

Ozawa, Masanao

Quantum set theory: transfer principle and De Morgan's laws. (English) [Zbl 07327745] [Ann. Pure Appl. Logic 172, No. 4, Article ID 102938, 42 p. \(2021\).](#)

Quantum logic was introduced by *G. Birkhoff* and *J. von Neumann* [Ann. Math. (2) 37, 823–843 (1936; Zbl 0015.14603)]. The study of quantum set theory was initiated by *G. Takeuti* [in: Current issues in quantum logic. Proceedings of the Workshop on Quantum Logic held in Erice, Sicily, December 2–9, 1979. New York - London: Plenum Press (1981; Zbl 0537.03044), p. 303–322]. The author [J. Symb. Log. 72, No. 2, 625–648 (2007; Zbl 1124.03045)] established the transfer principle for Takeuti's quantum set theory, which was generalized to complete orthomodular lattices [M. Ozawa, Rev. Symb. Log. 10, No. 4, 782–807 (2017; Zbl 1421.03026)].

Transfer Principle. For any Δ_0 -formula $\phi(x_1, \dots, x_n)$ in the language of set theory in ZFC and for any element u_1, \dots, u_n in the universe $V^{(\mathcal{Q})}$ with the \mathcal{Q} -valued truth value $\|\phi(u_1, \dots, u_n)\|$, we have

$$\|\phi(u_1, \dots, u_n)\| \geq \underline{\vee}(u_1, \dots, u_n)$$

where $\underline{\vee}(u_1, \dots, u_n)$ stands for the commutator.

Bounded universal and existential quantifications are evaluated by using implication \rightarrow and conjunction $*$.

$$\begin{aligned} \|(\forall x \in u) \phi(x)\| &= \vee_{u' \in \text{dom}(u)} (u(u') \rightarrow \|\phi(u')\|) \\ \|(\exists x \in u) \phi(x)\| &= \vee_{u' \in \text{dom}(u)} (u(u') * \|\phi(u')\|) \end{aligned}$$

Introducing a general class of binary operation \rightarrow for implication and one of binary operations $*$ for conjunction on complete orthomodular lattices, this paper determines which pairs $(\rightarrow, *)$ of an implication and a conjunction satisfy the transfer principle or both the transfer principle and the de Morgan laws.

De Morgan Laws.

$$\begin{aligned} \|\lceil (\forall x \in u) \phi(x)\| &= \|(\exists x \in u) \rceil \phi(x)\| \\ \|\lceil (\exists x \in u) \phi(x)\| &= \|(\forall x \in u) \rceil \phi(x)\| \end{aligned}$$

The pair of the Sasaki implication ($P \rightarrow Q$ for $P^\perp \vee (P \wedge Q)$, also called the Sasaki arrow) and the Sasaki conjunction ($P * Q$ for $(P \rightarrow Q^\perp)^\perp$, also called the Sasaki projection) [U. Sasaki, J. Sci. Hiroshima Univ., Ser. A 17, 293–302 (1954; Zbl 0055.25902)] satisfies both the transfer principle and de Morgan laws. It is shown that there are exactly 36 pairs of polynomially definable operations previous to the transfer principle, among which only 6 should obey the de Morgan laws as well.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

- 03E40 Other aspects of forcing and Boolean-valued models
- 03E70 Nonclassical and second-order set theories
- 03E75 Applications of set theory
- 03G12 Quantum logic
- 06C15 Complemented lattices, orthocomplemented lattices and posets
- 46L60 Applications of selfadjoint operator algebras to physics
- 81P10 Logical foundations of quantum mechanics; quantum logic (quantum-theoretic aspects)

Keywords:

quantum set theory; orthomodular-valued models; transfer principle; De Morgan's laws; quantum logic; orthomodular lattices; commutator; implication; Boolean-valued models; ZFC

Full Text: DOI

References:

- [1] Bell, J. L., Set Theory: Boolean-Valued Models and Independence Proofs (2005), Oxford Univ. Press: Oxford Univ. Press Oxford · [Zbl 1065.03034](#)
- [2] Birkhoff, G.; von Neumann, J., The logic of quantum mechanics, *Ann. Math.*, 37, 823-843 (1936) · [Zbl 62.1061.04](#)
- [3] Bruns, G.; Kalmbach, G., Some remarks on free orthomodular lattices, (Schmidt, J., Proc. Lattice Theory Conf. Houston. Proc. Lattice Theory Conf. Houston, U.S.A. (1973)), 397-408
- [4] Chevalier, G., Commutators and decompositions of orthomodular lattices, *Order*, 6, 181-194 (1989) · [Zbl 0688.06006](#)
- [5] Cohen, P. J., The independence of the continuum hypothesis I, *Proc. Nat. Acad. Sci. U.S.A.*, 50, 1143-1148 (1963) · [Zbl 0192.04401](#)
- [6] Cohen, P. J., Set Theory and the Continuum Hypothesis (1966), Benjamin: Benjamin New York · [Zbl 0182.01301](#)
- [7] D'Hooghe, B.; Pykacz, J., On some new operations on orthomodular lattices, *Int. J. Theor. Phys.*, 39, 641-652 (2000) · [Zbl 0963.06009](#)
- [8] Döring, A.; Eva, B.; Ozawa, M., A bridge between Q-worlds, *Rev. Symb. Log.*, 1-40 (2020)
- [9] Eda, K., On a Boolean power of a torsion free Abelian group, *J. Algebra*, 82, 84-93 (1983) · [Zbl 0538.20027](#)
- [10] Eva, B., Towards a paraconsistent quantum set theory, (Heunen, C.; Selinger, P.; Vicary, J., Proceedings of the 12th International Workshop on Quantum Physics and Logic. Proceedings of the 12th International Workshop on Quantum Physics and Logic, Oxford, U.K., July 15-17, 2015. Proceedings of the 12th International Workshop on Quantum Physics and Logic. Proceedings of the 12th International Workshop on Quantum Physics and Logic, Oxford, U.K., July 15-17, 2015, Electronic Proceedings in Theoretical Computer Science, vol. 195 (2015), Open Publishing Association), 158-169
- [11] Finch, P., Sasaki projections on orthocomplemented posets, *Bull. Aust. Math. Soc.*, 1, 319-324 (1969) · [Zbl 0176.28501](#)
- [12] Fourman, M. P.; Scott, D. S., Sheaves and logic, (Fourman, M. P.; Mulvey, C. J.; Scott, D. S., Applications of Sheaves. Applications of Sheaves, Lecture Notes in Math., vol. 753 (1979), Springer: Springer Berlin), 302-401 · [Zbl 0415.03053](#)
- [13] Georgacarakos, G. N., Orthomodularity and relevance, *J. Philos. Log.*, 8, 415-432 (1979) · [Zbl 0426.03017](#)
- [14] Grayson, R. J., Heyting-valued models for intuitionistic set theory, (Fourman, M. P.; Mulvey, C. J.; Scott, D. S., Applications of Sheaves. Applications of Sheaves, Lecture Notes in Math., vol. 753 (1979), Springer: Springer Berlin), 402-414 · [Zbl 0419.03033](#)
- [15] Hardegree, G. M., Material implication in orthomodular (and Boolean) lattices, *Notre Dame J. Form. Log.*, 22, 163-182 (1981) · [Zbl 0438.03060](#)
- [16] Husimi, K., Studies on the foundation of quantum mechanics I, *Proc. Phys. Math. Soc. Jpn.*, 19, 766-778 (1937) · [Zbl 63.1422.01](#)
- [17] Jech, T., Abstract theory of Abelian operator algebras: an application of forcing, *Transl. Am. Math. Soc.*, 289, 133-162 (1985) · [Zbl 0597.03030](#)
- [18] Johnstone, P. T., Topos Theory (1977), Academic Press: Academic Press London · [Zbl 0368.18001](#)
- [19] Kalmbach, G., Orthomodular Lattices (1983), Academic Press: Academic Press London · [Zbl 0512.06011](#)
- [20] Kotas, J., An axiom system for the modular logic, *Stud. Log.*, 21, 17-38 (1967) · [Zbl 0333.02023](#)
- [21] Kusraev, A. G.; Kutateladze, S. S., Boolean Valued Analysis (1999), Springer: Springer Berlin · [Zbl 0955.46046](#)
- [22] Marsden, E. L., The commutator and solvability in a generalized orthomodular lattice, *Pac. J. Math.*, 33, 357-361 (1970) · [Zbl 0234.06004](#)
- [23] Nishimura, H., Boolean valued Lie algebras, *J. Symb. Log.*, 56, 731-741 (1991) · [Zbl 0742.17021](#)
- [24] Olson, M. P., The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice, *Proc. Am. Math. Soc.*, 28, 537-544 (1971) · [Zbl 0215.20504](#)
- [25] Ozawa, M., Boolean valued analysis and type I AW*-algebras, *Proc. Jpn. Acad.*, 59A, 368-371 (1983) · [Zbl 0581.46047](#)
- [26] Ozawa, M., Boolean valued interpretation of Hilbert space theory, *J. Math. Soc. Jpn.*, 35, 609-627 (1983) · [Zbl 0526.46069](#)
- [27] Ozawa, M., A classification of type I AW*-algebras and Boolean valued analysis, *J. Math. Soc. Jpn.*, 36, 589-608 (1984) · [Zbl 0599.46083](#)
- [28] Ozawa, M., Nonuniqueness of the cardinality attached to homogeneous AW*-algebras, *Proc. Am. Math. Soc.*, 93, 681-684 (1985) · [Zbl 0614.46051](#)
- [29] Ozawa, M., A transfer principle from von Neumann algebras to AW*-algebras, *J. Lond. Math. Soc. (2)*, 32, 141-148 (1985) · [Zbl 0626.46052](#)
- [30] Ozawa, M., Forcing in nonstandard analysis, *Ann. Pure Appl. Log.*, 68, 263-297 (1994) · [Zbl 0812.03041](#)
- [31] Ozawa, M., Scott incomplete Boolean ultrapowers of the real line, *J. Symb. Log.*, 60, 160-171 (1995) · [Zbl 0819.03050](#)
- [32] Ozawa, M., Transfer principle in quantum set theory, *J. Symb. Log.*, 72, 625-648 (2007) · [Zbl 1124.03045](#)
- [33] Ozawa, M., Quantum reality and measurement: a quantum logical approach, *Found. Phys.*, 41, 592-607 (2011) · [Zbl 1210.81008](#)
- [34] Ozawa, M., Quantum set theory extending the standard probabilistic interpretation of quantum theory, *New Gener. Comput.*, 34, 125-152 (2016) · [Zbl 1396.81011](#)
- [35] Ozawa, M., Operational meanings of orders of observables defined through quantum set theories with different conditionals,

(Duncan, R.; Heunen, C., Quantum Physics and Logic 2016. Quantum Physics and Logic 2016, Electronic Proceedings in Theoretical Computer Science, vol. 236 (2017), Open Publishing Association), 127-144

- [36] Ozawa, M., Orthomodular-valued models for quantum set theory, *Rev. Symb. Log.*, 10, 782-807 (2017) · Zbl 1421.03026
- [37] Płaneta, A.; Stochel, J., Spectral order for unbounded operators, *J. Math. Anal. Appl.*, 389, 1029-1045 (2012) · Zbl 1262.47034
- [38] Pulmannová, S., Commutators in orthomodular lattices, *Demonstr. Math.*, 18, 187-208 (1985) · Zbl 0591.06011
- [39] Sasaki, U., Orthocomplemented lattices satisfying the exchange axiom, *J. Sci. Hiroshima Univ. A*, 17, 293-302 (1954) · Zbl 0055.25902
- [40] Scott, D., Boolean models and nonstandard analysis, (Luxemburg, W. A.J., Applications of Model Theory to Algebra, Analysis, and Probability (1969), Holt, Reinehart and Winston: Holt, Reinehart and Winston New York), 87-92 · Zbl 0187.27101
- [41] D. Scott, R. Solovay, Boolean-valued models for set theory, unpublished manuscript for Proc. AMS Summer Institute on Set Theory, Los Angeles, Univ. Cal., 1967, 1967.
- [42] Smith, K., Commutative regular rings and Boolean-valued fields, *J. Symb. Log.*, 49, 281-297 (1984) · Zbl 0589.03032
- [43] Takeuti, G., Two Applications of Logic to Mathematics (1978), Princeton Univ. Press: Princeton Univ. Press Princeton, NJ · Zbl 0393.03027
- [44] Takeuti, G., A transfer principle in harmonic analysis, *J. Symb. Log.*, 44, 417-440 (1979) · Zbl 0427.03047
- [45] Takeuti, G., Quantum set theory, (Beltrametti, E. G.; van Fraassen, B. C., Current Issues in Quantum Logic: Proceedings of the Workshop on Quantum Logic. Current Issues in Quantum Logic: Proceedings of the Workshop on Quantum Logic, December 2-9, 1979, Erice, Sicily, Italy (1981), Plenum: Plenum New York), 303-322
- [46] Takeuti, G., Von Neumann algebras and Boolean valued analysis, *J. Math. Soc. Jpn.*, 35, 1-21 (1983) · Zbl 0488.46052
- [47] Takeuti, G., Boolean simple groups and Boolean simple rings, *J. Symb. Log.*, 53, 160-173 (1988) · Zbl 0653.03034
- [48] Takeuti, G.; Zaring, W. M., Axiomatic Set Theory (1973), Springer: Springer New York · Zbl 0261.02038
- [49] Titani, S., Global Set Theory (2018), Society for Science and Education: Society for Science and Education Stockport, UK
- [50] Urquhart, A., Review, *J. Symb. Log.*, 48, 206-208 (1983)
- [51] von Neumann, J., Mathematical Foundations of Quantum Mechanics (1955), Princeton Univ. Press: Princeton Univ. Press Princeton, NJ, [Originally published: Mathematische Grundlagen der Quantenmechanik (Springer, Berlin, 1932)]
- [52] Ying, M., A theory of computation based on quantum logic (I), *Theor. Comput. Sci.*, 344, 134-207 (2005) · Zbl 1079.68035
- [53] Ying, M., Quantum computation, quantum theory and AI, *Artif. Intell.*, 174, 162-176 (2010)

This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.