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Theories, sites, toposes. Relating and studying mathematical theories through topos-theoretic 'bridges'. (English) Zbl 06638227

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This book is devoted to a general theory of geometric theories by exploiting the associated classifying toposes. The central theme of the book is a duality between the subtoposes of the classifying topos of a geometric theory $\mathbb{T}$ over a signature Σ and the quotients of the theory $\mathbb{T}$.

A synopsis of the book consisting of 10 chapters goes as follows. Chapter 1 provides the topos-theoretic background. Chapter 2 is divided into two sections.

- §2.1 reviews the fundamental notion of classifying topos of a geometric theory, discussing the appropriate kinds of interpretations between theories which induce morphisms between the associated classifying toposes. A characterization theorem for universal models of geometric theories inside classifying toposes is established.
- §2.2 explains the general unifying technique "toposes as bridges" allowing of extracting concrete information from the existence of different representations for the classifying topos of a geometric theory.

Chapter 3 establishes a duality theorem between the subtoposes of the classifying topos of a geometric theory and the quotients of the theory. The author discusses its proof-theoretic significance, obtaining an alternative proof of the theorem proof-theoretically. The chapter is concluded with a deduction theorem for geometric logic.By exploiting the main theorem of the previous chapter, Chapter 4 transfers many ideas and concepts of elementary topos theory to geometric logic. Chapter 5 is devoted to flat functors in relation to classifying toposes.

Chapter 6 gives a systematic investigation of the class of theories classified by a presheaf topos, particularly obtaining a characterization theorem (Theorem 6.3.1) on necessary and sufficient conditions for a theory to be of presheaf type in terms of the models of the theory in arbitary Grothendieck toposes. Chapter 7, consisting of two sections, introduces the concept of expansion of a geometric theory, developing some basic theory about it.

- §7.1 investigates expansions of geometric theories from a viewpoint of geometric morphisms between classifying toposes. In particular, the author introduces the notion of localic (respectively hyperconnected) expansion, showing that it naturally corresponds to the notion of localic (respectively hyperconnected) geometric morphism at the level of classifying toposes, which yields a syntactic description of the hyperconnected-localic factorization of a geometric morphism. The problem of expanding a given geometric theory $\mathbb{T}$ to a theory classified by the topos $[\mathrm{f.p.}\mathbb{T}-\mathrm{mod}\left(\mathbf{Set}\right), \mathbf{Set}]$ is then addressed, describing a general method for constructing such expansions.
- §7.2 investigates to what extent the conditions of Theorem 6.3.1 are preserved by faithful interpretations of geometric theories, notably examples being theories obtained by quotients and injectivizations. A general analysis on the relationship between finitely presentable and finitely generated models of a given geometric theory is carried out.

Chapter 8, consisting of two sections, investigates the quotients of a given theory of presheaf type $\mathbb{T}$ by means of Grothendieck topologies, establishing a semantic representation for the classifying topos of such a quotient as a subtopos of the classifying topos of $\mathbb{T}$.

- §8.1 addresses such semantic representations, introducing the notion of J-homogeneous $\mathbb{T}$-model provided with a theory of presheaf type $\mathbb{T}$ classified by a topos $[\mathcal{C}, \mathbf{Set}]$ and a Grothendieck topology J on the category $\mathcal{C}^{\mathrm{op}}$. It is shown that the models of the quotient of $\mathbb{T}$ corresponding to the subtopos

$$\mathbf{Sh}\left(\mathcal{C}^{\mathrm{op}}, J\right) \hookrightarrow [\mathcal{C}, \mathbf{Set}]$$

are precisely the $\mathbb{T}$-models which are J-homogeneous. Explicit axiomatizations for the J-homogeneous models in terms of J and of the formulas which present the finitely presentable $\mathbb{T}$-models are obtained. Restricting attention to the classifying toposes of quotients with enough set-based models, the author establishes, for any such quotient, a characterization of the Grothendieck topology corresponding to it in terms of its category of set-based models. Coherent quotients of a cartesian theory and the Grothendieck topologies corresponding to them are also discussed, showing that the lattice operations on Grothendieck topologies naturally restrict to the collection of finite-type ones and that one can effectively compute the lattice operations on the set of quotients of a given geometric theory by calculations in the associated lattices of Grothendieck topologies.

- §8.2 identifies two main independent conditions ensuring that a quotient of a given theory of presheaf type is again so. The author addresses the problem of finding a geometric theory classified by a given presheaf topos $[\mathcal{K}, \mathbf{Set}]$, establishing a general theorem claiming that if the category $\mathcal{K}$ is a full subcategory of the category of finitely presentable models of a theory of presheaf type $\mathbb{T}$, there exists a quotient of $\mathbb{T}$ classified by the topos $[\mathcal{K}, \mathbf{Set}]$, which is to be explicitly described in terms of $\mathbb{T}$ and $\mathcal{K}$ with the proviso of some natural conditions.

Chapter 9 discusses some classical as well as new examples of theories of presheaf type from the perspective of the theory established in the previous chapters. Chapter 10, consisting of 8 sections, is concerned with applications.

- §10.1 addresses the problem of restricting Morita equivalences to quotients of the two theories.
- §10.2 contains a solution to a problem posed by F. W. Lawvere concerning the effect of the boundary operation on subtoposes on quotients classified by them.
- §10.3 is concerned with a number of results connecting syntactic properties of quotients of theories of presheaf type and geometric properties of the Grothendieck topologies corresponding to them via the duality theorem of Chapter 3.
- §10.4 gives a topos-theoretic interpretation of Fraïssé's construction in model theory leading to new results on countably categorical theories,
- §10.5 describes a general theory of topological Galois representations.
- §10.6 presents the solution to I. Moerdijk's long-standing problem of finding an intrinsic semantic characterization of geometric logic in [*O. Caramello*, Ann. Pure Appl. Logic 162, No. 4, 318–321 (2011; Zbl 1247.03141)].
- §10.7 gives an analysis of the maximum spectrum of a commutative ring with unit from a viewpoint of the dualaity theorem in Chapter 3.
- §10.8 investigates compactness conditions for geometric theories allowing of identification of theories lying in smaller fragments of geometric logic.

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MSC:

03-02	Research exposition (monographs, survey articles) pertaining to mathematical logic and foundations
03G30	Categorical logic, topoi
18B25	Topoi
18C10	Theories (e.g., algebraic theories), structure, and semantics

Full Text: DOI