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**Ultrafilters, finite coproducts and locally connected classifying toposes.** (English)
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The principal objective in this paper is to describe a single category-theoretic result from which the notions of ultrafilter, ultrapower, tensor product of ultrafilters and Blass' category $\mathcal{UF}$ of ultrafilters, together with their interrelations, all flow naturally, and which, with only a little more effort, is able to speak towards the applications of ultrafilters in model theory. The first main result (Theorem 13 and Corollary 14) goes as follows.

Theorem. The category $\mathsf{FC}(\mathcal{S}\mathsf{et}, \mathcal{S}\mathsf{et})$ of finite-coproduct-preserving endofunctors of $\mathcal{S}\mathsf{et}$ is equivalent to the category $[\mathcal{UF}, \mathcal{S}\mathsf{et}]$ of functors on Blass' category of ultrafilters $\mathcal{UF}$. Under this equivalence, the ultrapower functor $(-)^{\mathcal{U}}$ corresponds to the representable functor at the ultrafilter $\mathcal{U}$.

The above result, claiming that once we know what a finite-coproduct-preserving endofunctor of $\mathcal{S}\mathsf{et}$ is, everything else is forced, is built on Börger's characterization [Zbl 0443.18005] of the functor

$$\beta : \mathcal{S}\mathsf{et} \to \mathcal{S}\mathsf{et}$$

sending a set $X$ to its set of ultrafilters as the terminal finite-coproduct-preserving endofunctor of $\mathcal{S}\mathsf{et}$. One thing that this theorem does not capture is the notion of ultraproduct, for which the second main result (Theorem 22) is obtained as the following generalization of the above theorem now dealing with ultrafilters on objects of an *extensive* [Zbl 0784.18001] category $\mathcal{C}$.

Theorem. Let $\mathcal{C}$ be extensive. The category $\mathsf{FC}(\mathcal{C}, \mathcal{S}\mathsf{et})$ of finite-coproduct-preserving functors from $\mathcal{C}$ to $\mathcal{S}\mathsf{et}$ is equivalent to the functor category $[\mathcal{UF}_{\mathcal{C}}, \mathcal{S}\mathsf{et}]$, where ultrafilters on the Boolean algebra give rise to a category $\mathcal{UF}_{\mathcal{C}}$ as a generalization of Blass' $\mathcal{UF}$.

Ultraproducts are to be recaptured from this theorem by taking

$$\mathcal{C} = \mathcal{S}\mathsf{et}^X$$

in which we have an equivalence

$$[\mathcal{UF}_{\mathcal{S}\mathsf{et}^X}, \mathcal{S}\mathsf{et}] \simeq \mathsf{FC}(\mathcal{S}\mathsf{et}^X, \mathcal{S}\mathsf{et})$$

so that the ultraproduct functors

$$\Pi_{\mathcal{U}} : \mathcal{S}\mathsf{et}^X \to \mathcal{S}\mathsf{et}$$

Taking $\mathcal{C}$ to be the classifying Boolean pretopos of a theory $\mathbb{T}$ of classical first-order logic, ultrafilters on $A \in \mathcal{C}$ correspond to model-theoretic types in context $A$, which allows of reconstructing a categorical treatment in [Zbl 0527.03042].

The third main result (Theorem 26) as a generalization of the first main result obtained by varying not only the domain category but also the codomain category goes as follows.

Theorem. Let $\mathcal{C}$ be extensive and $\mathcal{E}$ a locally connected Grothendieck topos$\partial$.The category $\mathsf{FC}(\mathcal{C}, \mathcal{E})$ of finite-coproduct-preserving functors from $\mathcal{C}$ to $\mathcal{E}$ is equivalent to the functor category $[\mathcal{UF}_{\mathcal{C}}, \mathcal{E}]$.

This theorem allows of reconstructing the indexed sum of ultrafilters by remarking that, for any sets $X$ and $Y$, we have an equivalence

$$[\mathcal{UF}_{\mathcal{S}\mathsf{et}^X}, \mathcal{S}\mathsf{et}^Y] \simeq \mathsf{FC}(\mathcal{S}\mathsf{et}^X, \mathcal{S}\mathsf{et}^Y)$$

and defining a *generalized ultraproduct* functor

$$\mathcal{S}\mathsf{et}^X \to \mathcal{S}\mathsf{et}^Y$$

which corresponds to a pointwise representable functor

$$\mathcal{UF}_{\mathcal{S}\mathsf{et}^X} \to \mathcal{S}\mathsf{et}^Y$$

to be represented as an *ultraspan*. The author has the idea of relating Makkai's *ultracategories* [Zbl 0649.03050; Zbl 0522.03006] to the theorem via the machinery of *enriched categories* [Zbl 0495.18009; Zbl 0478.18005], whose full development is relegated to a subsequent paper.

The final main result (Theorem 42) constructs the *locally connected classifying topos* of a suitable pretopos $\mathcal{C}$, the construction being similar to the *topos of types* [Zbl 0527.03042; Zbl 0521.03051] in that it provides a first-order analogue to the operation of canonical extension [Zbl 0045.31505; Zbl 0045.31601; Zbl 0855.06009] on propositional theories.

Theorem. Let $\mathcal{C}$ be a small De Morgan pretopos. The topos $\mathcal{S}\mathsf{h}(\mathcal{UF}_{\mathcal{C}})$ is a locally connected classifying topos for $\mathcal{C}$ and is itself De Morgan.

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### MSC:

03C20 Ultraproducts and related constructions
03G30 Categorical logic, topoi
18B25 Topoi
18A35 Categories admitting limits (complete categories), functors preserving limits, completions

### Keywords:

ultrafilter; ultracategory; canonical extension; topos of types; locally connected classifying topos

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