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Dyckerhoff, Tobias; Kapranov, Mikhail
Higher Segal spaces. (English) Zbl 1459.18001
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The theory of *Segal spaces* [*C. Rezk*, Trans. Am. Math. Soc. 353, No. 3, 973–1007 (2001; Zbl 0961.18008)], analyzing the role of Segal spaces as a model for the homotopy theory of $(\infty, 1)$-categories, has its roots in the classical work [*G. Segal*, Topology 13, 293–312 (1974; Zbl 0284.55016)] where the notion of a Γ-space was introduced and used to exhibit a slew of classifying spaces as infinite loop spaces. Given a simplicial set X, we have, for each $n \geq 1$, a natural map

$$f_n : X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

where the right-hand side is an n-fold fiber product. The condition that all maps f_n be bijective is called *Segal condition* and a simplicial set abiding by this condition is called *Segal*. The relevance of this condition comes from the fact that it characterizes the essential image of the fully faithful functor

$$\mathrm{N} : \mathcal{C}at \to \mathcal{S}et_\Delta$$

taking a small category to its nerve. Given a Segal simplicial set X, the corresponding category $\mathcal{C}$ is to be recovered with the set of objects being the set of vertices of X and morphisms between a pair of objects being the edges in X between the corresponding pair of vertices. The invertibility of f_2 admits of interpretation of the diagram

$$\mu : \left\{ X_1 \times_{X_0} X_1 \xleftarrow{f_2 = (\partial_2, \partial_0)} X_2 \xrightarrow{\partial_1} X_1 \right\}$$

as a composition law for $\mathcal{C}$, while the bijectivity of f_3 implies the associativity of this law. The theory of Segal spaces is to be viewed as a development of this idea within a homotopy theoretic framework, where simplicial sets are to be replaced by simplicial spaces.

The monograph, consisting of eleven chapters together with an appendix on bicategories, aims, together with a succeeding volume, to study a higher extension of Rezk's theory to what are called *d-Segal spaces*. A synopsis of the book goes as follows.

- Chapters 1–3 work in the elementary context of simplicial *topological spaces*, reducing to a minimum of background in homotopy theory required from the reader and being regarded as an extended introduction to the rest of the book. Particularly, the motivating example of the Waldhausen S-construction originating with *F. Waldhausen*'s work [Lect. Notes Math. None, 318–419 (1985; Zbl 0579.18006)] in algebraic K-theory is addressed in §2.4, where Quillen's concept of an exact category is generalized to a nonadditive setting, called *proto-exact*, to which the definition and properties of the classical Waldhausen S-construction are shown to extend.
- Chapter 3, consisting of 9 sections, axiomatizes of *associative multivalued compositions*, the 2-Segal condition being regarded as the associativity of μ in the sense of composition of correspondence, the only sense in which multivalued maps are to be meaningfully composed. §3.4 gives a procedure an associative mulltication in the usual sense on the linear envelope X_1 to a linear category $\mathcal{H}(X)$ called the *Hall category* of X. §3.5 establishes Theorem 3.5.7 showing how to categorify the Hall category construction one more time so as not to lose any information and to identify 2-Segal sets with certain bicategories. §3.6 shows that unital 2-Segal simplicial sets are to be identified with certain operads. Example 3.1.5 is concerned with examples relating to Bruhat-Tits complexes [*F. Bruhat* and *J. Tits*, Publ. Math., Inst. Hautes Étud. Sci. 41, 5–251 (1972; Zbl 0254.14017)]. §3.7 illustrates the results of §3.5 for an extreme, yet nontrivial class of 2-Segal sets, namely, examples relating to Bruhat-Tits complexes [loc. cit.], set-theoretic solutions of the pentagon equation [*R. M. Kashaev* and *S. M. Sergeev*, Commun. Math. Phys. 195, No. 2, 309–319 (1998; Zbl 0937.16045); *R. M. Kashaev* and *N. Reshetkhin*, Contemp. Math. 433, 267–279 (2007; Zbl 1177.17014)]. §3.8

describes a class of discrete 2-Segal spaces associated to almost complex manifolds, namely, pseudo-holomorphic polygons. §3.9 is concerned with examples of birational Segal schemes.

- In the main body of the text, the authors work in the general context of *combinatorial model categories*, which Chapter 4 recalls, understanding spaces combinatorially as simplicial sets.
- Chapter 5 constructs a model structure $\mathcal{S}_2$ on the category $\mathbb{S}_\Delta$ of simplicial spaces whose fibrant objects are exactly the Reedy fibrant 2-Segal spaces, constructing a chain of left Bousfield localizations
$$(\mathbb{S}_\Delta, \mathcal{I}) \to (\mathbb{S}_\Delta, \mathcal{S}_2) \to (\mathbb{S}_\Delta, \mathcal{S}_1)$$
where $\mathcal{I}$ denotes the Reedy model structure on $\mathbb{S}_\Delta$, while $\mathcal{S}_1$ is the model structure for 1-Segal spaces [*C. Rezk*, Trans. Am. Math. Soc. 353, No. 3, 973–1007 (2001; Zbl 0961.18008)].
- The main result of Chapter 6, consisting of 4 sections, is Theorem 6.3.2, which is concerned with the *path space criterion* expressing the 2-Segal condition for a simplicial object X in terms of 1-Segal conditions for simplicial analogues of the path space of X [*L. Illusie*, Complexe cotangent et déformations. II. Berlin-Heidelberg-New York: Springer-Verlag (1972; Zbl 0238.13017)]. Theorem 6.4.3 is a semi-simplicial variant of Theorem 6.3.2.
- Chapter 7 consists of 5 sections. Quasi-categories and complete 1-Segal spaces provide two equivalent approaches to formalizing the intuitive concept of $(\infty, 1)$-categories. §7.1 recalls a correspondence between the two models [*A. Joyal* and *M. Tierney*, Contemp. Math. 431, 277–326 (2007; Zbl 1138.55016)]. The remaining sections generalizes the Waldhausen S-construction from ordinary categories to quasi-categories.
- Chapter 8 explains how to extract associative algebras from 2-Segal objects by means of *theories with transfer*, which, applied to Waldhausen spaces, recovers variants of Hall algebras, such as classical Hall algebras [*O. Schiffmann*, Sémin. Congr. 24, 1–141 (2012; Zbl 1309.18012)], derived Hall algebras [*B. Toën*, Duke Math. J. 135, No. 3, 587–615 (2006; Zbl 1117.18011)] and motivic Hall algebras [*D. Joyce*, Adv. Math. 210, No. 2, 635–706 (2007; Zbl 1119.14005); *M. Kontsevich* and *Y. Soibelman*, "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations", Preprint, arXiv:0811.2435]. Applying a theory with transfer to other 2-Segal spaces, the authors obtain classically known algebras, such as Hecke algebras, but also new algebras, such as the ones associated to the cyclic nerve of a category.
- Chapter 9 constructs the *Hall monoidal* ∞*-category* associated to a given 2-Segal space X, which is to be interpreted as a categorification of the ordinary Hall algebra.
- Chapter 10 associates, to an ∞-category $\mathcal{C}$ with pullbacks, a new ∞-category $\mathrm{Span}'(\mathcal{C})$ called the ∞*-category of spans in* $\mathcal{C}$, which is an $(\infty, 1)$-categorical variant of the span category introduced in §3.3. Further, it generalizes the above span construction to a relative framework where families of ∞-categories varying in a Cartesian fibration are studied.
- Chapter 11 shows how 2-Segal spaces are to be naturally interpreted within the context of the $(\infty, 2)$-categorical theory of spans developed in the preceding chapter, functorially associating to a unital 2-Segal space X a monad in the $(\infty, 2)$-category of bispans in spaces, called *higher Hall monad* of X.
- Appendix is concerned with bicategories.

A complaint of the reviewer is that indexes are completely lacking.

In a sequel to this volume, the authors anticipate finding out that a Quillen adjunction

$$\mathbb{S}_\Delta \leftrightarrow \left(\mathcal{S}et_\Delta^+\right)/\mathrm{N}\left(\Delta\right)$$

between the category of simplicial spaces endowed with the 2-Segal model structure and the category of marked simplicial sets over $\mathrm{N}(\Delta)$ endowed with the model structre for quadratic operads is really a Quillen equivalence, therefore expecting to provide a natural higher categorical generalization of Deligne's theory of determinant functors [*P. Deligne*, Contemp. Math. 67, 93–117 (1987; Zbl 0629.14008)], and more generally, of the notion of a charade [*M. M. Kapranov*, Prog. Math. 131, 119–151 (1995; Zbl 0858.11062)] due to the second author.

Reviewer: Hirokazu Nishimura (Tsukuba)

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MSC:

Cited in **5** Reviews
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18-02 Research exposition (monographs, survey articles) pertaining to category theory
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19-02 Research exposition (monographs, survey articles) pertaining to K-theory
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