Streicher, Thomas; Weinberger, Jonathan
Simplicial sets inside cubical sets. (English) Zbl 07333619
Theory Appl. Categ. 37, 276-286 (2021).

Intensional Martin-Löf type theory should have a natural interpretation in weak ∞-groupoids [*M. Hofmann* and *T. Streicher*, "The groupoid model refutes uniqueness of identity proofs", in: Proceedings Ninth Annual IEEE Symposium on Logic in Computer Science. 208–212 (1994)]. In the first decade of this millennium it was observed that simplicial sets are a possible implementation of this idea, while around 2006 Voevodsky established that the universe in question validates the so-called Univalent Axiom (UA), which states, roughly speaking, that isomorphiic types are propositionally equal [*C. Kapulkin* and *P. LeFanu Lumsdaine*, "The simplicial model of univalent foundations (after Voevodsky)", Preprint, arXiv:1211.2851]. Coquand et al. developed Cubical Type Theory based on explicit box filling opeations from which UA is to be derived [*C. Cohen* et al., LIPIcs – Leibniz Int. Proc. Inform. 69, Article 5, 34 p. (2018; Zbl 1434.03036); *M. Bezem* et al., LIPIcs – Leibniz Int. Proc. Inform. 26, 107–128 (2014; Zbl 1359.03009)], where the type theory is interpreted in the topos of covariant presheaves over the category of finitely presented free de Morgan algebras. Clearly the standard intensional Martin-Löf type theory fragment together with UA is to be interpreted in the presheaf category $\boldsymbol{cSet}$ over the site $\square$ which is the full subcategory of $\boldsymbol{Poset}$ on finite powers of the 2 element lattice $\mathbb{I}$. This site is op-equivalent to the algebraic theory of distributive lattices [*B. Spitters*, "Cubical sets and the topological topos", Preprint, arXiv:1610.05270]. The topos $\boldsymbol{sSet}$ appears as a subtopos of $\boldsymbol{cSet}$ and actually as an essential subtopos [*K. Kapulkin* and *V. Voevodsky*, J. Topol. 13, No. 4, 1682–1700 (2020; Zbl 07311837); *C. Sattler*, "Idempotent completion of cubes in posets", Preprint, arXiv:1805.04126].

This paper constructs in $\boldsymbol{cSet}$ a fibrant univalent universe for those types that are sheaves, making it possible to consider $\boldsymbol{sSet}$ as a submodel of $\boldsymbol{cSet}$ for univalent Martin-Löf type theory. The authors, furthermore, address the question whether the type-theoretic Cisinski model structure considered on $\boldsymbol{cSet}$ coincides with the test model structure modelling the homotopy theory of spaces, giving a reformulation in terms of the adjoint functors at hand in place of providing an answer to this open problem.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

03B38 Type theory
03G30 Categorical logic, topoi
18F20 Presheaves and sheaves, stacks, descent conditions (category-theoretic aspects)
18N45 Categories of fibrations, relations to K-theory, relations to type theory

Keywords:

cubical sets; simplicial sets; universes; univalence axiom

Software:

cubicaltt

Full Text: Link

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