Masuda, Naruki; Thomas, Hugh; Tonks, Andy; Vallette, Bruno

The diagonal of the associahedra. (La diagonale de l'associaèdre.) (English. French summary)

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This paper has a threefold purpose.

1. to introduce a general machinery to solve the problem of the approximation of the diagonal of *face-coherent families of polytopes* (§2),
2. to give a complete proof for the case of the *associahedra* (Theorem 1), and
3. to popularize the resulting *magical formula* (Theorem 2).

The problem of the approximation of the diagonal of the associahedra lies at the crossroads of three clusters of domains.

1. There are mathematicians inclined to apply it in their work of computing the homology of fibered spaces in algebraic topology [*E. H. Brown jun.*, Ann. Math. (2) 69, 223–246 (1959; Zbl 0199.58201); *A. Prouté*, Repr. Theory Appl. Categ. 2011, No. 21, 99 p. (2011; Zbl 1245.55007)], to construct tensor products of string field theories [*M. R. Gaberdiel* and *B. Zwiebach*, Nucl. Phys., B 505, No. 3, 569–624 (1997; Zbl 0911.53044); Phys. Lett., B 410, No. 2–4, 151–159 (1997; Zbl 0911.53046)], or to consider the product of Fukaya $\mathcal{A}_\infty$-categories in symplectic geometry [*P. Seidel*, Fukaya categories and Picard-Lefschetz theory. Zürich: European Mathematical Society (EMS) (2008; Zbl 1159.53001); *L. Amorim*, Int. J. Math. 28, No. 4, Article ID 1750026, 38 p. (2017; Zbl 1368.53057)].
2. The analogous result is known, within the kingdom of operad theory and homotopical algebra, in the differential graded context [*S. Saneblidze* and *R. Umble*, Homology Homotopy Appl. 6, No. 1, 363–411 (2004; Zbl 1069.55015); *L. J. Billera* and *B. Sturmfels*, Ann. Math. (2) 135, No. 3, 527–549 (1992; Zbl 0762.52003)].
3. This result is appreciated conceptualy as a new development in the theory of *fiber polytopes* [loc. cit.] by combinatorists and discrete geometers.

The possible ways of iteraring a binary product are to be encoded by planar binary trees, the associativity relation being interpreted as an order relation, which encouraged Dov Tamari to introduce the so-called *Tamari lattice* [*D. Tamari*, Nieuw Arch. Wiskd., III. Ser. 10, 131–146 (1962; Zbl 0109.24502)]. These lattices are to be realized by associahedra in the sense that their 1-skeleton is the Hasse diagram of the Tamari lattice [*C. Ceballos* et al., Combinatorica 35, No. 5, 513–551 (2015; Zbl 1389.52013)]. For loop spaces, composition fails to be strictly associative due to the different parametrizations, but this failure is governed by an infinite sequence of higher homotopies, which was made precise by *J. D. Stasheff* [Trans. Am. Math. Soc. 108, 275–292, 293–312 (1963; Zbl 0114.39402)], introducing a family of curvilinear polytopes called the *Stascheff polytopes*, whose combinatorics coincides with the associahedra. Staschеff's work opened the door to the study of homotopical algebra by means of operad-like objects, summoning [*J. M. Boardman* and *R. M. Vogt*, Homotopy invariant algebraic structures on topological spaces. Berlin-Heidelberg-New York: Springer-Verlag (1973; Zbl 0285.55012)] and [*J. P. May*, The geometry of iterated loop spaces. Berlin-Heidelberg-New York: Springer-Verlag (1972; Zbl 0244.55009)] in particular, the latter of which introduced the *little disks* operads playing a key role in many domains nowadays. In dimension 1, this gives the *little intervals* operad, a finite-dimensional topological operad pervious to Staschеff's theory, whose operad structure is given by scaling a configuration of intervals in order to insert it into another interval. So far there has been no operad structure on any family of convex polytopal realizations of the associahedra in the literature, though there should be a rainbow bridge between operad theory as well as homotopy theories on the one hand and combinatorics as well as discrete geometry on the other.

Since the set-theoretic diagonal of a polytope fails to be cellular in general, the authors need to find a *cellular approximation to the diagonal*, that is to say, a cellular map from the polytope to its cartesian square homotopic to the diagonal. For a coherent family of polytopes, it is highly challenging to find a family of diagonals compatible with the combinatorics of faces. In the case of the first face-coherent family of polytopes, the geometric simplices, such a diagonal map is given by the classical *Alexander-*

Whitney map [*S. Eilenberg* and *J. A. Zilber*, Am. J. Math. 75, 200–204 (1953; Zbl 0050.17301); *N. E. Steenrod*, Ann. Math. (2) 48, 290–320 (1947; Zbl 0030.41602)]. For the next family given by cubes, a coassociative approximation to the diagonal is straightforward [*J.-P. Serre*, Ann. Math. (2) 54, 425–505 (1951; Zbl 0045.26003)]. The associahedra form the face-coherent family of polytopes coming next in terms of further truncations of the simplices or of combinatorial complexity. While a face of a simplex or a cube is a simplex or a cube of lower dimension, a face of an associahedra is a product of associahedra of lower dimensions, which makes the problem of the approximation of the diagonal in this turn highly intricate. The two-fold principal result of the paper (Theorem 1) is an explicit operad structure on the *Loday realizations* of the associahedra together with a compatible approximation to the diagonal.

There is a dichotomy between pointwise and cellular formulas. To investigate their relationship and to make precise the various face-coherent properties, the authors introduce the *category of polytopes with subdivision*. The definition of the diagonal maps comes from the theory of fiber polytopes so that an induced polytopal subdivision of the associahedra is obtained, for which the authors establishes a magical formula in the verbalism of Jean-Louis Loday. It is made up of the pairs of cells of matching dimensions and comparable order under the Tamari order (Theorem 2).

A synopsis of the paper consisting of four sections goes as follows. §1 recalls the main relevant notions, introducing the category of polytopes in which the authors work. §2 gives a canonical definition of the diagonal map for positively oriented polytopes, addressing their cellular properties. §3 endows the family of Loday realizations of the associahedra with a nonsymmetric operad structure compatible with the diagonal maps. §4 establishes the magical celluar formula for the diagonal map of the associahedra.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18M75 Topological and simplicial operads
18M70 Algebraic operads, cooperads, and Koszul duality
52B11 n-dimensional polytopes
55P35 Loop spaces
06A07 Combinatorics of partially ordered sets

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