

Sumano, Elhoim

The 2-nerve of a 2-group and Deligne’s determinant functors. (English) Zbl 1458.18014
Theory Appl. Categ. 37, 227–265 (2021).

The principal objective in this paper is to establish that the bisimplicial set obtained by application of the 2-nerve functor [*S. Lack and S. Paoli, K-Theory* 38, No. 2, 153–175 (2008; [Zbl 1155.18006](#))] to a 2-group [*J. C. Baez and A. D. Lauda, Theory Appl. Categ.* 12, 423–491 (2004; [Zbl 1056.18002](#))] seen as a bicategory with a single object is a fibrant object in the universal simplicial replacement of *D. Dugger* [*Trans. Am. Math. Soc.* 353, No. 12, 5003–5027 (2001; [Zbl 0974.55011](#))] of the model category of reduced homotopy 2-types.

The paper consists of four sections. §2 addresses, for every $0 \leq n \leq \infty$, two Quillen equivalent simplicial model categories modelling the connected and pointed homotopy n -types. Proposition 2.1 recalls the simplicial model category structure on the category of reduced simplicial sets \mathbf{sSet}_0 , where the cofibrations are the injections, the weak equivalences are the weak homotopy n -equivalences $v^*\mathbf{W}_n$, and the mapping space is the restriction to \mathbf{sSet}_0 of the usual mapping space for pointed simplicial sets. Proposition 2.5 gives a generating set of trivial cofibrations for the family of fibrations between fibrant objects of the model category $(\mathbf{sSet}_0, v^*\mathbf{W}_n)$. On the other hand, §2.8 gives, by applying the construction of *D. Dugger* [*Trans. Am. Math. Soc.* 353, No. 12, 5003–5027 (2001; [Zbl 0974.55011](#))] to the model category $(\mathbf{sSet}_0, v^*\mathbf{W}_n)$, a simplicial model category structure on the category of reduced bisimplicial sets \mathbf{ssSet}_0 , where the cofibrations are the injections, the weak equivalences are the diagonal weak homotopy n -equivalences $d^*\mathbf{W}_n$ and the mapping space is induced from the functor

$$p_1^*(\Delta^*) : \Delta \rightarrow \mathbf{ssSet}_0$$

defined by

$$p_1^*(\Delta^*)_{p,q} = \Delta_p^n$$

Corollary 2.10 gives sufficient conditions for a reduced bisimplicial set to be a fibrant object in the model category $(\mathbf{ssSet}_0, d^*\mathbf{W}_n)$.

§3 begins with some notations about the 2-category $\mathbf{cat}_{Nlax}^\otimes$ of monoidal categories, normal lax monoidal functors and monoidal natural transformations as well as the full 2-subcategory 2-Grp of $\mathbf{cat}_{Nlax}^\otimes$ whose objects are 2-groups. §3.5 defines the geometrical nerve functor

$$\mathcal{N} : \mathbf{cat}_{Nlax}^\otimes \rightarrow \mathbf{sSet}_0$$

as in [*R. Street, J. Pure Appl. Algebra* 49, 283–335 (1987; [Zbl 0661.18005](#))], while §3.8 defines a geometrical nerve \mathbf{ssSet}_0 -valued functor

$$\mathcal{N}^2 : \mathbf{cat}_{Nlax}^\otimes \rightarrow \mathbf{ssSet}_0$$

taking a simplicial resolution of the functor \mathcal{N} . The main result of the paper is Theorem 3.9 claiming that $\mathcal{N}^2(\mathcal{G})$ is a fibrant object of the model category $(\mathbf{ssSet}_0, d^*\mathbf{W}_2)$ for every 2-group \mathcal{G} . It is shown that the functor \mathcal{N}^2 is full and faithful, having an extension to a full and faithful simplicial functor.

§4 deduces, as an application of the main theorem, a well-known theorem about non-symmetric determinant function for Waldhausen categories of derivators [*P. Deligne, Contemp. Math.* 67, 93–117 (1987; [Zbl 0629.14008](#)); *F. Muro and A. Tonks, Adv. Math.* 216, No. 1, 178–211 (2007; [Zbl 1125.19001](#)); *F. Muro et al., Publ. Mat., Barc.* 59, No. 1, 137–233 (2015; [Zbl 1316.19002](#))].

Reviewer: [Hirokazu Nishimura \(Tsukuba\)](#)

MSC:

- 18N50 Simplicial sets, simplicial objects
- 55P15 Classification of homotopy type
- 55U35 Abstract and axiomatic homotopy theory in algebraic topology
- 55P05 Homotopy extension properties, cofibrations in algebraic topology
- 18G45 2-groups, crossed modules, crossed complexes
- 18F25 Algebraic K -theory and L -theory (category-theoretic aspects)

Keywords:

reduced homotopy n -type; geometric nerve for monoidal categories; 2-group; determinant functor; simplicial model category

Full Text: [Link](#)

References:

- [1] J. Adámek and J. Rosický, Locally presentable and accessible categories, London Mathematical Society Lecture Note Series, Cambridge University Press, Vol. 189, 1994. · [Zbl 0795.18007](#)
- [2] J. C. Baez and A. D. Lauda, Higher-dimensional algebra V: 2-Groups, Theory and Applications of Categories, Vol. 12 (14) pp. 423-491, 2004. · [Zbl 1056.18002](#)
- [3] G. Biedermann, On the homotopy theory of n -types, Homology, Homotopy and Applications, Vol. 10 (1), pp. 305-325, 2008. · [Zbl 1138.55015](#)
- [4] F. Borceux, Handbook of Categorical Algebra: Volume 1, Basic Category Theory, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1994. · [Zbl 0911.18001](#)
- [5] 264ELHOIM SUMANO
- [6] D-C. Cisinski, Les préfaisceaux comme modèles des types d'homotopie, Astérisque, Société Mathématique de France, Vol. 308, 2006.
- [7] P. Deligne, Le déterminant de la cohomologie, Current Trends in Arithmetical Algebraic Geometry, Contemporary Mathematics, American Mathematical Society, Vol. 67, pp. 93-177, 1987.
- [8] D. Dugger, Replacing model categories with simplicial ones, Transactions of the American Mathematical Society, Vol. 353 (12), page 5003-5027, 2001. · [Zbl 0974.55011](#)
- [9] J. W. Duskin, Simplicial matrices and the nerves of weak n -categories I: Nerves of bicategories, Theory and Applications of Categories, Vol. 9 (10), pp. 198-308, 2002. · [Zbl 1046.18009](#)
- [10] F. Muro and A. Tonks, The 1-type of a Waldhausen K -theory spectrum, Advances in Mathematics, Vol. 216 (1), pp. 178-211, 2007. · [Zbl 1125.19001](#)
- [11] F. Muro, A. Tonks and M. Witte, On Determinant Functors and K -theory, Publicacions Matemàtiques, Vol. 59 (1), pp 137-233, 2015. · [Zbl 1316.19002](#)
- [12] P. G. Glenn, Realization of cohomology classes in arbitrary exact categories, Journal of Pure and Applied Algebra, Vol. 25 (1), pp. 33-105, 1982. · [Zbl 0487.18015](#)
- [13] P. Goerss and J. Jardine, Simplicial homotopy theory, Progress in Mathematics, Birkhäuser Basel, Vol. 174, 1999. · [Zbl 0949.55001](#)
- [14] P. S. Hirschhorn, Model categories and their localizations, Mathematical Surveys and Monographs, American Mathematical Society, Vol. 99, 2002. · [Zbl 1017.55001](#)
- [15] M. Hovey, Model categories, Mathematical Surveys and Monographs, American Mathematical Society, Vol. 63, 1999.
- [16] A. Joyal and R. Street, Braided tensor categories, Advances in Mathematics, Vol. 102 (1), pp. 20-78, 1993. · [Zbl 0817.18007](#)
- [17] G. M. Kelly, On MacLane's conditions for coherence of natural associativities, commutativities, etc., Journal of Algebra, Vol. 1 (4), pp. 397-402, 1964. · [Zbl 0246.18008](#)
- [18] F. F. Knudsen, Determinant functors on exact categories and their extensions to categories of bounded complexes, The Michigan Mathematical Journal, Vol. 50 (2), pp. 407-445, 2002. · [Zbl 1023.18012](#)
- [19] S. Lack and S. Paoli, 2-nerves for bicategories, K -Theory, Vol. 38 (2), pp.153-175, 2008. · [Zbl 1155.18006](#)
- [20] J. Lurie, Higher topos theory, Annals of mathematics studies, Princeton University Press, Vol. 170, 2009.
- [21] G. Maltsiniotis, La K -théorie d'un dérivateur triangulaire, Categories in Algebra, Geometry and Mathematical Physics, Contemporary Mathematics, American Mathematical Society, Vol. 431, pp. 341-368, 2007.
- [22] J. P. May, Simplicial objects in algebraic topology, Chicago Lectures in Mathematics, University of Chicago Press, 1967. · [Zbl 0769.55001](#)
- [23] B. Noohi, Notes on 2-groupoids, 2-groups, and crossed-modules, Homology, Homotopy and Applications, Vol. 9 (1), pp. 75-106, 2007. · [Zbl 1221.18002](#)
- [24] A. Stanculescu, Constructing model categories with prescribed fibrant objects, Theory and applications of categories, Vol. 29 (23), pp. 635-653, 2014. · [Zbl 1302.18016](#)

- [25] R. Street, The algebra of oriented simplexes, *Journal of Pure and Applied Algebra*, Vol. 49 (3), pp. 283-335, 1987. · [Zbl 0661.18005](#)
- [26] F. Waldhausen, Algebraic K-theory for spaces, *Algebraic and Geometric Topology, Lecture Notes in Mathematics*, Springer-Verlag, Vol. 1126, pp. 318-419, 1983.
- [27] Facultad de Ciencias, UNAM. Mexico.
- [28] Email: sumano@ciencias.unam.mx
- [29] This article may be accessed at <http://www.tac.mta.ca/tac/>
- [30] THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles that significantly advance
- [31] the study of categorical algebra or methods, or that make significant new contributions to mathematical
- [32] science using categorical methods. The scope of the journal includes: all areas of pure category theory,
- [33] including higher dimensional categories; applications of category theory to algebra, geometry and topology
- [34] and other areas of mathematics; applications of category theory to computer science, physics and other
- [35] mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
- [36] Articles appearing in the journal have been carefully and critically refereed under the responsibility of
- [37] for publication.
- [38] Subscription information Individual subscribers receive abstracts of articles by e-mail as they
- [39] are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. Full
- [40] text of the journal is freely available at <http://www.tac.mta.ca/tac/>.
- [41] Information for authors LATEX2 ϵ is required. Articles may be submitted in PDF by email
- [42] directly to a Transmitting Editor following the author instructions at
- [43] <http://www.tac.mta.ca/tac/authinfo.html>.
- [44] Managing editor. Geoff Cruttwell, Mount Allison University: gcruttwell@mta.ca
- [45] TEXnical editor. Michael Barr, McGill University: michael.barr@mcgill.ca
- [46] Assistant TEX editor. Gavin Seal, Ecole Polytechnique Fédérale de Lausanne:
- [47] gavin.seal@fastmail.fm
- [48] Transmitting editors.
- [49] Clemens Berger, Université de Nice-Sophia Antipolis: cberger@math.unice.fr
- [50] Julie Bergner, University of Virginia: jeb2md@virginia.edu
- [51] Richard Blute, Université d'Ottawa: rblute@uottawa.ca
- [52] Gabriella Böhm, Wigner Research Centre for Physics: bohm.gabriella@wigner.mta.hu
- [53] Valeria de Paiva, Nuance Communications Inc: valeria.depaiva@gmail.com
- [54] Richard Garner, Macquarie University: richard.garner@mq.edu.au
- [55] Ezra Getzler, Northwestern University: getzler@northwestern.edu
- [56] Kathryn Hess, Ecole Polytechnique Fédérale de Lausanne: kathryn.hess@epfl.ch
- [57] Dirk Hofmann, Universidade de Aveiro: dirk@ua.pt
- [58] Pieter Hofstra, Université d'Ottawa: phofstra@uottawa.ca
- [59] Anders Kock, University of Aarhus: kock@math.au.dk
- [60] Joachim Kock, Universitat Autònoma de Barcelona: kock@mat.uab.cat
- [61] Stephen Lack, Macquarie University: steve.lack@mq.edu.au
- [62] Tom Leinster, University of Edinburgh: Tom.Leinster@ed.ac.uk
- [63] Matias Menni, Conicet and Universidad Nacional de La Plata, Argentina: matias.menni@gmail.com
- [64] Ieke Moerdijk, Utrecht University: i.moerdijk@uu.nl
- [65] Susan Niefield, Union College: niefiels@union.edu
- [66] Kate Ponto, University of Kentucky: kate.ponto@uky.edu
- [67] Robert Rosebrugh, Mount Allison University: rrosebrugh@mta.ca
- [68] Jiri Rosicky, Masaryk University: rosicky@math.muni.cz
- [69] Giuseppe Rosolini, Università di Genova: rosolini@disi.unige.it
- [70] Alex Simpson, University of Ljubljana: Alex.Simpson@fmf.uni-lj.si
- [71] James Stasheff, University of North Carolina: jds@math.upenn.edu
- [72] Ross Street, Macquarie University: ross.street@mq.edu.au
- [73] Tim Van der Linden, Université catholique de Louvain: tim.

This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.