**Sumano, Elhoim**

**The 2-nerve of a 2-group and Deligne's determinant functors.** (English) Zbl 1458.18014
Theory Appl. Categ. 37, 227-265 (2021).

The principal objective in this paper is to establish that the bisimplicial set obtained by application of the 2-nerve functor [*S. Lack* and *S. Paoli*, *K*-Theory 38, No. 2, 153–175 (2008; Zbl 1155.18006)] to a 2-group [*J. C. Baez* and *A. D. Lauda*, Theory Appl. Categ. 12, 423–491 (2004; Zbl 1056.18002)] seen as a bicategory with a single object is a fibrant object in the universal simplicial replacement of *D. Dugger* [Trans. Am. Math. Soc. 353, No. 12, 5003–5027 (2001; Zbl 0974.55011)] of the model category of reduced homotopy 2-types.

The paper consists of four sections. §2 addresses, for every $0 \leq n \leq \infty$, two Quillen equivalent simplicial model categories modelling the connected and pointed homotopy $n$-types. Proposition 2.1 recalls the simplicial model category structure on the category of reduced simplicial sets $\boldsymbol{sSet}_0$, where the cofibrations are the injections, the weak equivalences are the weak homotopy $n$-equivalences $\upsilon^* \boldsymbol{W}_n$, and the mapping space is the restriction to $\boldsymbol{sSet}_0$ of the usual mapping space for pointed simplicial sets. Proposition 2.5 gives a generating set of trivial cofibrations for the family of fibrations between fibrant objects of the model category $(\boldsymbol{sSet}_0, \upsilon^* \boldsymbol{W}_n)$. On the other hand, §2.8 gives, by applying the construction of *D. Dugger* [Trans. Am. Math. Soc. 353, No. 12, 5003–5027 (2001; Zbl 0974.55011)] to the model category $(\boldsymbol{sSet}_0, \upsilon^* \boldsymbol{W}_n)$, a simplicial model category structure on the category of reduced bisimplicial sets $\boldsymbol{ssSet}_0$, where the cofibrations are the injections, the weak equivalences are the diagonal weak homotopy $n$-equivalences $d^* \boldsymbol{W}_n$ and the mapping space is induced from the functor

$$p_1^* (\Delta^*) : \Delta \to \boldsymbol{ssSet}_0$$

defined by

$$p_1^* (\Delta^*)_{p,q} = \Delta_p^n$$

Corollary 2.10 gives sufficient conditions for a reduced bisimplicial set to be a fibrant object in the model category $(\boldsymbol{ssSet}_0, d^* \boldsymbol{W}_n)$.

§3 begings with some notations about the 2-category $\underline{\boldsymbol{cat}}^{\otimes}_{Nlax}$ of monoidal categories, normal lax monoidal functors and monoidal natural transformations as well as the full 2-subcategory $2-\boldsymbol{Grp}$ of $\underline{\boldsymbol{cat}}^{\otimes}_{Nlax}$ whose objects are 2-groups. §3.5 defines the geometrical nerve functor

$$\mathcal{N} : \boldsymbol{cat}^{\otimes}_{Nlax} \to \boldsymbol{sSet}_0$$

as in [*R. Street*, J. Pure Appl. Algebra 49, 283–335 (1987; Zbl 0661.18005)], while§3.8 defines a geometrical nerve $\boldsymbol{ssSet}_0$-valued functor

$$\mathcal{N}^2 : \boldsymbol{cat}^{\otimes}_{Nlax} \to \boldsymbol{ssSet}_0$$

taking a simplicial resolution of the functor $\mathcal{N}$. The main result of the paper is Theorem 3.9 claiming that $\mathcal{N}^2(\mathcal{G})$ is a fibrant object of the model category $(\boldsymbol{ssSet}_0, d^* \boldsymbol{W}_2)$ for every 2-group $\mathcal{G}$. It is shown that the functor $\mathcal{N}^2$ is full and faithful, having an extension to a full and faithful simplicial functor.

§4 deduces, as an application of the main theorem, a well-known theorem about non-symmetric determinant function for Waldhansen categories of derivators [*P. Deligne*, Contemp. Math. 67, 93–117 (1987; Zbl 0629.14008); *F. Muro* and *A. Tonks*, Adv. Math. 216, No. 1, 178–211 (2007; Zbl 1125.19001); *F. Muro* et al., Publ. Mat., Barc. 59, No. 1, 137–233 (2015; Zbl 1316.19002)].

Reviewer: Hirokazu Nishimura (Tsukuba)

**MSC:**

18N50 Simplicial sets, simplicial objects
55P15 Classification of homotopy type
55U35 Abstract and axiomatic homotopy theory in algebraic topology
55P05 Homotopy extension properties, cofibrations in algebraic topology
18G45 2-groups, crossed modules, crossed complexes
18F25 Algebraic $K$-theory and $L$-theory (category-theoretic aspects)

**Keywords:**

reduced homotopy $n$-type; geometric nerve for monoidal categories; 2-group; determinant functor; simplicial model category

**Full Text:** Link

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[27] Facultad de Ciencias, UNAM. Mexico.

[28] Email:sumano@ciencias.unam.mx

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