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A topological groupoid representing the topos of presheaves on a monoid. (English summary)

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It was shown in [C. Butz and I. Moerdijk, J. Pure Appl. Algebra **130** (1998), no. 3, 223–235; MR1633759] that, for every (Grothendieck) topos $\mathcal{T}$ with enough points, we can find a topological groupoid G with

$$\mathcal{T} \simeq \mathrm{Sh}(G),$$

where $\mathrm{Sh}(G)$ is the category of sheaves on G. This paper gives an alternative construction in the case that $\mathcal{T} = M$-Sets for M a monoid with M-Sets the category of sets endowed with a left M-action as objects and mappings preserving left M-actions as morphisms. Recently people have been interested in M-Sets [A. Connes and C. Consani, C. R. Math. Acad. Sci. Paris **352** (2014), no. 12, 971–975; MR3276804; I. Pirashvili, Math. Proc. Cambridge Philos. Soc. **169** (2020), no. 1, 31–74; MR4120784; M. Rogers, "Toposes of discrete monoid actions", preprint, arXiv:1905.10277]. In a previous paper [J. Number Theory **204** (2019), 155–184; MR3991417] the author studied the topos $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets with

$$\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) = \{a \in \mathrm{M}_2(\mathbb{Z}) : \det(a) \neq 0\}.$$

A synopsis of the paper, consisting of four sections, goes as follows. §2 investigates topological spaces X with a continuous right action of a discrete group G, giving some sufficient conditions for equivalence of the associated topos $\mathrm{Sh}_G(X)$ of G-equivariant sheaves on X to the topos M-Sets for some monoid M (Theorem 2). If X has a minimal basis, it is shown that the topos of sheaves $\mathrm{Sh}(X)$ can be described completely in terms of posets via the duality between Alexandrov-discrete spaces and preorders, while, similarly, $\mathrm{Sh}_G(X)$ can be described in terms of posets with an order-preserving right G-action.

§3 shows (Theorem 11) as a converse to Theorem 2 that, provided that M can be embedded in a group G, one can construct a topological space X_P with a continuous right G-action with

$$M\text{-Sets} \simeq \mathrm{Sh}_G(X_P).$$

§3.2 explicitly writes down the G-equivariant sheaf on X_P corresponding to a certain M-set S.

§4 constructs a topological groupoid G with

$$M\text{-Sets} \simeq \mathrm{Sh}(G)$$

for M an arbitrary monoid. *Hirokazu Nishimura*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.