Anel, Mathieu; Biedermann, Georg; Finster, Eric; Joyal, André
A generalized Blakers-Massey theorem. (English) Zbl 1456.18017
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The classical *Blakers-Massey theorem* [*A. L. Blakers* and *W. S. Massey*, Proc. Natl. Acad. Sci. USA 35, 322–328 (1949; Zbl 0040.25801); Ann. Math. (2) 53, 161–205 (1951; Zbl 0042.17301); Ann. Math. (2) 55, 192–201 (1952; Zbl 0046.40604); Ann. Math. (2) 58, 409–417 (1953; Zbl 0053.12901)], aka the *homotopy excision theorem*, is one of the most fundamental facts in homotopy theory, claiming that, given a homotopy pushout diagram of spaces

$$\begin{array}{ccc} A & \overset{g}{\to} & C \\ f \downarrow & \lrcorner & \downarrow \\ B & \to & D \end{array}$$

such that the map f is m-connected and the map g is n-connected, the canonical map $A \to B \times_D C$ to the homotopy pullback is in fact $(m+n)$-connected, giving rise to the *Freudenthal suspension theorem* [*H. Freudenthal*, Compos. Math. 5, 299–314 (1937; Zbl 0018.17705)] and therefore paving the way to *stable homotopy theory*.

A new proof of this theorem [*K.-B. Hou (Favonia)* et al., in: Proceedings of the 2016 31st annual ACM/IEEE symposium on logic in computer science, LICS 2016, New York City, NY, USA, July 5–8, 2016. New York, NY: Association for Computing Machinery (ACM). 565–574 (2016; Zbl 1395.55011)] was found in the context of *homotopy type theory* providing an elementary axiomatization of homotopy-theoretic reasoning [*The Univalent Foundations Program*, Homotopy type theory. Univalent foundations of mathematics. Princeton, NJ: Institute for Advanced Study; Raleigh, NC: Lulu Press (2013; Zbl 1298.03002)], which is generally thought to serve as an internal language for the ∞-topoi as developed by Rezk [`https://faculty.math.illinois.edu/\char126\relaxrezk/homotopy-topos-sketch.pdf`] and [*J. Lurie*, Higher topos theory. Princeton, NJ: Princeton University Press (2009; Zbl 1175.18001)]. The original proof in [*K.-B. Hou (Favonia)* et al., in: Proceedings of the 2016 31st annual ACM/IEEE symposium on logic in computer science, LICS 2016, New York City, NY, USA, July 5–8, 2016. New York, NY: Association for Computing Machinery (ACM). 565–574 (2016; Zbl 1395.55011)] was translated into the language of higher category theory by Rezk [`https://faculty.math.illinois.edu/\char126\relaxrezk/freudenthal-and-blakers-massey.pdf`]. The main result of this paper is a much generalized theorem, applying not only to spaces, but to an arbitrary ∞-topos. Indeed, as was shown in [*M. Anel* et al., J. Topol. 11, No. 4, 1100–1132 (2018; Zbl 1423.18009)], the generalized theorem is to be applied to an appropriate presheaf topos, yielding an analogue of the Blakers-Massey theorem in the context of Goodwille's calculus of functors.

The authors observe that the n-connected maps form the left class of a *factorization system* $(\mathfrak{L}, \mathfrak{R})$ on the category of spaces with the additional property that the left class $\mathfrak{L}$ is stable under base change, referring to a factorization system obedient to this condition as a *modality*. The main result of this paper goes as follows.

Theorem. Let $\mathcal{E}$ be an ∞-topos and $(\mathfrak{L}, \mathfrak{R})$ a modality on $\mathcal{E}$. Write $\Delta h : A \to A \times_B A$ for the diagonal of a map $h : A \to B \in \mathcal{E}$ and $-\Box_Z-$ for the pushout product in the slice category $\mathcal{E}_{/Z}$. Given a pushout square

$$\begin{array}{ccc} Z & \overset{g}{\to} & Y \\ f \downarrow & \lrcorner & \downarrow \\ X & \to & W \end{array}$$

in $\mathcal{E}$ with $\Delta f \Box_Z \Delta g \in \mathfrak{L}$, the canonical map $(f, g) : Z \to X \times_W Y$ is also in $\mathfrak{L}$.

A similar generalization of the Blakers-Massey theorem was obtained by *W. Chachólski* et al. [Ann. Inst. Fourier 66, No. 6, 2641–2665 (2016; Zbl 1368.55006)], though their method involves the manipulation of *weak cellular inequalities* of spaces, as introduced in [*E. Dror Farjoun*, Cellular spaces, null spaces and homotopy localization. Berlin: Springer-Verlag (1995; Zbl 0842.55001)], whereas the authors focus on ∞-topos-theoretic tools such as descent.

A synopsis of the paper consisting of four sections goes as follows. §2 fixes higher-categorical conventions and recalls some elementary facts. §3 proceeds as follows.

- It begins by introducing the notion of a factorization system as well as the pushout product and pullback hom.
- It then gives a short treatment of the n-connected/n-truncated factorization system in an ∞-topos.
- It introduces the notion of a modality itself, providing a number of examples and deriving some elementary properties, including the dual Blakers-Massey theorem.
- It concludes with the descent theorem for $\mathfrak{L}$-cartesian squares.

§4 turns to the proof of the generalized Blakers-Massey theorem itself, finishing with the derivation of the classical theorem as well as that of Chacholski-Scherner-Werndli.

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MSC:

18N20 Tricategories, weak n-categories, coherence, semi-strictification
18B25 Topoi
18N45 Categories of fibrations, relations to K-theory, relations to type theory
55U35 Abstract and axiomatic homotopy theory in algebraic topology
03B38 Type theory

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