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The Moore complex of a simplicial cocommutative Hopf algebra. (English) Zbl 07323982

Theory Appl. Categ. 37, 189-226 (2021).

A *simplicial group* is a simplicial object in the category of groups [*J. P. May*, Simplicial objects in algebraic topology. Princeton, N.J.-Toronto-London-Melbourne: D. van Nostrand Company, Inc. (1968; Zbl 0165.26004)]. The *Moore complex* of a simplicial group is its normalized chain complex [Séminaire Henri Cartan: Algèbres d'Eilenberg-MacLane et homotopie. 7e année 1954/55. Vols. I, II. Exposés I–XI, XII–XXII. École Normale Supérieure. Paris: Secrétariat mathématique. (1955; Zbl 0067.15601); Séminaire Henri Cartan: Algèbres d'Eilenberg-MacLane et homotopie. 7e année 1954/55. 2e éd., revue et corrigée. Paris: Secrétariat mathématique (1956; Zbl 0075.32002)], yielding a Moore complex functor from the simplicial abelian groups to the category of chain complexes of abelian groups, which is an equivalence called *Dold-Kan correspondence* [*A. Dold*, Ann. Math. (2) 68, 54–80 (1958; Zbl 0082.37701); *D. M. Kan*, Trans. Am. Math. Soc. 87, 330–346 (1958; Zbl 0090.39001)] and which was generalized to abelian categories in [*A. Dold* and *D. Puppe*, Ann. Inst. Fourier 11, 201–312 (1961; Zbl 0098.36005)], to *semi-abelian categories* in [*D. Bourn*, Contemp. Math. 431, 105–124 (2007; Zbl 1143.18013)], and to more general source categories and settings rather than simplicial objects in [*S. Lack* and *R. Street*, J. Pure Appl. Algebra 219, No. 10, 4343–4367 (2015; Zbl 1317.18016); ibid. 224, No. 3, 1364–1366 (2020; Zbl 1423.18032)]. The author focuses on Bourn's treatment of semi-abelian categories considering the Moore complex structure on semi-abelian categories, of which the category of cocommutative Hopf algebras as well as the category of groups and that of Lie algebras is an instance. Bourn's work is based on [*T. Everaert* and *T. Van der Linden*, Theory Appl. Categ. 12, 1–33 (2004; Zbl 1065.18011); ibid. 12, 195–224 (2004; Zbl 1065.18012)], while semi-abelian categories were introduced in [*G. Janelidze* et al., J. Pure Appl. Algebra 168, No. 2–3, 367–386 (2002; Zbl 0993.18008)]. For brevity, the author uses the word "semi-abelian category" for a category with binary coproducts which is pointed, Barr exact and Bourn protomodular. It was shown in [*P. Carrasco* and *A. M. Cegarra*, J. Pure Appl. Algebra 75, No. 3, 195–235 (1991; Zbl 0742.55003)] that the Moore complex construction yields a functor from the category of simplicial groups to the category of *hypercrossed complexes* of groups. The same thing holds for the Lie algebra case [*P. Carrasco* and *A. M. Cegarra*, Theory Appl. Categ. 32, 1165–1212 (2017; Zbl 1377.55014)]. Hypercrossed complexes, capturing crossed modules and 2-crossed modules for dimensions 1 and 2, respectively, are strongly related to this paper.

A *crossed module* is a group homomorphism

$$\partial : E \to G$$

together with an action $\triangleright$ of G on E by automorphisms obeying

$$\partial\left(g \triangleright e\right) = g\partial\left(e\right)g^{-1} \ \text{(equivalence)}$$

and

$$\partial\left(e\right)f = efe^{-1} \ \text{(Peiffer condition)}$$

for all $e, f \in E$ and $g \in G$. The notion was introduced in [Zbl 0040.38801] as an algebraic model of connected homotopy 2-group [*R. Brown* and *C. B. Spencer*, Nederl. Akad. Wet., Proc., Ser. A 79, 296–302 (1976; Zbl 0333.55011)]. Categorically speaking, crossed modules are equivalent to internal categories of groups [*J.-L. Loday*, J. Pure Appl. Algebra 24, 179–202 (1982; Zbl 0491.55004)]. 2-*crossed modules* of groups were introduced in [*D. Conduché*, J. Pure Appl. Algebra 34, 155–178 (1984; Zbl 0554.20014)]. In the light of the close relationship between groups and Lie algebras, the Lie algebraic case of the whole 2-crossed module theory was given in [*G. J. Ellis*, J. Aust. Math. Soc., Ser. A 54, No. 3, 393–419 (1993; Zbl 0824.17022)].

Hopf algebras [*M. E. Sweedler*, Hopf algebras. New York: W.A. Benjamin, Inc. (1969; Zbl 0194.32901)] are to be thought of as a unification of groups and Lie algebras as being the group algebra of a group and the universal enveloping algebra of a Lie group. Conversely, one has the functors *Gl* and *Prim* from the category of Hopf algebras to that of groups and of Lie algebras assigning group-like and primitive elements, respectively. It is important to note that both the group algebra and the universal enveloping algebra turn to a specific type of Hopf algebras called *cocommutative* . One has a coreflection functor [*H.-E. Porst*, J. Algebra 328, No. 1, 254–267 (2011; Zbl 1232.18008)]

$$R : \{\mathrm{Hopf}\} \to \{\mathrm{Hopf}^{\mathrm{cc}}\}$$

from the category of all Hopf algebras to the full and replete subcategory of cocommutative Hopf algebras, which is semi-abelian. There exists a category $\{\mathrm{Grp} \ltimes \mathrm{Lie}\}$ whose objects are triples $(G, L, \triangleright)$, where G is a group, L is a Lie algebra and $\triangleright$ is a representation of G on L by Lie algebra maps. One has functors

$$\{\mathrm{Grp} \ltimes \mathrm{Lie}\} \to \{\mathrm{Hopf}^{\mathrm{cc}}\}$$
$$\{\mathrm{Hopf}^{\mathrm{cc}}\} \to \{\mathrm{Grp} \ltimes \mathrm{Lie}\}$$

where one has to recall that the group of group-like elements acts on the set of primitive elements by conjugation. Thus one is now on the scene of the *Cartier-Gabriel-Kostant-Milnor-Moore theorem.*

The principal objective in this paper is to define 2-crossed modules of cocommutative Hopf algebras, unifying the theory of 2-crossed modules of groups and of Lie algebras when the functors Gl and $Prim$ are taken into consideration. There is no generally accepted definition of crossed modules of Hopf algebras, but the author adopts *S. Majid*'s in ["Strict quantum 2-groups", Preprint, arXiv:1208.6265]. As for the group and Lie algebra case, the functorial relationship between simplicial objects and 2-crossed modules within the category of cocommutative Hopf algebras is found out. To this end, the explicit definition of a Moore complex of a simplicial cocommutative Hopf algebra is given, being constructed via Hopf kernels. This unifies the Moore complex of groups and Lie algebras in the sense of the same functors. One then obtains a 2-crossed module structure from a simplicial cocommutative Hopf algebra with Moore complex of length 2 with the iterated Peiffer pairings, getting the functor

$$\left\{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq 2}\right\} \to \{\mathrm{X}_2\mathrm{Hopf}^{\mathrm{cc}}\}$$

On the other hand, one already has the functor

$$\left\{\mathrm{SimpGrp}_{\leq 2}\right\} \to \{\mathrm{X}_2\mathrm{Grp}\}$$

from the category of simplicial groups with Moore complex of length 2 to the category of 2-crossed modules of groups [*A. Mutlu* and *T. Porter*, Theory Appl. Categ. 4, 148–173 (1998; Zbl 0917.18006)], while similarly one has the functor

$$\left\{\mathrm{SimpLie}_{\leq 2}\right\} \to \{\mathrm{X}_2\mathrm{Lie}\}$$

for the category of Lie algebras [*G. J. Ellis*, J. Aust. Math. Soc., Ser. A 54, No. 3, 393–419 (1993; Zbl 0824.17022)]. One finally has the following comutative diagram

$$\begin{array}{ccccc} \left\{\mathrm{SimpGrp}_{\leq 2}\right\} & \leftarrow & \left\{\mathrm{SimpHopf}^{\mathrm{cc}}_{\leq 2}\right\} & \rightarrow & \left\{\mathrm{SimpLie}_{\leq 2}\right\} \\ \downarrow & & \downarrow & & \downarrow \\ \{\mathrm{X}_2\mathrm{Grp}\} & \leftarrow & \{\mathrm{X}_2\mathrm{Hopf}^{\mathrm{cc}}\} & \rightarrow & \{\mathrm{X}_2\mathrm{Lie}\} \end{array}$$

in which the horizontal arrows are extended from Gl and $Prim$, respectively. The outer vertical arrows are known to be equivalence of categories, while the auther adds the middle vertical arrow as an equivalence as well. This ensures the coherence of the 2-crossed module definition.

This study is the first step towards enhancing understanding of Hopf algebras in terms of category theory and algebraic topology for higher dimensions with crossed modules considered as 1-dimensional categorical objects, paving the way towards unification of higher dimensional structures for groups and Lie algebras within the category of cocommutative Hopf algebras.

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MSC:

16T05 Hopf algebras and their applications
16S40 Smash products of general Hopf actions
18G45 2-groups, crossed modules, crossed complexes
55U10 Simplicial sets and complexes in algebraic topology
55U15 Chain complexes in algebraic topology

Keywords:

Hopf algebra; simplicial object; Moore complex; 2-crossed module

Full Text: Link

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