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Every elementary higher topos has a natural number object. (English) Zbl 07333622
Theory Appl. Categ. 37, 337-377 (2021).

The principal objective in this paper is to establish that, not within the 1-categorical setting but within the higher categorical setting, the existence of a natural number object follows from the axioms of an elementary $(\infty, 1)$ -topos. The proof is divided into three steps.

- [1] The first step is to generalize the construction of the loop space of the circle from the category of spaces to $(\infty, 1)$ -categories (§2).
- [2] The second step is, by using the knowledge of elementary toposes, to realize that one can construct a natural number object in the underlying elementary topos of the $(\infty, 1)$ -category (§3.8).
- [3] The final step goes as follows. First, the construction of the *space of partial maps* is explicitly parameterized over the natural number object (§5.4), by which the desired retract is to be constructed in an $(\infty, 1)$ -setting (§5.16). The $(\infty, 1)$ -categorical analogue of the *object of contractibility* is used in order to establish the contractibility of the space of partial maps (Theorem 5.10). Finally, it is shown, by using the object of contractibility, that the various fibers are compatible (Proposition 5.15).

The existence of natural number objects in elementary $(\infty, 1)$ -toposes has the following consequences.

- One can study infinite colimits (§6.7, §6.10).
- One can show that not every elementary topos is to be lifted to an elementary $(\infty, 1)$ -topos, leading to a natural question whether every elementary topos with a natural number object is to be in fact lifted (§6.5).
- The author studies truncations in an elementary $(\infty, 1)$ -topos [*N. Rasekh*, “An elementary approach to truncations”, Preprint, [arXiv:1812.10527](https://arxiv.org/abs/1812.10527)].
- Using a filter construction for $(\infty, 1)$ -categories, the author can construct elementary $(\infty, 1)$ -toposes with non-standard natural number objects [*N. Rasekh*, “Filter quotients and non-presentable $(\infty, 1)$ -toposes”, Preprint, [arXiv:2001.10088](https://arxiv.org/abs/2001.10088)].

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

03G30 Categorical logic, topoi

18B25 Topoi

18N60 $(\infty, 1)$ -categories (quasi-categories, Segal spaces, etc.); ∞ -topoi, stable ∞ -categories

55U35 Abstract and axiomatic homotopy theory in algebraic topology

Keywords:

elementary topos theory; higher category theory; natural number objects

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