $\mathfrak{g}^{\mathsf{X}}$ of $\mathfrak{g}$. This completes the proof. $\square$

Let $\mathsf{G} = \mathrm{Spec}_\delta(H)$ be an affine algebraic δ-K-group with $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$, as above. Recall from (3.1) that $\mathfrak{g} = \mathrm{Der}_\epsilon(H, K)$.

Proposition 3.6. *Regard H merely as the trivial right H-Galois K-algebra, forgetting δ on it.*

(1) *Given an element $D \in \mathfrak{g}$, define*

$$\delta_D : H \to H, \quad \delta_D(h) = \delta h + D(h_{(1)})h_{(2)}, \tag{3.15}$$

where δ denotes the original operator on H. Then this is a δ-operator with which H is made into a right H-Galois δ-K-algebra. Conversely, such a δ-operator uniquely arises in this way.

(2) *Given an element $D \in \mathfrak{g}$, let X_D denote the right δ-K-torsor for G which is represented by the right H-Galois δ-K-algebra (H, δ_D) obtained above. Then the twisted form $\mathfrak{g}^{\mathsf{X}_D}$ of $\mathfrak{g}$ is the K-Lie algebra $\mathfrak{g}$ equipped with the new δ-operator*

$$\delta + \mathsf{ad}(D) : \mathfrak{g} \to \mathfrak{g}, \quad z \mapsto \delta z + [D, z],$$

where δ denotes the original operator on $\mathfrak{g}$. Thus $\mathfrak{g}^{\mathsf{X}_D}$ is quasi-isomorphic to the original $\mathfrak{g}$; see Definition 1.3.

Proof. (1) Suppose that δ_1 is a desired operator, or namely, (H, δ_1) is a right H-Galois δ-K-algebra. Then one sees that $\delta_1 - \delta : H \to H$ is a K-linear derivation and is at the same time a right H-comodule map. It follows that δ_1 is necessarily of the form δ_D with $D \in \mathfrak{g}$ uniquely determined. Such δ_D is seen to be a desired operator for any D, indeed.

(2) Let $H' = H^{(H,\delta_D)}$. Then $\mathfrak{g}^{\mathsf{X}_D} = \mathrm{Der}_\epsilon(H', K)$. Using the uniqueness of the δ-K-Hopf algebra H' in general, which was discussed in Section 3.4, we see that the present H' is the K-Hopf algebra H equipped with the δ-operator

$$H \to H, \quad h \mapsto \delta h + D(h_{(1)})h_{(2)} - h_{(1)}D(h_{(2)}).$$

This implies the desired result. □

A simple consequence of the proposition above is the following.

Corollary 3.7. *Let $\mathfrak{g}$ be a δ-K-Lie algebra of finite K-dimension. Once the Lie algebra* $\mathrm{Lie}(\mathsf{G})$ *of some affine algebraic δ-K-group G is shown to be a twisted form of $\mathfrak{g}$, then every δ-K-Lie algebra quasi-isomorphic to* $\mathrm{Lie}(\mathsf{G})$ *is a twisted form of $\mathfrak{g}$, as well.*

3.6 Differential δ-K-objects arising from C-linear objects

Let K be a δ-field of characteristic zero. Let

$$C = C_K \, (= \{ x \in K \mid \delta x = 0 \})$$

denote the field of constants in K, which is necessarily of characteristic zero. In this subsection we let $\otimes$ denote the tensor product $\otimes_C$ over C.

Let A_0 be a C-linear object. We can and do regard the base extension $A_0 \otimes K$ as a δ-K-object with respect to the operator δ_0 defined by

$$\delta_0 : A_0 \otimes K \to A_0 \otimes K, \quad a \otimes x \mapsto a \otimes \delta x.$$

For every δ-K-algebra R, $A_0 \otimes R$ is similarly a δ-R-object, and is a base extension of the δ-K-object $A_0 \otimes K$ above.

Proposition 3.8. *If the automorphism-group functor* $\mathsf{Aut}(A_0)$ *of A_0 happens to be an affine C-group, represented by a C-Hopf algebra H_0, then the δ-automorphism-group functor* $\mathsf{Aut}_\delta(A_0 \otimes K)$ *of the δ-K-object $(A_0 \otimes K, \delta_0)$ is an affine δ-K-group, represented by the δ-K-Hopf algebra $H_0 \otimes K$.*

Proof. Let $R \in \delta\text{-}K\text{-}\mathtt{Algebras}$. One sees that every automorphism of the δ-R-object $A_0 \otimes R$ restricts to an automorphism of $A_0 \otimes C_R$ over the C-algebra C_R of constants in R, and so it is uniquely presented as the base extension of the restriction. This shows $\mathrm{Aut}_{\delta\text{-}R}(A_0 \otimes R) = \mathrm{Aut}_{C_R}(A_0 \otimes C_R)$; this last is naturally isomorphic to $\mathrm{Spec}_C(H_0)(C_R) = \mathrm{Spec}_{\delta\text{-}K}(H_0 \otimes K)(R)$. This proves the proposition. □

We remark that the proposition follows from the proof of Proposition 3.3 if $(\dim_K(A_0 \otimes K) =)\, \dim_C A_0 < \infty$. For the relation (3.3) turns into $\delta X = O$ since $D = O$.

Corollary 3.9. *If $\dim_C A_0 < \infty$, then $\mathsf{Aut}_\delta(A_0 \otimes K)$ is an affine algebraic δ-K-group.*

Proof. This follows from the proposition above, since $\mathsf{Aut}(A_0)$ is an affine algebraic C-group under the assumption; see [15, Section 7.6]. □

The following result would be worth presenting, though it is not essentially used in this paper.

Proposition 3.10. *Assume $\dim_C A_0 < \infty$, and that the field C is algebraically closed. Then every twisted form of the δ-K-object $(A_0 \otimes K, \delta_0)$ is split by some (finitely generated) Picard-Vessiot extension L over K.*

Proof. By the preceding results the first assumption implies $\mathsf{Aut}(A_0) = \mathrm{Spec}_C(H_0)$ and $\mathsf{Aut}_\delta(A_0 \otimes K) = \mathrm{Spec}_{\delta\text{-}K}(H_0 \otimes K)$, where H_0 is a finitely generated C-Hopf algebra.

Let B be a right $(H_0 \otimes K)$-Galois δ-K-algebra, and regard it as a twisted form of the right $(H_0 \otimes K)$-comodule δ-K-algebra $H_0 \otimes K$. We should prove that this twisted form B is split by some L/K as above. It suffices to prove that there exists a δ-K-algebra map from B to such an L, since B is split by B, itself.

We have the δ-B-algebra isomorphism

$$\tilde{\rho} : B \otimes_K B \xrightarrow{\simeq} B \otimes_K (H_0 \otimes K) = B \otimes H_0$$

as in (3.2). Choose a simple quotient δ-K-algebra R of B, as in the proof of Lemma 3.2. Then R is an integral domain by [12, Lemma 1.17], as before. This is finitely generated as a K-algebra since B is such. The quotient field $L = Q(R)$ of R uniquely turns into a δ-field, which is necessarily a finitely generated extension over K. The second assumption above, combined with [2, Lemma 4.2], implies that the field C_L of constants in L equals C. This L/K will be proved to be a desired Picard-Vessiot extension by [2, Definition 1.8 and Theorem 3.11], if one sees that the canonical δ-R-algebra map $R \otimes C_{R\otimes_K R} \to R \otimes_K R$ which arises from the embedding $C_{R\otimes_K R} \hookrightarrow R \otimes_K R$ of the constants into $R \otimes_K R$ is surjective. (By [2, Proposition 6.7] this canonical map is injective, though this fact is not needed here.) Indeed, the desired surjectivity is seen from the commutative diagram

$$\begin{tikzcd}
B \otimes H_0 \arrow[r, "\tilde{\rho}^{-1}", "\simeq"'] \arrow[d] & B \otimes_K B \arrow[d, two heads] \\
R \otimes C_{R\otimes_K R} \arrow[r] & R \otimes_K R.
\end{tikzcd}$$

Here the vertical arrow on the left-hand side naturally arises from the composite of $\tilde{\rho}^{-1}|_{H_0} : H_0 \to B \otimes_K B$ with the natural surjection $B \otimes_K B \to R \otimes_K R$, which clearly takes values in $C_{R\otimes_K R}$. □

4 Proof of the theorem and computations

Throughout in this section we let $K := \mathbb{C}(t)$, and write $\otimes$ for $\otimes_K$.

Suppose that we are in the situation of Section 1. Let $\mathfrak{g}_0$ be a complex simple Lie algebra, and let $\mathfrak{g} = \mathfrak{g}_0(K)$ denote the δ-K-Lie algebra as in (1.1).

4.1 Two key facts

One key fact for us is the following description of the automorphism-group scheme $\mathsf{Aut}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$. Recall that the finite group $\Gamma = \mathsf{Out}(\mathfrak{g}_0)$ of outer-automorphisms of $\mathfrak{g}_0$ is explicitly given by (1.2); this Γ will be identified with the associated, finite constant group scheme. Let G_0° be the adjoint simple $\mathbb{C}$-group associated with $\mathfrak{g}_0$. A natural action by Γ on G_0° constitutes an affine algebraic $\mathbb{C}$-group

$$\mathsf{G}_0 = \mathsf{G}_0^\circ \rtimes \Gamma$$

of semi-direct product, so that

$$\mathrm{Lie}(\mathsf{G}_0) = \mathrm{Lie}(\mathsf{G}_0^\circ) = \mathfrak{g}_0,$$

and the adjoint action by G_0 on $\mathfrak{g}_0$ gives an isomorphism

$$\mathsf{Ad} : \mathsf{G}_0 \xrightarrow{\simeq} \mathsf{Aut}(\mathfrak{g}_0) \tag{4.1}$$

of affine algebraic $\mathbb{C}$-groups. By restriction this Ad induces the identity $\Gamma = \mathsf{Out}(\mathfrak{g}_0)$. Note that G_0° is the connected component of G_0 containing the identity element. See [11, Chapter 4, Section 4, 1°].

Suppose $\mathsf{G}_0 = \mathrm{Spec}_{\mathbb{C}}(H_0)$, and define

$$\mathsf{G} = \mathrm{Spec}_{\delta\text{-}K}(H_0 \otimes_{\mathbb{C}} K).$$

Then one sees $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$. Moreover, it follows from (4.1) and Proposition 3.8 that the adjoint action by G on $\mathfrak{g}$ gives an isomorphism

$$\mathsf{Ad} : \mathsf{G} \xrightarrow{\simeq} \mathsf{Aut}_\delta(\mathfrak{g})$$

of affine algebraic δ-K-groups. This together with Proposition 3.5 prove the following.

Proposition 4.1. *Every twisted form of the δ-K-Lie algebra $\mathfrak{g}$ uniquely arises, as described by* (3.11)*, from a right δ-K-torsor for* G.

Another key fact is the cohomology vanishing of the (non-differential) Amitsur 1st cohomology due to Steinberg (see Serre [14, III, 2.3, Theorem 1′]),

$$H^1(K, \mathsf{F}) = 0, \tag{4.2}$$

where F is a connected affine algebraic K-group. This is proved more generally when K is replaced by a perfect field, say K', of dimension ≤ 1 [14, Definitoion on Page 78], and in addition, F is assumed to be smooth if $\mathrm{char}\, K' > 0$; note that every affine algebraic K-group is necessarily smooth since $\mathrm{char}\, K = 0$. One sees that $K\,(= \mathbb{C}(t))$ is a (C_1)-field by Tsen's Theorem, whence K is of dimension ≤ 1 by [14, Corollary on Page 80].

4.2 Proof of Theorem 1.4, Part 1: Case $\Gamma = \{1\}$

In this case, G, regarded as an affine K-group, is connected. By (4.2) applied to this G, it follows that every right δ-K-torsor for G, regarded as a right K-torsor, is trivial. Propositions 3.6 and 4.1 conclude the proof.

4.3 Galois descent

To proceed to Parts 2 and 3, suppose that we are now in Case $\Gamma \neq \{1\}$.

Note that Γ, regarded as a finite constant $\mathbb{C}$-group scheme, is represented by the dual $(\mathbb{C}\Gamma)^*$ of the group algebra $\mathbb{C}\Gamma$; this $(\mathbb{C}\Gamma)^*$ is the separable part $\pi_0(H_0)$ of H_0 [15, Page 49], that is, the largest separable subalgebra (in fact, Hopf subalgebra) of the $\mathbb{C}$-Hopf algebra H_0. Suppose $\mathsf{G}_0^\circ = \mathrm{Spec}_{\mathbb{C}}(J_0)$, and define

$$H := H_0 \otimes_{\mathbb{C}} K, \quad J := J_0 \otimes_{\mathbb{C}} K, \quad Z := (\mathbb{C}\Gamma)^* \otimes_{\mathbb{C}} K \, (= (K\Gamma)^*),$$

which are naturally δ-K-Hopf algebras, such that $\mathsf{G} = \mathrm{Spec}_\delta(H)$, in particular. One sees that $Z \subset H$ is a δ-K-Hopf subalgebra, and

$$J = H/(Z^+), \tag{4.3}$$

where (Z^+) is the ideal (in fact, δ-stable Hopf ideal) generated by the augmentation ideal $Z^+ = \mathrm{Ker}(\epsilon : Z \to K)$ of Z, that is, the kernel of the counit. Since Γ acts innerly on $\mathsf{G}_0^\circ\,(\subset \mathsf{G}_0)$ from the right, it acts from the left on J as δ-K-Hopf-algebra automorphisms. The action gives rise by adjoint to the co-action $J \to J \otimes Z$ by $Z = (K\Gamma)^*$, so that the associated smash coproduct $Z \blacktriangleright\!\!< J$ (see [9, Definition 10.6.1]) coincides with H. Here one should recall $\mathsf{G}_0 = \Gamma \ltimes \mathsf{G}_0^\circ \,(= \mathsf{G}_0^\circ \rtimes \Gamma)$.

Choose arbitrarily a right δ-K-torsor $\mathsf{X} = \mathrm{Spec}_\delta(B)$ for $\mathsf{G} = \mathrm{Spec}_\delta(H)$. In view of Proposition 4.1 we wish to describe the δ-K-Lie algebra $\mathfrak{g}^{\mathsf{X}}\,(= \mathrm{Lie}(\mathsf{G}^{\mathsf{X}}))$. Let $H' := H^B$, or in other words, suppose $\mathsf{G}^{\mathsf{X}} = \mathrm{Spec}_\delta(H')$, so that B is an (H', H)-bi-Galois δ-K-algebra. We are going to prove the following.

Proposition 4.2. *There is a finite-dimensional δ-K-Hopf subalgebra Z' of H' such that*

(i) *Z' is separable as a K-algebra;*

(ii) *the associated quotient δ-K-Hopf algebra*

$$J' := H'/(Z'^+) \quad (\textit{cf.}\ (4.3)) \tag{4.4}$$

has the trivial separable part, $\pi_0(J') = K$, or in other words, it includes no non-trivial separable K-subalgebra.

This implies that the affine K-group $\mathrm{Spec}(H')$ includes $\mathrm{Spec}(J')$ as the connected component containing the identity element, and thereby concludes

$$\mathfrak{g}^{\mathsf{X}} = \text{ the Lie algebra of the affine } \delta\text{-}K\text{-group } \mathrm{Spec}_\delta(J') \tag{4.5}$$

as δ-K-Lie algebras. Therefore, we aim first to prove the proposition above, and then to describe the $\mathfrak{g}^{\mathsf{X}}$ above.

Let $\rho : B \to B \otimes H$ denote the structure map on B, and define

$$R := \rho^{-1}(B \otimes Z).$$

Then this R is a right δ-K-Galois algebra for Z, or in other words, $\mathrm{Spec}_\delta(R)$ is a right δ-K-torsor for the finite constant δ-K-group scheme Γ_K given by Γ; it arises from the the right δ-K-torsor $\mathsf{X} = \mathrm{Spec}_\delta(B)$ for G through the restriction map $H^1_\delta(K, \mathsf{G}) \to H^1_\delta(K, \Gamma_K)$ which is defined in the differential situation, as well, just as in the ordinary situation. Note that R is naturally a δ-K-algebra of finite K-dimension, and is a *Galois K-algebra* with Galois group Γ in the classical sense that the K-algebra map $R \rtimes \Gamma \to \mathrm{End}_K(R)$ which arises from the natural module-action on R by the semi-direct product $R \rtimes \Gamma$ is an isomorphism. Note that $R \rtimes \Gamma$ is naturally a non-commutative δ-K-algebra with $\Gamma\,(= \{1\} \times \Gamma)$ included in constants. A *δ-$(R \rtimes \Gamma)$-module* is thus an R-module M equipped with an additive operator δ and a Γ-action of K-linear automorphisms, such that

$$\delta(\gamma m) = \gamma(\delta m), \quad \delta(am) = (\delta a)m + a(\delta m), \quad \gamma(am) = (\gamma a)(\gamma m),$$

where $\gamma \in \Gamma$, $a \in R$ and $m \in M$. We call this a *(δ, Γ)-R-module*, to treat δ and Γ on an equality, and let (δ, Γ)-R-$\mathsf{Modules}$ denote the category of those modules. The classical Galois Descent Theorem (see [7, Section 18]) tells us that the functor $M \mapsto M^\Gamma$, Γ-invariants in M, gives the category equivalence

$$(\delta, \Gamma)\text{-}R\text{-}\mathsf{Modules} \xrightarrow{\approx} \delta\text{-}K\text{-}\mathsf{Modules},$$

whose quasi-inverse is given by the base-extension functor $\otimes_K R$. In fact, this is a symmetric tensor equivalence, so that there is induced the category equivalence between their (commutative-)algebra objects, or between any other kind of linear objects. The category on the left-hand side has the tensor product $\otimes_R$, the unit object R and the obvious symmetry, while the category on the right-hand side has the tensor product $\otimes_K$, the unit object K and the obvious symmetry. A commutative algebra in (δ, Γ)-R-$\mathsf{Modules}$ will be called a *(δ, Γ)-R-algebra*; it descends to a δ-K-algebra by the category equivalence above. Similarly, a *(δ, Γ)-R-Hopf* or *Lie algebra* is defined, and it descends to a δ-K-object.

We have the commutative diagram

$$\begin{tikzcd}
B \otimes B \arrow[r, "\tilde\rho", "\simeq"'] & B \otimes H \\
R \otimes R \arrow[u, hook] \arrow[r, "\simeq"'] \arrow[dr, "\mathrm{mult}"'] & R \otimes Z \arrow[u, hook] \arrow[d, "\mathrm{id}_R \otimes \epsilon"] \\
 & R
\end{tikzcd}$$

of δ-K-algebras, where the upper horizontal arrow indicates the isomorphism $\tilde\rho$ (see (3.2)) associated with the structure map $\rho : B \to B \otimes H$ on B, and the lower one is the analogous isomorphism for the right Z-Galois δ-K-algebra R. In addition, $\mathrm{mult} : R \otimes R \to R$ indicates the multiplication $x \otimes y \mapsto xy$. By the base extensions along the two diagonal arrows $\mathrm{mult} : R \otimes R \to R$ and $\mathrm{id}_R \otimes \epsilon : R \otimes H \to R$, the $\tilde{\rho}$ induces the isomorphism

$$B \otimes_R B \xrightarrow{\simeq} B \otimes J = B \otimes_R (J \otimes R). \tag{4.6}$$

Recall that Γ acts on J as δ-K-Hopf algebra automorphisms. Then one sees that $J \otimes R$ is a (δ, Γ)-R-Hopf algebra, and hence descends to a δ-K-Hopf algebra

$$\mathcal{J} := (J \otimes R)^{\Gamma}.$$

The composite

$$B \to B \otimes H \to B \otimes J = B \otimes_R (J \otimes R)$$

of the structure map on B with the natural surjection onto $B \otimes_R (J \otimes R)$ is a (δ, Γ)-R-algebra map, and hence descends to a δ-K-algebra map $B^{\Gamma} \to B^{\Gamma} \otimes \mathcal{J}$, which we call ϱ.

Lemma 4.3. *B^{Γ} is a right $\mathcal{J}$-Galois δ-K-algebra by the ϱ above.*

Proof. One sees that ϱ satisfies the co-associativity and the counit property since the last composite does. One sees that (4.6) is an isomorphism of (δ, Γ)-R-algebras, and descends to $\tilde{\varrho} : B^{\Gamma} \otimes B^{\Gamma} \to B^{\Gamma} \otimes \mathcal{J}$, which is, therefore, an isomorphism. $\square$

Recall $H' = H^B$. Define $Z' := Z^R$, so that R is a (Z', Z)-bi-Galois δ-K-algebra.

Lemma 4.4. *Z' is a finite-dimensional δ-K-Hopf subalgebra of H' which has the property* (i) *of Proposition 4.2, that is, Z' is separable as a K-algebra.*

Proof. By (3.13) we have $Z' \subset H'$. This inclusion is compatible with the Hopf-algebra structure maps, as is seen from the construction of H^B given in [13, Theorem 3.5]. To verify this here only for the coproduct, recall from (3.14) that $H' \subset B \otimes B$ gives rise to a left B-linear isomorphism $B \otimes H' = B \otimes B$. Therefore, we have

$$H' \otimes H' \subset B \otimes H' \otimes H' = B \otimes B \otimes H' = B \otimes B \otimes B.$$

The construction cited above tells us that the coproduct on H' is the restriction of

$$B \otimes B \to B \otimes B \otimes B, \quad b \otimes c \mapsto b \otimes 1 \otimes c.$$

This, combined with the analogous restriction of $R \otimes R \to R \otimes R \otimes R$ to the coproduct $Z' \to Z \otimes Z'$, shows the desired compatibility, as is verified by a commutative diagram in cube.

The K-algebras Z, R and Z' turn to be mutually isomorphic after base extension to some algebraically closed field. It follows that Z' is finite-dimensional separable, since Z is. $\square$

Define $J' := H'/(Z'^{+})$, as in (4.4). The proof of Proposition 4.2 completes by proving the next lemma. The following proposition describes the δ-K-Lie algebra $\mathfrak{g}^{\mathsf{X}}$; see (3.10).

Lemma 4.5. *B^Γ is a $(J', \mathcal{J})$-bi-Galois δ-K-algebra, and J' has the property* (ii) *of Proposition 4.2, that is, $\pi_0(J') = K$.*

Proof. The same argument as proving Lemma 4.3 shows that B^Γ is a left J'-Galois δ-K-algebra. Here one should notice that Γ acts (or Z co-acts) trivially on H', and hence on J'. Indeed, B^Γ is bi-Galois, since the structure maps

$$H' \otimes B \leftarrow B \rightarrow B \otimes H$$

on B commute with each other (see (3.7)), and hence those on B^Γ do.

Note that $\pi_0(J)\,(= \pi_0(J_0) \otimes_{\mathbb{C}} K)$ equals K. This is equivalent to saying that the K-algebra J contains no non-trivial idempotent even after base extension to some (or any) algebraically closed field. It follows that $\mathcal{J}$ and J' have the same property, since J and $\mathcal{J}$, as well as $\mathcal{J}$ and J', are mutually isomorphic after base extension such as above. □

Since $\mathfrak{g}_0(R) = \mathfrak{g}_0 \otimes_{\mathbb{C}} R$, on which Γ acts diagonally, is a (δ, Γ)-R-Lie algebra, it descends to $\mathfrak{g}_0(R)^\Gamma$, a δ-K-Lie algebra. Our aim of this subsection is achieved by the following.

Proposition 4.6. *The δ-K-Lie algebra $\mathfrak{g}^{\mathsf{X}}$ is quasi-isomorphic to $\mathfrak{g}_0(R)^\Gamma$.*

Proof. Recall (4.5) and the result of Proposition 4.2 that B^Γ is a $(J', \mathcal{J})$-bi-Galois δ-K-algebra. By Steinberg's Cohomology-Vanishing (4.2) applied to the connected affine K-group Spec$(\mathcal{J})$, we see that the right $\mathcal{J}$-Galois K-algebra B^Γ is isomorphic to $\mathcal{J}$. This together with Proposition 3.6 prove the desired result. □

We add an important consequence.

Corollary 4.7. *The twisted forms of $\mathfrak{g}_0(K)$ are precisely the δ-K-Lie algebras quasi-isomorphic to $\mathfrak{g}_0(R)^\Gamma$, where R ranges over all right $(K\Gamma)^*$-Galois δ-K-algebras.*

Proof. By Propositions 4.1 and 4.6, every twisted form is quasi-isomorphic to some $\mathfrak{g}_0(R)^\Gamma$. Conversely, any $\mathfrak{g}_0(R)^\Gamma$ is clearly a twisted form, whence any one that is quasi-isomorphic to $\mathfrak{g}_0(R)^\Gamma$ is, as well, by (4.5) and Corollary 3.7. □

Before proceeding we make the following remark: given an integer $n \geq 2$, let $\Lambda_n = K^\times/(K^\times)^n$ denote the quotient group of the multiplicative group $K^\times$ by the subgroup of all n-th powers. This is an infinite group, as will be seen below. Removing the identity element, let $\Lambda_n^+ = \Lambda_n \setminus \{1\}$. Needed here is the set only in $n = 2, 3$. The set Λ_2^+ parametrizes the quadratic field extensions over K, while the set Λ_3^+ modulo the equivalence relation $x \sim x^{\pm 1}$ parametrizes the cubic Galois field extensions over K. Therefore, these two classes of field extensions both consist of infinitely many ones.

We show that Λ_n^+, $n \geq 2$, is infinite. To the contrary suppose that it has finitely many generators $a_1, ..., a_r$, each of which may be supposed to be a polynomial by multiplying the denominator by the n-th power of some polynomial. We can choose a complex number α which is not a root of any a_k, $1 \leq k \leq r$. Since $t - \alpha \in < a_1, ..., a_r >= \Lambda_n^+$, we have ${a_1}^{i_1} ... {a_r}^{i_r} b^n = (t - \alpha)c^n$ in $\mathbb{C}[t]$ for some non-zero polynomials b, c. The multiplicity of $t - \alpha$ in ${a_1}^{i_1} ... {a_r}^{i_r} b^n$ is a multiple of n, while the one in $(t - \alpha)c^n$ is congruent to 1 modulo n. This contradicts the last equation.

4.4 Proof of Theorem 1.4, Part 2: Case $\Gamma = \mathbb{Z}_2$

In this case, the right $(K\Gamma)^*$-Galois δ-K-algebras R are precisely

(i) the trivial one $(K\Gamma)^*$ (equipped with the obvious δ-operator), and

(ii) the quadratic field extensions over K (equipped with the δ-operator uniquely extending the one on K).

By Corollary 4.7 it remains to show that for $R = (K\Gamma)^*$ in (i), we have $\mathfrak{g}_0(R)^\Gamma \simeq \mathfrak{g}_0(K)$. Let $\mathrm{Map}(\Gamma, \mathfrak{g}_0(K))$ denote the Γ-set of all maps $\Gamma \to \mathfrak{g}_0(K)$, equipped with the action

$$\gamma f : \gamma' \mapsto f(\gamma'\gamma),$$

where $\gamma, \gamma' \in \Gamma$ and $f \in \mathrm{Map}(\Gamma, \mathfrak{g}_0(K))$. Regard this naturally as the direct product of $\#\Gamma$-copies of the δ-K-Lie algebra $\mathfrak{g}_0(K)$. Then we see that associating to $x \otimes a \in \mathfrak{g}_0 \otimes_{\mathbb{C}} (K\Gamma)^*$, the map $\gamma \mapsto \gamma x \otimes a(\gamma)$ gives a Γ-equivariant isomorphism

$$\mathfrak{g}_0(R) \xrightarrow{\simeq} \mathrm{Map}(\Gamma, \mathfrak{g}_0(K))$$

of δ-K-Lie algebras, whose restriction to the Γ-invariants is the desired $\mathfrak{g}_0(R)^\Gamma \simeq \mathfrak{g}_0(K)$. This completes the proof.

4.5 Proof of Theorem 1.4, Part 3: Case $\Gamma = \mathfrak{S}_3$

In this case, let R be a right $(K\Gamma)^*$-Galois δ-K-algebra. In view of Corollary 4.7 we wish to show that $\mathfrak{g}_0(R)^\Gamma$ is such as in Part 3 of the theorem. This is obvious when R is either trivial or a Galois field extension L/K with $\Gamma = \mathrm{Gal}(L/K)$; notice from the preceding case that $\mathfrak{g}_0(R)^\Gamma = \mathfrak{g}_0(K)$ if R is trivial. We may thus exclude these two cases.

To describe R, note that R is artinian as a ring, and Γ-simple in the sense that it does not include any non-trivial Γ-stable ideal. Since the action by Γ on R commutes with the δ-operator, R is a module algebra over the $\mathbb{C}$-Hopf algebra $\mathbb{C}\Gamma \otimes_{\mathbb{C}} \mathbb{C}[\delta]$, which is *artinian simple* or *AS* in the sense of [2, Definition 11.6]; see the original [1, Definition 2.6] as an alternate. This $\mathbb{C}$-Hopf algebra is the group algebra $\mathbb{C}\Gamma$ tensored with the polynomial algebra $\mathbb{C}[\delta]$ in which δ is primitive. Choose arbitrary a maximal (or equally, minimal) ideal $\mathfrak{m}$ of R, and let Γ' be the subgroup of Γ consisting of all elements that stabilize $\mathfrak{m}$. By [2, Proposition 11.5] we have

$$\text{(a) } \Gamma' \simeq \mathbb{Z}_2 \quad \text{or} \quad \text{(b) } \Gamma' \simeq \mathbb{Z}_3,$$

with the extremal cases being excluded. Moreover, there exists a δ-K-field L such that R is naturally isomorphic to the (δ, Γ)-K-algebra $\mathrm{Map}(\Gamma' \backslash \Gamma, L)$ consisting of all maps from the set of right cosets $\Gamma' \backslash \Gamma$ to L. This $\mathrm{Map}(\Gamma' \backslash \Gamma, L)$ is naturally isomorphic to the direct product of $[\Gamma : \Gamma']$-copies of L, as δ-K-algebra, and possesses the Γ-action presented below. Suppose that $\mathbb{Z}_2$ is an arbitrarily chosen subgroup of Γ of order 2, and $\mathbb{Z}_3$ is the unique subgroup of Γ of order 3, so that we have $\Gamma = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$.

Case (a). We may suppose $\Gamma' = \mathbb{Z}_2$ (see [2, Proposition 11.5 (1)]) and $\Gamma'\backslash\Gamma = \mathbb{Z}_3$. If $\gamma \in \Gamma$, $\gamma' \in \mathbb{Z}_3$ and $f \in \mathrm{Map}(\mathbb{Z}_3, L)$, then we have

$$\gamma f : \gamma' \mapsto \begin{cases} f(\gamma\gamma'), & \text{if } \gamma \in \mathbb{Z}_3 \\ \gamma f(\gamma'^{-1}), & \text{if } 0 \neq \gamma \in \mathbb{Z}_2. \end{cases}$$

Case (b). We have $\Gamma' = \mathbb{Z}_3$, and we may suppose $\Gamma'\backslash\Gamma = \mathbb{Z}_2$. If $\gamma \in \Gamma$, $\gamma' \in \mathbb{Z}_2$ and $f \in \mathrm{Map}(\mathbb{Z}_2, L)$, then we have

$$\gamma f : \gamma' \mapsto \begin{cases} f(\gamma\gamma'), & \text{if } \gamma \in \mathbb{Z}_2; \\ \gamma f(\gamma'), & \text{if } \gamma \in \mathbb{Z}_3,\ \gamma' = 0 \text{ in } \mathbb{Z}_2; \\ \gamma^{-1} f(\gamma'), & \text{if } \gamma \in \mathbb{Z}_3,\ \gamma' \neq 0 \text{ in } \mathbb{Z}_2. \end{cases}$$

In either case, since R is right $(K\Gamma)^*$-Galois, Γ' must act non-trivially on L, so that L/K is a Galois field extension with $\Gamma' = \mathrm{Gal}(L/K)$. Conversely, if L/K is such, then R is seen to be a right $(K\Gamma)^*$-Galois δ-K-algebra, being split by L. Moreover, $\mathfrak{g}_0(R)$ is naturally isomorphic to the (δ, Γ)-R-Lie algebra $\mathrm{Map}(\Gamma'\backslash\Gamma, \mathfrak{g}_0(L))$ equipped with the obviously induced structure. We see

$$\begin{aligned} \mathfrak{g}_0(R)^\Gamma &\simeq \mathrm{Map}(\Gamma'\backslash\Gamma, \mathfrak{g}_0(L))^\Gamma = \left(\mathrm{Map}(\Gamma'\backslash\Gamma, \mathfrak{g}_0(L))^{\mathbb{Z}_3}\right)^{\mathbb{Z}_2} \\ &= \begin{cases} \{\text{all constant maps } \mathbb{Z}_3 \to \mathfrak{g}_0(L)\}^{\mathbb{Z}_2} & \text{in Case (a)} \\ \mathrm{Map}(\mathbb{Z}_2, \mathfrak{g}_0(L)^{\mathbb{Z}_3})^{\mathbb{Z}_2} & \text{in Case (b)} \end{cases} \\ &= \mathfrak{g}_0(L)^{\Gamma'}, \end{aligned}$$

which completes the proof.

4.6 Explicit non-trivial twisted forms

Let us describe explicitly (up to quasi-isomorphism) the non-trivial twisted forms of $\mathfrak{g}_0(K)$ listed in (ii) of Part 2 and (ii)–(iv) of Part 3 of the theorem, separately for four types. For all those, quadratic field extensions are needed. Such an extension L/K is of the form

$$L = K(\sqrt{\alpha}) = \{a + b\sqrt{\alpha} \mid a, b \in K\},$$

where $\alpha \in K^\times \setminus (K^\times)^2$. The generator of $\mathrm{Gal}(L/K)\,(= \mathbb{Z}_2)$ sends each element $x = a + b\sqrt{\alpha}$ to

$$\overline{x} := a - b\sqrt{\alpha}. \tag{4.7}$$

We will use this symbol $\overline{x}$, regardless of α.

4.6.1 Type A_ℓ ($\ell \geq 2$)

We have $\mathfrak{g}_0 = \mathfrak{sl}_n(\mathbb{C})$, where $n = \ell + 1 \geq 3$. The order 2 outer-automorphism is conjugate to $X \mapsto -{}^tX$. For a quadratic extension field $L = K(\sqrt{\alpha})$ over K as above, the generator of $\Gamma\,(= \mathbb{Z}_2)$

may supposed to act on $\mathfrak{g}_0(L)$ by $X = \big(x_{ij}\big)_{i,j} \mapsto -{}^t\overline{X} = \big(-\overline{x}_{ji}\big)_{i,j}$; see [4, Chapter IX, Theorem 5]. We see

$$\mathfrak{g}_0(L)^\Gamma = \mathfrak{o}_n(K) \;\oplus \sqrt{\alpha}\big(\operatorname{Sym}_n(K) \cap \mathfrak{sl}_n(K)\big),$$

where $\operatorname{Sym}_n(K)$ (resp., $\mathfrak{o}_n(K)$) denotes the K-subspace of $\mathfrak{gl}_n(L)$ consisting of all matrices X with entries in K that are symmetric (resp., skew-symmetric, ${}^tX = -X$).

4.6.2 Type D_ℓ ($\ell \geq 5$)

Let $m = 2\ell$. We have $\mathfrak{g}_0 = \mathfrak{o}_m(\mathbb{C})$, which consists of all skew-symmetric $m \times m$ complex matrices. The order 2 outer-automorphism is conjugate to $X \mapsto DXD$, where $D = \operatorname{diag}(-1, 1, \ldots, 1)$; see [4, Chapter IX, Theorem 6]. For a quadratic extension field $L = K(\sqrt{\alpha})$ over K as above, we see

$$\mathfrak{g}_0(L)^\Gamma = \left\{ \begin{pmatrix} 0 & -\sqrt{\alpha}\,{}^tX \\ \sqrt{\alpha}\,X & Y \end{pmatrix} \;\middle|\; X \in K^{m-1},\ Y \in \mathfrak{o}_{m-1}(K) \right\}, \tag{4.8}$$

where by writing $X \in K^{m-1}$, we mean that X is an $(m-1)$-columned vector with entries in K.

4.6.3 Type E_6

Here we follow Jacobson [5, Section 7] for the construction. Let $\mathfrak{J}$ be the exceptional central simple Jordan algebra over $\mathbb{C}$, and let $\mathfrak{J}^+$ denote the subspace of $\mathfrak{J}$ which consists of the elements a with trace zero, $T(a) = 0$. We have the general linear complex Lie algebra $\mathfrak{gl}(\mathfrak{J})$ on the $\mathbb{C}$-vector space $\mathfrak{J}$. Given an element $a \in \mathfrak{J}^+$, we have an element $R_a \in \mathfrak{gl}(\mathfrak{J})$ given by $R_a(x) = xa\,(= ax)$, $x \in \mathfrak{J}$. Let $R_{\mathfrak{J}^+}$ be the subspace of $\mathfrak{gl}(\mathfrak{J})$ which consists of all R_a, $a \in \mathfrak{J}^+$. The complex simple Lie algebra $\mathfrak{g}_0$ of type E_6 is the Lie subalgebra of $\mathfrak{gl}(\mathfrak{J})$ generated by $R_{\mathfrak{J}^+}$. We have

$$\mathfrak{g}_0 = R_{\mathfrak{J}^+} \oplus\ \mathfrak{f}_0,$$

where we set $\mathfrak{f}_0 := [R_{\mathfrak{J}^+}, R_{\mathfrak{J}^+}]$; this is a Lie subalgebra of $\mathfrak{g}_0$ such that $[R_{\mathfrak{J}^+}, \mathfrak{f}_0] = R_{\mathfrak{J}^+}$, and is in fact the complex simple Lie algebra of type E_6. The order 2 outer-automorphism of $\mathfrak{g}_0$ is conjugate to $X \mapsto -X^*$, where X^* denotes the operator adjoint to X with respect to the trace form $(a, b) \mapsto T(ab)$. More explicitly this is given by

$$X \mapsto \begin{cases} -X & \text{if } X \in R_{\mathfrak{J}^+}; \\ \ X & \text{if } X \in \mathfrak{f}_0. \end{cases}$$

Therefore, we have

$$\mathfrak{g}_0(L)^\Gamma = (R_{\mathfrak{J}^+} \otimes_{\mathbb{C}} K\sqrt{\alpha}) \oplus\ \mathfrak{f}_0(K).$$

4.6.4 Type D_4

The complex Lie algebra $\mathfrak{g}_0$ is the Lie algebra $\mathfrak{o}_8(\mathbb{C})$ of skew-symmetric 8×8 complex matrices. We follow É. Cartan [3] for the explicit description of outer-automorphisms. We discuss for each group action, separately as in Part 3 of the theorem.

(ii) **Action by** $\mathbb{Z}_2$. The argument above for D_ℓ ($\ell \geq 5$) works for $\ell = 4$, as well, so that $\mathfrak{g}_0(L)^{\mathbb{Z}_2}$ is given by the right-hand side of (4.8) with $m = 8$.

(iii) **Action by** $\mathbb{Z}_3$. Choose a generator σ of the group. The relevant Galois extension is a cubic one, and it is of the form $L = K(\sqrt[3]{\beta})$, where $\beta \in K^\times \setminus (K^\times)^3$. The generator σ acts on L so that $\sqrt[3]{\beta} \mapsto \omega\sqrt[3]{\beta}$, where ω is a primitive 3rd root of 1.

We suppose that the rows and the columns of matrices in $\mathfrak{g}_0\,(= \mathfrak{o}_8(\mathbb{C}))$ are indexed by the eight integers $0, 1, \cdots, 7$. Given a matrix $X = \big(x_{ij}\big)_{0\leq i,j\leq 7}$ in $\mathfrak{g}_0$, we define seven vectors in $\mathbb{C}^4$ by

$$X_i = {}^t\big(x_{0,i},\ x_{i+1,i+5},\ x_{i+4,i+6},\ x_{i+2,i+3}\big), \quad i = 1, 2, \ldots, 7, \tag{4.9}$$

where the index $i+p$ greater than 7 is understood to be $i+p-7$; e.g., the third entry $x_{i+4,i+6}$ in $i = 3$ in understood to be $x_{7,2}\,(= -x_{2,7})$. Then every matrix X as above is uniquely determined by these seven vectors. This holds when $\mathfrak{g}_0$ is replaced by its base extension. We will use in (4.13)–(4.15) the notation X_i for the seven vectors which are associated as above with a matrix X in such a base extension.

The action by $\mathbb{Z}_3$ on $\mathfrak{g}_0$ is (up to conjugation) such that σ acts on the vectors above as the $\mathbb{C}$-linear automorphisms given by the matrix

$$S = \frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{4.10}$$

which is seen to have $1, 1, \omega$ and ω^2 as eigen-values; see [3, Section 4]. Set $\sqrt{-3} := 1 + 2\omega$, a square root of -3. Then we have the eigen-vectors

$$\boldsymbol{v}_1^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \ \boldsymbol{v}_1^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \ \boldsymbol{v}_\omega = \begin{pmatrix} \sqrt{-3} \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ \boldsymbol{v}_{\omega^2} = \begin{pmatrix} -\sqrt{-3} \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.11}$$

of S which are associated with 1, 1, ω and ω^2, respectively; these form a basis of $\mathbb{C}^4$. Let L^4 denote the L-vector space of all 4-columned vectors with entries in L. Define a 4-dimensional K-subspace of L^4 by

$$\Xi_{L/K} = K\boldsymbol{v}_1^{(1)} + K\boldsymbol{v}_1^{(2)} + K\sqrt[3]{\beta}\,\boldsymbol{v}_{\omega^2} + K(\sqrt[3]{\beta})^2\boldsymbol{v}_\omega. \tag{4.12}$$

We see now easily

$$\mathfrak{g}_0(L)^\Gamma = \{\, X \in \mathfrak{g}_0(L) \mid X_i \in \Xi_{L/K},\ 1 \leq i \leq 7 \,\}. \tag{4.13}$$

(iv) **Action by** $\Gamma\,(= \mathfrak{S}_3)$. The relevant Galois extension is described by the following.

Lemma 4.8. *A Galois extension field over K with Galois group $\Gamma\,(= \mathfrak{S}_3)$ is the same as a field L of the form $L = K(\sqrt{\alpha}, \sqrt[3]{\beta})$, where*

(a) *$\alpha \in K^\times \setminus (K^\times)^2$, so that $K(\sqrt{\alpha})/K$ is a quadratic field extension,*

(b) $\beta \in M^\times \setminus (M^\times)^3$, *where* $M = K(\sqrt{\alpha})$, *and*

(c) $\beta\overline{\beta} \in (K^\times)^3$, *where* $\overline{\beta}$ *is such as given by* (4.7).

For such an L, *we have*

(x) *an order* 3 *element* σ *and an order* 2 *element* τ *of* Γ, *which necessarily generate* Γ, *satisfying* $\sigma\tau = \tau\sigma^2$,

(y) *a primitive 3rd root* ω *of* 1, *and*

(z) *an element* γ *of* $K^\times$ *such that* $\gamma^3 = \beta\overline{\beta}$ *(see* (c) *above)*,

with which the action by Γ *is presented as*

$$\sigma : \sqrt{\alpha} \mapsto \sqrt{\alpha}, \;\; \sqrt[3]{\beta} \mapsto \omega \sqrt[3]{\beta}; \quad \tau : \sqrt{\alpha} \mapsto -\sqrt{\alpha}\,(=\overline{\sqrt{\alpha}}), \;\; \sqrt[3]{\beta} \mapsto \frac{\gamma}{\sqrt[3]{\beta}}.$$

Proof. Given β such as in (b), we have a cubic extension $M(\sqrt[3]{\beta})/M$. One sees that $\beta\overline{\beta} \in (M^\times)^3$ if and only if $M(\sqrt[3]{\beta}) = M(\sqrt[3]{\overline{\beta}})$. If $\gamma \in M^\times$ and $\gamma^3 = \beta\overline{\beta}$, then $\gamma/\sqrt[3]{\beta}$ is a 3rd root of $\overline{\beta}$. A point is only to see that $\sqrt[3]{\beta} \mapsto \gamma/\sqrt[3]{\beta}$ gives an involution which extends $M \to M, x \mapsto \overline{x}$ if and only if $\gamma \in K^\times$. □

Example 4.9. Recall $K = \mathbb{C}(t)$. One can prove that

$$\alpha = 1 - t^3 \quad \text{and} \quad \beta = 1 + \sqrt{1 - t^3}$$

satisfy the conditions above.

To verify (c), one has to prove that there exists no pair of elements a and b in $\mathbb{C}(t)$ that satisfy $a^3 + 3ab^2\alpha = 1$, $3a^2b + b^3\alpha = 1$, where $\alpha = 1 - t^3$. Alternatively, one may prove that there exist no triple of polynomials p, q and r in $\mathbb{C}[t]$ with r *monic* that satisfy (i) $p^3 + 3pq^2\alpha = r^3$, (ii) $3p^2q + q^3\alpha = r^3$, by expressing a, b with a common denominator, so as $a = p/r$, $b = q/r$. Suppose we had such p, q and r. To show a contradiction let $|p|, |q|, |r|$ denote the degrees of p, q, r, and let c_p, c_q denote the top-term coefficients of p, q. Comparing the degrees in (i), (ii) shows (a) $3|r| = 3|p| > |p| + 2|q| + 3$ or $3|p| < |p| + 2|q| + 3 = 3|r|$ and (b) $3|r| = 2|p| + |q| > 3|q| + 3$ or $2|p| + |q| < 3|q| + 3 = 3|r|$, since any of these inequalities cannot be replaced by equality, as is seen by mod 2 reduction. It follows that (c) $|p| = |q| = |r|$ or (d) $|p| = |q| = |r| - 1$. We see from (i), (ii) that $8p^3q = (3p - q)r^3$, $8pq^3\alpha = (3q - p)r^3$. For the latter, (c) is impossible. In Case (d) one has $3c_p - c_q = 0$, $-8c_p^3c_q = 3c_q - c_p$, whence $c_p^3 = -1/3$, $c_q^3 = -9$. But this last contradicts $-c_q^3 = 1$, as is seen from (ii).

The result shows that there exists a Galois extension L/K with $\mathrm{Gal}(L/K) = \mathfrak{S}_3$.

Let $L = K(\sqrt{\alpha}, \sqrt[3]{\beta})$, $M = K(\sqrt{\alpha})$, σ, τ, ω and γ be as in Lemma 4.8. Recall from the proof of the lemma that $\gamma/\sqrt[3]{\beta}$ is a 3rd root of $\overline{\beta}$, and denote it by $\sqrt[3]{\overline{\beta}}$, so that one has

$$\tau(\sqrt[3]{\beta}) = \sqrt[3]{\overline{\beta}}, \quad \tau(\sqrt[3]{\overline{\beta}}) = \sqrt[3]{\beta}.$$

The action by Γ on $\mathfrak{g}_0$ $(= \mathfrak{o}_8(\mathbb{C}))$ is (up to conjugation) such that the generators σ and τ act on the seven vectors in (4.9) as the K-linear automorphisms given by the matrix S in (4.10) and the diagonal matrix $D = \mathrm{diag}(-1, 1, 1, 1)$, respectively. The latter action by τ on $\mathfrak{g}_0$ coincides with the above mentioned outer-automorphism $X \mapsto DXD$ for type D_ℓ, when $\ell = 4$.

Note $L = M(\sqrt[3]{\beta})$, and apply the previous result for the action by $\mathbb{Z}_3$ to the action by $\langle\sigma\rangle$ (on L/M). Then, by using the M-subspace $\Xi_{L/M}$ of L^4 defined by (4.12) (modified into the present situation), we have

$$\mathfrak{g}_0(L)^{\langle\sigma\rangle} = \{\, X \in \mathfrak{g}_0(L) \mid X_i \in \Xi_{L/M},\ 1 \le i \le 7 \,\}. \tag{4.14}$$

By using the vectors given in (4.11) we define a 4-dimensional K-subspace of L^4 by

$$\begin{aligned}\Theta_{L/K} = K\boldsymbol{v}_1^{(1)} + K\boldsymbol{v}_1^{(2)} + K\Big(\sqrt[3]{\beta}\,\boldsymbol{v}_{\omega^2} + \sqrt[3]{\overline{\beta}}\,\boldsymbol{v}_\omega\Big)\\ +K\sqrt{\alpha}\Big(\sqrt[3]{\beta}\,\boldsymbol{v}_{\omega^2} - \sqrt[3]{\overline{\beta}}\,\boldsymbol{v}_\omega\Big).\end{aligned}$$

We see now easily

$$\mathfrak{g}_0(L)^\Gamma = (\mathfrak{g}_0(L)^{\langle\sigma\rangle})^{\langle\tau\rangle} = \{\, X \in \mathfrak{g}_0(L) \mid X_i \in \Theta_{L/K},\ 1 \le i \le 7 \,\}. \tag{4.15}$$

5 Cohomology non-vanishing

We prove that a differential analogue of Steinberg's cohomology-vanishing theorem does not hold.

Theorem 5.1. *Suppose that* $\mathsf{G}_0 = Spec(H_0)$ *is a non-trivial connected affine algebraic* $\mathbb{C}$*-group, and let*

$$\mathsf{G} = Spec_{\delta\text{-}\mathbb{C}(t)}(H_0 \otimes_{\mathbb{C}} \mathbb{C}(t))$$

denote the naturally associated, affine algebraic δ-$\mathbb{C}(t)$*-group. Then the cohomology*

$$H^1_\delta(\mathbb{C}(t), \mathsf{G})$$

defined by (3.6) *does not vanish.*

Proof. By the solution [10, Theorem1] to the inverse problem of differential Galois theory, G_0 is realized as the Galois group of a Picard-Vessiot extension, say, $L/\mathbb{C}(t)$. The Picard-Vessiot ring R of $L/\mathbb{C}(t)$ represents a right δ-$\mathbb{C}(t)$-torsor $X = Spec_\delta(R)$. Since R is simple as a δ-ring, and clearly $R \ne \mathbb{C}(t)$, it follows that X is not trivial, or namely, $X \not\simeq (\mathsf{G}_0)_{\mathbb{C}(t)}$. This prove the theorem. ☐

One may wonder if mutually isomorphic differential Lie algebras are necessarily isomorphic. But this is not the case, as is seen from the following.

Corollary 5.2. *Let* $\mathfrak{g}_0$ *be a complex simple Lie algebra such that the automorphism group scheme* $\mathsf{Aut}(\mathfrak{g}_0)$ *of* $\mathfrak{g}_0$ *is connected; see* (1.2). *then every twisted form of the* δ-$\mathbb{C}(t)$*-Lie algebra* $\mathfrak{g}_0(\mathbb{C}(t))$ *is quasi-isomorphic to* $\mathfrak{g}_0(\mathbb{C}(t))$*, but there exists a twisted form of* $\mathfrak{g}_0(\mathbb{C}(t))$ *which is not isomorphic to* $\mathfrak{g}_0(\mathbb{C}(t))$*.*

Proof. The first half ("preceding but") follows from 1.4, Case $\Gamma = \{1\}$; see section 4.2. The second half follows by Theorem 5.1 for G_0 applied to $\mathsf{Aut}(\mathfrak{g}_0)$, since the cohomology classifies by isomorphism the twisted forms of $\mathfrak{g}_0(\mathbb{C}(t))$.

$\square$

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