Gambino, Nicola; **Lobbia, Gabriele**

On the formal theory of pseudomonads and pseudodistributive laws. (English) Zbl 1457.18023

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Monads are one of the fundamental notions of category theory. *Beck's theorem* on *distributive laws* between monads [*J. Beck*, Lect. Notes Math. 80, 119–140 (1969; Zbl 0186.02902)] describes concisely the structure that is necessary and sufficient so as to combine two algebraic structures so that the operations of one distribute over those of the other. The formal theory of monads, having been introduced in [*R. Street*, J. Pure Appl. Algebra 2, 149–168 (1972; Zbl 0241.18003)] and having later been developed in [*S. Lack* and *R. Street*, J. Pure Appl. Algebra 175, No. 1–3, 243–265 (2002; Zbl 1019.18002)], offers an elegant account of the theory of monads, starting with the observation that the notion of a monad is to be defined within any 2-category and providing a characterization of the existence of categories of Eilenberg-Moore algebras as a completeness property and a simple account of Beck's theorem on distributive laws.

Pseudomonads are the counterparts of monads in 2-dimensional category theory obtained by requiring the axioms for a monad to hold only up to coherent isomorphisms rather than strictly [*M. C. Bunge*, Trans. Am. Math. Soc. 197, 355–390 (1974; Zbl 0358.18004)]. Knowing how the formal theory of monads offers a simple proof of Beck's theorem on distributive laws, it is natural to try to develop a formal theory of pseudomonads, confronting a notoriously hard problem that, just as the formal theory of monads is formulated within 2-dimensional category theory [*G. M. Kelly* and *R. Street*, Lect. Notes Math. 420, 75–103 (1974; Zbl 0334.18016)], the formal theory of pseudomonads is to be developed within 3-dimensional category theory [*R. Gordon* et al., Coherence for tricategories. Providence, RI: American Mathematical Society (AMS) (1995; Zbl 0836.18001); *N. Gurski*, Coherence in three-dimensional category theory. Cambridge: Cambridge University Press (2013; Zbl 1314.18002); *T. Leinster*, Bull. Lond. Math. Soc. 47, No. 3, 550–553 (2015; Zbl 1321.00040)], where it is convenient to work with *Gray-categories*, i.e., semistrict tricategories. So far there has not been a direct counterpart of the 2-category $\boldsymbol{Mnd}(\mathcal{K})$ of monads, monad morphisms and monad transformations.

This paper aims to take some further steps in the development of the formal theory of pseudomonads. The main results are as follows.

- Theorem 2.5 answers the question in [*S. Lack*, Adv. Math. 152, No. 2, 179–202 (2000; Zbl 0971.18008)], claiming that, for every Gray-category $\mathcal{K}$, there is a Gray-category $\boldsymbol{Psm}(\mathcal{K})$ of pseudomonads, pseudomonad morphisms, pseudomonad transformations and pseudomonad modifications in $\mathcal{K}$.
- Theorem 3.4 is the analogue of a fundamental result of the formal theory of monads, assering that $\boldsymbol{Psm}(\mathcal{K})$ is equivalent, in a suitable 3-categorical sense, to the Gray-category $\boldsymbol{Lift}(\mathcal{K})$.
- Proposition 4.4 gives an identification of pseudodistributive laws with an object in $\boldsymbol{Psm}(\boldsymbol{Psm}(\mathcal{K}))$.
- Following naturally from Theorem 2.5 and Theorem 3.4,Theorem 4.5 claims the equivalence between pseudodistributive laws and liftings of pseudomonads to 2-categories of pseudoalgebras, which has already been established in [*F. Marmolejo*, Theory Appl. Categ. 5, 91–147 (1999; Zbl 0919.18004), Theorems 6.2, 9.3 and 10.2], The authors take a modular, abstract approach to the verification of the coherence conditions, avoiding completely the notion of a composite of pseudomonads with compatible structure.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18N10	2-categories, bicategories, double categories
18C15	Monads (= standard construction, triple or triad), algebras for monads, homology and derived functors for monads
18C20	Eilenberg-Moore and Kleisli constructions for monads

Keywords:

pseudomonads; distributive laws; Gray-categories

Full Text: Link

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[2] School of Mathematics, University of Leeds

[3] Woodhouse Lane, Leeds LS2 9JT, United Kingdom

[4] Email:n.gambino@leeds.ac.uk mmgl@leeds.ac.uk