Rasekh, Nima

Every elementary higher topos has a natural number object. (English) Zbl 07333622

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The principal objective in this paper is to establish that, not within the 1-categorical setting but within the higher categorical setting, the existence of a natural number object follows from the axioms of an elementary $(\infty, 1)$-topos. The proof is divided into three steps.

[1] The first step is to generalize the construction of the loop space of the circle from the category of spaces to $(\infty, 1)$-categories (§2).

[2] The second step is, by using the knowledge of elementary toposes, to realize that one can construct a natural number object in the underlying elementary topos of the $(\infty, 1)$-category (§3.8).

[3] The final step goes as follows. First, the construction of the *space of partial maps* is explicitly parameterized over the natural number object (§5.4), by which the desired retract is to be constructed in an $(\infty, 1)$-setting (§5.16). The $(\infty, 1)$-categorical analogue of the *object of contractibility* is used in order to establish the contractibility of the space of partial maps (Theorem 5.10). Finally, it is shown, by using the object of contractibility, that the various fibers are compatible (Proposition 5.15).

The existence of natural number objects in elementary $(\infty, 1)$-toposes has the following consequences.

- One can study infinite colimits (§6.7, §6.10).
- One can show that not every elementary topos is to be lifted to an elementary $(\infty, 1)$-topos, leading to a natural question whether every elementary topos with a natural number object is to be in fact lifted (§6.5).
- The author studies truncations in an elementary $(\infty, 1)$-topos [*N. Rasekh*, "An elementary approach to truncations", Preprint, arXiv:1812.10527].
- Using a filter construction for $(\infty, 1)$-categories, the author can construct elementary $(\infty, 1)$-toposes with non-standard natural number objects [*N. Rasekh*, "Filter quotients and non-presentable $(\infty, 1)$-toposes", Preprint, arXiv:2001.10088].

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

03G30	Categorical logic, topoi
18B25	Topoi
18N60	$(\infty, 1)$-categories (quasi-categories, Segal spaces, etc.); ∞-topoi, stable ∞-categories
55U35	Abstract and axiomatic homotopy theory in algebraic topology

Keywords:

elementary topos theory; higher category theory; natural number objects

Full Text: Link

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[26] Ecole Polytechnique F´´ed´erale de Lausanne, SV BMI UPHESS, Station 8, CH-1015 Lau

[27] sanne, Switzerland

[28] Email:nima.rasekh@epfl.ch

[50] Julie Bergner, University of Virginia:jeb2md (at) virginia.edu

[51] Richard Blute, Universit´e d' Ottawa:rblute@uottawa.ca

[52] Gabriella B¨ohm, Wigner Research Centre for Physics:bohm.gabriella (at) wigner.mta.hu

[53] Valeria de Paiva: Nuance Communications Inc:valeria.depaiva@gmail.com

[54] Richard Garner, Macquarie University:richard.garner@mq.edu.au

[55] Ezra Getzler, Northwestern University:getzler (at) northwestern(dot)edu

[56] Kathryn Hess, Ecole Polytechnique F´ed´erale de Lausanne:kathryn.hess@epfl.ch

[57] Dirk Hofmann, Universidade de Aveiro:dirk@ua.pt

[58] Pieter Hofstra, Universit´e d' Ottawa:phofstra (at) uottawa.ca

[59] Anders Kock, University of Aarhus:kock@math.au.dk

[60] Joachim Kock, Universitat Aut'onoma de Barcelona:kock (at) mat.uab.cat

[61] Stephen Lack, Macquarie University:steve.lack@mq.edu.au

[62] Tom Leinster, University of Edinburgh:Tom.Leinster@ed.ac.uk

[63] Matias Menni, Conicet and Universidad Nacional de La Plata, Argentina:matias.menni@gmail.com

[64] Ieke Moerdijk, Utrecht University:i.moerdijk@uu.nl

[65] Susan Niefield, Union College:niefiels@union.edu

[66] Kate Ponto, University of Kentucky:kate.ponto (at) uky.edu

[67] Robert Rosebrugh, Mount Allison University:rrosebrugh@mta.ca

[68] Jiri Rosicky, Masaryk University:rosicky@math.muni.cz

[69] Giuseppe Rosolini, Universit'a di Genova:rosolini@disi.unige.it

[70] Alex Simpson, University of Ljubljana:Alex.Simpson@fmf.uni-lj.si

[71] James Stasheff, University of North Carolina:jds@math.upenn.edu

[72] Ross Street, Macquarie University:ross.street@mq.edu.au

[73] Tim Van der Linden, Universit´e catholique de Louvain:tim.vanderlinden@uclouvain