Schopieray, Andrew
Norm, trace, and formal codegrees of fusion categories. (English) Zbl 1456.18013
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Fusion categories together with their many variants such as tensor, spherical, braided and modular ones are a vast generalization of the representation theory of finite groups and finite-dimensional Hopf algebras. The *formal codegrees* of a fusion category C, which are a finite collection of numerical invariants associated to representations of the underlying Grothendieck ring, have proven to be critical, including the *Frobenius-Perron dimension* $\mathrm{FP}\dim(C)$ and *global dimension* $\dim(C)$ for spherical fusion categories. Formal codegrees are highly restrictive from a number-theoretic viewpoint, since they are examples of *totally positive cyclotomic integers*, and the less-familiar *algebraic d-numbers* [*V. Ostrik*, Math. Res. Lett. 16, No. 5–6, 895–901 (2009; Zbl 1204.18003), Definition 1.1].

This paper consisting of five sections aims to expand the general theory of formal codegrees of fusion categories mainly contained in [*V. Ostrik*, Math. Res. Lett. 16, No. 5–6, 895–901 (2009; Zbl 1204.18003); Contemp. Math. 728, 169–180 (2019; Zbl 1423.18023)]. It is shown (Theorem 3.1) that for each $m \in \mathbb{Z}_{\geq 1}$ there exist finitely many fusion categories C up to equivalence with $N(\dim(C)) = m$, where, for a formal codegree $f \in N(f)$ denotes the *norm* of f, or the product of its Galois conjugates. This norm finiteness holds also for $\mathrm{FP}\dim(C)$ but because the set of all Frobenius-Perron dimensions of fusion categories is a discrete subset of the positive real numbers [*P. Bruillard* et al., J. Am. Math. Soc. 29, No. 3, 857–881 (2016; Zbl 1344.18008), Corollary 3.13]. §4 observes that divisibility of algebraic d-numbers is equivalent to divisibility of their norms. It is demonstrated (Theorem 4.4) that if f is a formal codegree of a fusion category C which is not divisible by any rational integer, then the formal codegrees of C are precisely the Galois orbits of f. There is merely one family of this type known, namely, $\mathcal{C}(\mathfrak{sl}_2, \kappa-2)_{\mathrm{ad}}$ for prime $\kappa \in \mathbb{Z}_{\geq 3}$ coming from the representation theory of $\mathcal{U}_q(\mathfrak{sl}_2)$ with q a root of unity [*A. Schopieray*, Contemp. Math. 747, 1–26 (2020; Zbl 1436.18018)]. It is also demonstarated (Theorem 4.13) that if C is a spherical fusion category with a formal degree f of square-free norm, then $f \in \mathbb{Z}$ or $f = \frac{1}{2}\left(5 \pm \sqrt{5}\right)$. A spherical fusion category possessing either $\frac{1}{2}\left(5 \pm \sqrt{5}\right)$ as a formal codegree is equivalent to one of the rank 2 spherical fusion categories Fib or its Galois conjugate Fib^{σ} [*V. Ostrik*, Math. Res. Lett. 10, No. 2–3, 177–183 (2003; Zbl 1040.18003)].

§5 is concerned with applications of the above general theory to classification results of spherical fusion categoriesspherical fusion categories. It was asked in [*Z. Yu*, "Pre-modular fusion categories of small global dimensions", Preprint, `arXiv:2001.00785`, Question 38] whether a spherical braided fusion category of prime global dimension is pointed or equivalent to $\mathrm{Fib} \boxplus \mathrm{Fib}^{\sigma}$. Theorem 5.12 answers this question affirmatively, generalizing the result to arbitary number fields, which is to say that any spherical braided fusion category whose global dimension has prime norm is pointed with the exception of Fib, Fib^{σ} and $\mathrm{Fib} \boxplus \mathrm{Fib}^{\sigma}$. The reason why these exceptional cases can occur is that the categorical dimensions of their simple objects are exceptional cases of a classical result of *J. W. S. Cassels* [J. Reine Angew. Math. 238, 112–131 (1969; Zbl 0179.35203), Lemma 3] on cyclotomic integers α such that the absolute trace of $|\alpha|\mathrm{I}^2$ is less than 2. In Theorem 5.5, the author removes the assumption of a braiding for spherical fusion categories with global dimension whose norm is a *safe prime*, which is of the form $p = 2q+1$ with $q \in \mathbb{Z}_{\geq 1}$ being also prime. It is expected, but currently not established, that there exist infinitely many safe primes. The reason why the *exceptional* spherical fusion categories can occur in this case is that all of their formal codegrees are of the form $p \cdot u$ where u is one of the three totally positive algebraic integers of smallest absolute trace, namely, 1 or $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$. For any $\lambda < 2$, the proof of finiteness, and classification of totally positive algebraic integers of absolute trace strictly less than λ is known as the *Schur-Siegel-Smyth trace problem* [*J. Aguirre* and *J. C. Peral*, Lond. Math. Soc. Lect. Note Ser. 352, 1–19 (2008; Zbl 1266.11113)], for which *I. Schur* [Math. Z. 1, 377–402 (1918; JFM 46.0128.03), Satz VIII] solved the finiteness problem for $\lambda = \sqrt{e}$ and *C. L. Siegel* [Ann. Math. (2) 46, 302–312 (1945; Zbl 0063.07009)] settled the classification for $\lambda = \frac{3}{2}$, satisfying the incumbent purposes in this paper.

This paper lies in the tradition of the Schur-Siegel-Smyth trace problem and the results on absolute trace of Cassels' having been exploited in relation to fusion categories in the literature.

- *S. Gelaki* et al. [Algebra Number Theory 3, No. 8, 959–990 (2009; Zbl 1201.18006), Proposition

6.2] used Siegel's initial trace bound to investigate the number of zeros in the S-matrix of a weakly integral modular tensor category, which can be seen as an extension of simple results for zeros of characters of finite groups [*F. Stan* and *A. Zaharescu*, J. Reine Angew. Math. 637, 217–234 (2009; Zbl 1242.11078)].

- Calegari, Morrison and Snyder [Zbl 1220.18004] improved upon Cassels result [*J. W. S. Cassels*, J. Reine Angew. Math. 238, 112–131 (1969; Zbl 0179.35203), Lemma 3] as a means to classify the smallest possible Frobenius-Perron dimensions of objects in fusion categories.
- *F. Calegari* and *Z. Guo* [Trans. Am. Math. Soc. 370, No. 9, 6515–6533 (2018; Zbl 1429.11195)] made further improvements. This paper illuminates the necessity for algebraic number theory in the study of fusion categories.

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MSC:

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