Kang, Seok-Jin; Kashiwara, Masaki; Kim, Myungho; Oh, Se-Jin
Monoidal categorification of cluster algebras. (English) Zbl 06836098
J. Am. Math. Soc. 31, No. 2, 349-426 (2018).

The principal objective in this paper is to provide a monoidal categorification of the quantum cluster algebra structure on the unipotent quantum coordiante ring $A_q((\mathfrak{n}(w)))$ associated with a symmetric Kac-Moody algebra $\mathfrak{g}$ and a Weyl group element w.

Cluster algebras were introduced by *S. Fomin* and *A. Zelevinsky* [J. Am. Math. Soc. 15, No. 2, 497–529 (2002; Zbl 1021.16017)] for studying total positivity and upper global bases, having found a lot of connections and applications in various fields of mathematics such as representation theory, Teichmüller theory, tropical geometry, integrable systems and Poisson geometry. A *cluster algebra* is a $\mathbb{Z}$-subalgebra of a rational function field by a set of generators called the *cluster variables*, which are grouped into possibly overlapping subsets, called the clusters, being defined inductively by what is called *mutaion* from the *initial cluster* $\{X_i\}_{1\leq i\leq r}$ under the control of an exchange matrix $\widetilde{B}$. A monomial in each cluster is called a *cluster monomial*. Fomin and Zelevinsky established that every cluster variable is a Laurant polynomial of the initial cluster $\{X_i\}_{1\leq i\leq r}$, conjecturing that this Laurent polynomial has positive coefficients. This *positivity conjecture* was settled for *skew-symmetric* cluster algebras in [*K. Lee* and *R. Schiffler*, Ann. Math. (2) 182, No. 1, 73–125 (2015; Zbl 1350.13024)], while the *linearly independence conjecture* on cluster monomials was established also for *skew-symmetric* cluster algebras in [*G. Cerulli Irelli* et al., Compos. Math. 149, No. 10, 1753–1764 (2013; Zbl 1288.18011)].

Quantum cluster algebras as a q-analogue of cluster algebras were introduced by *A. Berenstein* and *A. Zelevinsky* [Adv. Math. 195, No. 2, 405–455 (2005; Zbl 1124.20028)]. The commutation relation among the cluster variables is determined by a skew-symmetric matrix L, while every cluster variable belongs to $\mathbb{Z}\left[q^{\pm 1/2}\right]\left[X_i^{\pm 1}\right]_{1\leq i\leq r}$, being expected to be an element of $\mathbb{Z}_{\geq 0}\left[q^{\pm 1/2}\right]\left[X_i^{\pm 1}\right]_{1\leq i\leq r}$ (the *quantum positivity conjecture* [*B. Davison* et al., Compos. Math. 151, No. 10, 1913–1944 (2015; Zbl 1345.14014), Conjecture 4.7]). *Y. Kimura* and *F. Qin* [Adv. Math. 262, 261–312 (2014; Zbl 1331.13016)] established the quantum positivity conjecture for quantum cluster algebras with acyclic seed and specific coefficients. The *unipotent quantum coordinate rings* $A_q(\mathfrak{n})$ and $A_q(\mathfrak{n}(w))$ are examples of quantum cluster algebras stemming from Lie theory, where the algebra $A_q(\mathfrak{n})$ is a q-deformation of the coordinate ring $\mathbb{C}[N]$ of the unipotent subgroup, being isomorphic to the negative half $U_q^-(\mathfrak{g})$ of the quantum group as $\mathbb{Q}(q)$-algebras, while the algebra $A_q(\mathfrak{n}(w))$ is a $\mathbb{Q}(q)$-subalgebra of $A_q(\mathfrak{n})$ generated by a set of the *dual Poincaré-Birkhoff-Witt basis elements* associated with a Weyl group element w. The unipotent quantum coordinate ring $A_q(\mathfrak{n})$ is of a very interesting basis, the so-called *upper global basis* $\boldsymbol{B}^{\mathrm{up}}$. It was shown in [*C. Geiß* et al., Adv. Math. 228, No. 1, 329–433 (2011; Zbl 1232.17035); Ann. Sci. Éc. Norm. Supér. (4) 38, No. 2, 193–253 (2005; Zbl 1131.17006); Sel. Math., New Ser. 19, No. 2, 337–397 (2013; Zbl 1318.13038)] that the unipotent quantum coordinate ring $A_q(\mathfrak{n}(w))$ is of a skew-symmetric quantum cluster algebra structure whose initial cluster consists of what are called *unipotent quantum minors*. *Y. Kimura* [Kyoto J. Math. 52, No. 2, 277–331 (2012; Zbl 1282.17017)] established that $A_q(\mathfrak{n}(w))$ is compatible with the upper global basis $\boldsymbol{B}^{\mathrm{up}}$ of $A_q(\mathfrak{n})$, encouraging one to anticipate that every cluster monomial of is contained in the upper global basis $\boldsymbol{B}^{\mathrm{up}}(w)$. Therefore we have

Conjecture. ([*C. Geiß* et al., Sel. Math., New Ser. 19, No. 2, 337–397 (2013; Zbl 1318.13038), Conjecture 12.9], [*Y. Kimura*, Kyoto J. Math. 52, No. 2, 277–331 (2012; Zbl 1282.17017), Conjecture 1.1 (2)]). When $\mathfrak{g}$ is a symmetric Kac-Moody algebra, every quantum cluster monomial in

$$A_{q^{1/2}}(\mathfrak{n}(w)) = \mathbb{Q}\left(q^{1/2}\right)\otimes A_q(\mathfrak{n}(w))$$

belongs the upper global basis $\boldsymbol{B}^{\mathrm{up}}$ up to a power $q^{1/2}$.

The conjecture is to be put down as a reformulation of Berenstein-Zelevinsky's ideas on the multiplicative properties of $\boldsymbol{B}^{\mathrm{up}}$. There are already some partial results of this conjecture. It was established for $\mathfrak{g}=A_2$, A_3, A_4 and $A_q(\mathfrak{n}(w)) = A_q(\mathfrak{n})$ in [*A. Berenstein* and *A. Zelevinsky*, Adv. Sov. Math. 16, 51–89 (1993; Zbl 0794.17007)] and [*C. Geiss* et al., Compos. Math. 143, No. 5, 1313–1334 (2007; Zbl 1132.17004), §12].

It was shown in [*P. Lampe*, Int. Math. Res. Not. 2011, No. 13, 2970–3005 (2011; Zbl 1273.17020); *P. Lampe*, Proc. Lond. Math. Soc. (3) 108, No. 1, 1–43 (2014; Zbl 1284.13031)] that, when $\mathfrak{g} = A_1^{(1)}$, A_n and w is a square of a Coxeter element, the cluster variables belong to the upper global basis. When $\mathfrak{g}$ is symmetric and w is a squareof a Coxeter element, the conjecture was established in [*Y. Kimura* and *F. Qin*, Adv. Math. 262, 261–312 (2014; Zbl 1331.13016)]. *F. Qin* [Duke Math. J. 166, No. 12, 2337–2442 (2017; Zbl 1454.13037)] has recently provided a proof of the conjecture for a large class with a condition on the Weyl element.

This paper establishes the above conjecture perfectly by demonstrating existence of a *monoidal categorification of* $A_{q^{1/2}}(\mathfrak{n}(w))$. *D. Hernandez* and *B. Leclerc* [in: Symmetries, integrable systems and representations. Proceedings of the conference on infinite analysis: frontier of integrability, Tokyo, Japan, July 25–29, 2011 and the conference on symmetries, integrable systems and representations, Lyon, France, December 13–16, 2011. London: Springer. 175–193 (2013; Zbl 1317.13052)] introduced the notion of *monoidal categorification* of cluster algebras, establishing that certain categories of modules over symmetric quantum affine algebras $U'(\mathfrak{g})$ give monoidal categorification of some cluster algebras. *H. Nakajima* [Kyoto J. Math. 51, No. 1, 71–126 (2011; Zbl 1223.13013)] has extended this result to cluster algebras of types A, D and E. This paper extends Hernandez and Leclere's notion of monoidal categorification to quantum cluster algebras.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

Cited in **1** Review

13F60 Cluster algebras
16Gxx Representation theory of associative rings and algebras
17B37 Quantum groups (quantized enveloping algebras) and related deformations
18M05 Monoidal categories, symmetric monoidal categories
81R50 Quantum groups and related algebraic methods applied to problems in quantum theory

Keywords:

cluster algebra; quantum cluster algebra; monoidal categorification; Khovanov-Lauda-Rouquier algebra; unipotent quantum coordinate ring; quantum affine algebra

Full Text: DOI arXiv

References:

[1] Beĭlinson, A. A.; Bernstein, J.; Deligne, P., Faisceaux pervers. Analysis and topology on singular spaces, I, Luminy, 1981, Astérisque 100, 5\textendash171 pp., (1982), Soc. Math. France, Paris

[2] Berenstein, Arkady; Zelevinsky, Andrei, String bases for quantum groups of type A_r. I. M. Gel\cprime fand Seminar, Adv. Soviet Math. 16, 51\textendash89 pp., (1993), Amer. Math. Soc., Providence, RI · Zbl 0794.17007

[3] Berenstein, Arkady; Zelevinsky, Andrei, Quantum cluster algebras, Adv. Math., 195, 2, 405\textendash455 pp., (2005) · Zbl 1124.20028

[4] Cerulli Irelli, Giovanni; Keller, Bernhard; Labardini-Fragoso, Daniel; Plamondon, Pierre-Guy, Linear independence of cluster monomials for skew-symmetric cluster algebras, Compos. Math., 149, 10, 1753\textendash1764 pp., (2013) · Zbl 1288.18011

[5] Davison, Ben; Maulik, Davesh; Sch\"urmann, J\"org; Szendrői, Balázs, Purity for graded potentials and quantum cluster positivity, Compos. Math., 151, 10, 1913\textendash1944 pp., (2015) · Zbl 1345.14014

[6] Fomin, Sergey; Zelevinsky, Andrei, Cluster algebras. I. Foundations, J. Amer. Math. Soc., 15, 2, 497\textendash529 pp., (2002) · Zbl 1021.16017

[7] Geiss, Christof; Leclerc, Bernard; Schröer, Jan, Semicanonical bases and preprojective algebras, Ann. Sci. Éc. Norm. Supér. (4), 38, 2, 193\textendash253 pp., (2005) · Zbl 1131.17006

[8] Geiß, Christof; Leclerc, Bernard; Schröer, Jan, Kac–Moody groups and cluster algebras, Adv. Math., 228, 1, 329\textendash433 pp., (2011) · Zbl 1232.17035

[9] GLS07 Christof Geiß, Bernard Leclerc, and Jan Schröer, \emph Cluster algebra structures and semicanonical bases for unipotent groups, arXiv:0703039\/v4 [math.RT].

[10] Geiss, Christof; Leclerc, Bernard; Schröer, Jan, Factorial cluster algebras, Doc. Math., 18, 249\textendash274 pp., (2013) · Zbl 1275.13018

[11] Geiß, C.; Leclerc, B.; Schröer, J., Cluster structures on quantum coordinate rings, Selecta Math. (N.S.), 19, 2, 337\textendash397 pp., (2013) · Zbl 1318.13038

[12] Hernandez, David; Leclerc, Bernard, Cluster algebras and quantum affine algebras, Duke Math. J., 154, 2, 265\textendash341

pp., (2010) · Zbl 1284.17010

[13] Hernandez, David; Leclerc, Bernard, Monoidal categorifications of cluster algebras of type A and D. Symmetries, integrable systems and representations, Springer Proc. Math. Stat. 40, 175\textendash193 pp., (2013), Springer, Heidelberg · Zbl 1317.13052

[14] Kang, Seok-Jin; Kashiwara, Masaki; Kim, Myungho; Oh, Se-Jin, Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras IV, Selecta Math. (N.S.), 22, 4, 1987\textendash2015 pp., (2016) · Zbl 1354.81030

[15] Kang, Seok-Jin; Kashiwara, Masaki; Kim, Myungho; Oh, Se-jin, Simplicity of heads and socles of tensor products, Compos. Math., 151, 2, 377\textendash396 pp., (2015) · Zbl 1366.17014

[16] Kashiwara, M., On crystal bases of the Q-analogue of universal enveloping algebras, Duke Math. J., 63, 2, 465\textendash516 pp., (1991) · Zbl 0739.17005

[17] Kashiwara, Masaki, Global crystal bases of quantum groups, Duke Math. J., 69, 2, 455\textendash485 pp., (1993) · Zbl 0774.17018

[18] Kashiwara, Masaki, The crystal base and Littelmann's refined Demazure character formula, Duke Math. J., 71, 3, 839\textendash858 pp., (1993) · Zbl 0794.17008

[19] Kashiwara, Masaki, Crystal bases of modified quantized enveloping algebra, Duke Math. J., 73, 2, 383\textendash413 pp., (1994) · Zbl 0794.17009

[20] Kashiwara, Masaki, On crystal bases. Representations of groups, Banff, AB, 1994, CMS Conf. Proc. 16, 155\textendash197 pp., (1995), Amer. Math. Soc., Providence, RI

[21] Khovanov, Mikhail; Lauda, Aaron D., A diagrammatic approach to categorification of quantum groups. I, Represent. Theory, 13, 309\textendash347 pp., (2009) · Zbl 1188.81117

[22] Khovanov, Mikhail; Lauda, Aaron D., A diagrammatic approach to categorification of quantum groups II, Trans. Amer. Math. Soc., 363, 5, 2685\textendash2700 pp., (2011) · Zbl 1214.81113

[23] Kimura, Yoshiyuki, Quantum unipotent subgroup and dual canonical basis, Kyoto J. Math., 52, 2, 277\textendash331 pp., (2012) · Zbl 1282.17017

[24] Kimura, Yoshiyuki; Qin, Fan, Graded quiver varieties, quantum cluster algebras and dual canonical basis, Adv. Math., 262, 261\textendash312 pp., (2014) · Zbl 1331.13016

[25] Kuniba, Atsuo; Nakanishi, Tomoki; Suzuki, Junji, T-systems and Y-systems in integrable systems, J. Phys. A, 44, 10, 103001, 146 pp., (2011) · Zbl 1222.82041

[26] Lampe, Philipp, A quantum cluster algebra of Kronecker type and the dual canonical basis, Int. Math. Res. Not. IMRN, 13, 2970\textendash3005 pp., (2011) · Zbl 1273.17020

[27] Lampe, P., Quantum cluster algebras of type A and the dual canonical basis, Proc. Lond. Math. Soc. (3), 108, 1, 1\textendash43 pp., (2014) · Zbl 1284.13031

[28] Lauda, Aaron D.; Vazirani, Monica, Crystals from categorified quantum groups, Adv. Math., 228, 2, 803\textendash861 pp., (2011) · Zbl 1246.17017

[29] Leclerc, B., Imaginary vectors in the dual canonical basis of $U_q(\mathfrak{n})$, Transform. Groups, 8, 1, 95\textendash104 pp., (2003) · Zbl 1044.17009

[30] Lee, Kyungyong; Schiffler, Ralf, Positivity for cluster algebras, Ann. of Math. (2), 182, 1, 73\textendash125 pp., (2015) · Zbl 1350.13024

[31] Lusztig, G., Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc., 3, 2, 447\textendash498 pp., (1990) · Zbl 0703.17008

[32] Lusztig, G., Canonical bases in tensor products, Proc. Natl. Acad. Sci. USA, 89, 17, 8177\textendash8179 pp., (1992) · Zbl 0760.17011

[33] Lusztig, George, Introduction to quantum groups, Progress in Mathematics 110, xii+341 pp., (1993), Birkhäuser Boston, Inc., Boston, MA · Zbl 0788.17010

[34] McNamara, Peter J., Representations of Khovanov–Lauda–Rouquier algebras III: symmetric affine type, Math. Z., 287, 1-2, 243\textendash286 pp., (2017) · Zbl 1388.16041

[35] Nakajima, Hiraku, Cluster algebras and singular supports of perverse sheaves. Advances in representation theory of algebras, EMS Ser. Congr. Rep., 211\textendash230 pp., (2013), Eur. Math. Soc., Zürich · Zbl 1353.13024

[36] Nakajima, Hiraku, Quiver varieties and cluster algebras, Kyoto J. Math., 51, 1, 71\textendash126 pp., (2011) · Zbl 1223.13013

[37] Qin, Fan, Triangular bases in quantum cluster algebras and monoidal categorification conjectures, Duke Math. J., 166, 12, 2337\textendash2442 pp., (2017) · Zbl 1454.13037

[38] R08 R.~Rouquier, \emph 2-Kac–Moody algebras, arXiv:0812.5023\/v1.

[39] Rouquier, Raphaël, Quiver Hecke algebras and 2-Lie algebras, Algebra Colloq., 19, 2, 359\textendash410 pp., (2012) · Zbl 1247.20002

[40] Varagnolo, M.; Vasserot, E., Canonical bases and KLR-algebras, J. Reine Angew. Math., 659, 67\textendash100 pp., (2011) · Zbl 1229.17019