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Di Liberti, Ivan (CZ-AOS); **Ramos González, Julia** (B-UA2S-M)
Gabriel-Ulmer duality for topoi and its relation with site presentations. (English summary)
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It is well known that different sites can give rise to the same Grothendieck topos and that, given a Grothendieck topos, there is no canonical choice of small site to present it. Ideally one would like to have a way of presenting Grothendieck topoi by nice small categories functorially, avoiding having to check independence of presentations. It is well known that Grothendieck topoi are indeed locally presentable categories, and Gabriel-Ulmer duality [P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Mathematics, Vol. 221, Springer, Berlin, 1971; MR0327863] claims that locally κ-presentable categories are completely determined by their small subcategory of κ-presentable objects, providing a natural presentation of locally κ-presentable categories in terms of κ-complete small categories. The principal objective in the paper under review is to restrict, for each regular cardinal κ, the 2-categorical generalization of the classical Gabriel-Ulmer duality to a duality for locally κ-presentable Grothendieck topoi with κ-accessible geometric functors so as to understand which κ-cocomplete categories appear as their natural presentations, and eventually interpret those in terms of classical site presentations.

A synopsis of the paper, which consists of six sections, goes as follows. §2 reviews the 2-categorical version of Gabriel-Ulmer duality theorem applied to locally κ-presentable categories [C. Centazzo and E. M. Vitale, Theory Appl. Categ. **10** (2002), No. 20, 486–497 (Theorem 3.1); MR1942329].

§3 focuses on the restriction of Gabriel-Ulmer duality to Grothendieck topoi at the level of objects, characterizing the locally κ-presentable categories which are Grothendieck topoi in terms of their subcategories of κ-presentable objects. The question was first posed for the case of locally finitely presentable categories [*Théorie des topos et cohomologie étale des schémas. Tome 1*, Lecture Notes in Mathematics, Vol. 269, Springer, Berlin, 1972; MR0354652], having been answered for the first time in [A. Carboni, M. C. Pedicchio and J. Rosický, J. Pure Appl. Algebra **161** (2001), no. 1-2, 65–90; MR1834079] with a characterization of the small categories C with finite colimits whose Ind-completion $\mathrm{Ind}(\mathsf{C})$ is a Grothendieck topos. The authors provide a generalization of this characterization to higher cardinalities (Theorem 3.17), where Ind_κ denotes the free κ-directed colimit completion, usually referred to as Ind_κ-completion.

Theorem 1. Let κ be a regular cardinal and let C be a small category closed under κ-small colimits. The following are equivalent:

(1) C is κ-extensive and pro-exact;
(2) $\mathrm{Ind}_\kappa(\mathsf{C})$ is a Grothendieck topos.

Therefore, a locally κ-presentable category $\mathcal{A}$ is a Grothendieck topos iff its full subcategory of κ-presentable objects $\mathrm{Pres}_\kappa\mathcal{A}$ is κ-extensive and pro-exact. It is said that a small category C with κ-small colimits is a κ-*prototopos* if it satisfies one of the two equivalent properties indicated in Theorem 1.

§4 focuses on the restriction of Gabriel-Ulmer duality at the level of 1-cells, presenting the desired restriction of the result of Gabriel and Ulmer (Theorem 4.5).

Theorem 2. Let κ be a regular cardinal. There is a biequivalence of 2-categories

$$\mathrm{Ind}_\kappa\colon \mathit{Prototopoi}_\kappa^{\mathrm{coop}} \rightleftarrows \mathit{GrTopoi}_\kappa :\mathrm{Pres}_\kappa$$

where $\mathit{Prototopoi}_\kappa$ is the 2-category of κ-prototopoi with κ-small colimit preserving functors which are Ind_κ-flat.

§5 analyses up to what point the understanding of the family of 2-categories

$$\{\mathit{GrTopoi}_\kappa\}_{\kappa\in\mathrm{RegCard}}$$

(where RegCard denotes the set of small regular cardinals) provided by this topoi-prototopoi duality is to help one comprehend the 2-category *GrTopoi* of Grothendieck topoi with geometric morphisms and natural transformations.

§6 investigates the relationship between prototopos presentations of Grothendieck topoi and the classical site presentations, establishing the following theorem (Theorem 6.7).

Theorem 3. The 2-category $\mathit{GrTopoi}_\kappa$ is a reflective sub-bicategory in the 2-category $\mathbf{Sites}_\kappa^{\mathrm{coop}}$, where the left adjoint to the reflection is provided by taking sheaves.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.