Publ. RIMS, Kyoto Univ.
19 (1983), 305-316

A Cut-Free Sequential System for the Propositional Modal Logic of Finite Chains

By

Hirokazu NISHIMURA*

Abstract

The main purpose of this paper is to give a cut-free Gentzen-type sequential system for **K4.3G** of finite chains. The cut-elimination theorem is proved both model-theoretically and proof-theoretically.

§1. Introduction

There are thousands of modal logics, only a bit of which enjoy Gentzen-type sequential formulations. Modal logics with cut-free sequential systems are even fewer and it is often a challenging problem to find out such pleasant formulations to a given modal logic. See Zeman [7] for the general reference and Sato [5] for an example of recent such attempts. The main purpose of this paper is to give a cut-free sequential system for **K4.3G** of Gabbay [2, §25].

Formulas (of **K4.3G**) are constructed from propositional variables p and $\bot$ (falsity) by using $\supset$ (implication) and $\Box$ (necessity). Other connectives like $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation) can be introduced as defined symbols in the usual manner. A *structure* (for **K4.3G**) is a quadruple (S, R, s_0, D_s), where

(1) S is a nonempty finite set.
(2) R is an irreflexive transitive binary relation on S such that either xRy or yRx for any distinct $x, y \in S$.
(3) $s_0 \in S$.
(4) For any $s \in S$, D_s assigns a truth-value 0 or 1 to every propositional variable.

Given a structure (S, R, s_0, D_s), the truth-value $\|A\|_s$ of a formula A at $s \in S$ is defined inductively as follows:

Received May 24, 1982.
* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

(1) $\|p\|_s=D_s(p)$ for any propositional variable p.
(2) $\|\perp\|_s=0$.
(3) $\|A\supset B\|_s=1$ iff $\|A\|_s=0$ or $\|B\|_s=1$.
(4) $\|\Box A\|_s=1$ iff for any $t\in S$ such that sRt, $\|A\|_t=1$.

If $\|A\|_{s_0}=1$ for any structure (S, R, s_0, D_s), then A is called *valid*, notation: $\models A$.

K4.3G can be axiomatized by the classical propositional calculus plus the following axioms and inference rules.

(A1) $\Box(A\supset B)\supset(\Box A\supset\Box B)$
(A2) $\Box A\supset\Box\Box A$
(A3) $\Box(\Box A\supset A)\supset\Box A$
(A4) $\Box(\Box A\supset B)\vee\Box(B\wedge\Box B\supset A)$
(R1) $\dfrac{A}{\Box A}$

We write $\vdash_{\mathbf{K4.3G}}A$ if A is provable in the above formal system.

Theorem 1.1. *For any formula A, $\vdash_{\mathbf{K4.3G}}A$ iff $\models A$.*

In the next section we present our sequential system **SK4.3G** and establish its cut-freeness semantically while its purely syntactic proof is given in Section 3. Finally we admit that this paper was inspired by a cut-free sequential system of Leivant [4] for the modal logic **K4G** of finite partial orders but its subtle error in the proof of the cut-elimination theorem is corrected in our more general context.

§ 2. Cut-free System for K4.3G

A *sequent* is an ordered pair (Γ, Δ) of (possibly empty) finite sets of formulas, which we usually denote by $\Gamma\rightarrow\Delta$. We use such self-explanatory notations as $A, \Gamma\rightarrow\Delta, B$ for $\{A\}\cup\Gamma\rightarrow\Delta\cup\{B\}$ and $\Box\Gamma$ for $\{\Box A: A\in\Gamma\}$ freely.

Our sequential formal system **SK4.3G** ("S" for "Sequential") consists of the following axioms and inference rules:

Axioms: $A\rightarrow A$

$\perp\rightarrow$

Rules: $\dfrac{\Gamma\rightarrow\Delta}{\Pi, \Gamma\rightarrow\Delta, \Lambda}$ (thin)

$\dfrac{\Gamma\rightarrow\Delta, A \quad B, \Pi\rightarrow\Lambda}{A\supset B, \Gamma, \Pi\rightarrow\Delta, \Lambda}$ ($\supset$L)

$\dfrac{A, \Gamma\rightarrow\Delta, B}{\Gamma\rightarrow\Delta, A\supset B}$ ($\supset$R)

$$\frac{\{\Gamma,\ \Box\Gamma,\ \Box\Pi\to\Pi,\ \Box\Lambda :\ \Pi\cup\Lambda=\Delta,\ \Pi\cap\Lambda=\emptyset \text{ and } \Pi\neq\emptyset\}}{\Box\Gamma\to\Box\Delta}\ (\text{GL4.3})$$

, where $\Delta\neq\emptyset$ in (GL4.3).

It is easy to show that the following rules are admissible in **SK4.3G**.

$$\frac{\Gamma\to\Delta,\ A}{\neg A,\ \Gamma\to\Delta}\ (\neg\text{L})$$

$$\frac{A,\ \Gamma\to\Delta}{\Gamma\to\Delta,\ \neg A}\ (\neg\text{R})$$

$$\left.\begin{array}{c}\dfrac{A,\ \Gamma\to\Delta}{A\wedge B,\ \Gamma\to\Delta}\\[2ex] \dfrac{B,\ \Gamma\to\Delta}{A\wedge B,\ \Gamma\to\Delta}\end{array}\right\}\ (\wedge\text{L})$$

$$\frac{\Gamma\to\Delta,\ A \quad \Gamma\to\Delta,\ B}{\Gamma\to\Delta,\ A\wedge B}\ (\wedge\text{R})$$

$$\frac{A,\ \Gamma\to\Delta \quad B,\ \Gamma\to\Delta}{A\vee B,\ \Gamma\to\Delta}\ (\vee\text{L})$$

$$\left.\begin{array}{c}\dfrac{\Gamma\to\Delta,\ A}{\Gamma\to\Delta,\ A\vee B}\\[2ex] \dfrac{\Gamma\to\Delta,\ B}{\Gamma\to\Delta,\ A\vee B}\end{array}\right\}\ (\vee\text{R})$$

If $\Gamma\to\Delta$ is provable in **SK4.3G**, we write $\vdash_{\mathbf{SK4.3G}}\Gamma\to\Delta$. We notice that the rule (GL4.3) has the variable number of upper sequents, depending on the number $|\Delta|$. If $|\Delta|=1$, our rule (GL4.3) degenerates into the rule (GL) of Leivant [4].

$$\frac{\Gamma,\ \Box\Gamma,\ \Box A\to A}{\Box\Gamma\to\Box A}\ (\text{GL})$$

If $|\Delta|=2$, then the rule (GL4.3) goes as follows:

$$\frac{\Gamma,\ \Box\Gamma,\ \Box A,\ \Box B\to A,\ B \qquad \Gamma,\ \Box\Gamma,\ \Box A\to A,\ \Box B \qquad \Gamma,\ \Box\Gamma,\ \Box B\to\Box A,\ B}{\Box\Gamma\to\Box A,\ \Box B}$$

To deepen the reader's understanding of the rule (GL4.3), we shall show that $\vdash_{\mathbf{SK4.3G}}\to\Box(\Box A\supset B)\vee\Box(B\wedge\Box B\supset A)$.

We have the following proof π_1 of the sequent $\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A)\to\Box A\supset B,\ B\wedge\Box B\supset A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{B\to B}{\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A),\ \Box A,\ B\to A,\ B}\ (\text{thin})}{\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A),\ \Box A,\ B\wedge\Box B\to A,\ B}\ (\wedge\text{L})}{\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A),\ \Box A\to B,\ B\wedge\Box B\supset A}\ (\supset\text{R})}{\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A)\to\Box A\supset B,\ B\wedge\Box B\supset A}\ (\supset\text{R})$$

We have the following proof π_2 of the sequent $\Box(\Box A\supset B)\to\Box A\supset B,\ \Box(B\wedge\Box B\supset A)$.

$$\begin{array}{cl}
A\to A & \\ \hline
\Box(\Box A\supset B),\ \Box A\supset B,\ \Box A,\ A,\ \Box(B\wedge\Box B\supset A),\ B\wedge\Box B\to A & (\text{thin}) \\ \hline
\Box(\Box A\supset B),\ \Box A\supset B,\ \Box A,\ A,\ \Box(B\wedge\Box B\supset A)\to B\wedge\Box B\supset A & (\supset\text{R}) \\ \hline
\Box(\Box A\supset B),\ \Box A\to\Box(B\wedge\Box B\supset A) & (\text{GL4.3}) \\ \hline
\Box(\Box A\supset B),\ \Box A\to B,\ \Box(B\wedge\Box B\supset A) & (\text{thin}) \\ \hline
\Box(\Box A\supset B)\to\Box A\supset B,\ \Box(B\wedge\Box B\supset A) & (\supset\text{R})
\end{array}$$

We have the following proof π_3 of the sequent $\Box(B\wedge\Box B\supset A)\to\Box(\Box A\supset B),\ B\wedge\Box B\supset A$.

$$\begin{array}{cl}
B\to B & \\ \hline
\Box(B\wedge\Box B\supset A),\ B\wedge\Box B\supset A,\ \Box B,\ B,\ \Box(\Box A\supset B),\ \Box A\to B & (\text{thin}) \\ \hline
\Box(B\wedge\Box B\supset A),\ B\wedge\Box B\supset A,\ \Box B,\ B,\ \Box(\Box A\supset B)\to\Box A\supset B & (\supset\text{R}) \\ \hline
\Box(B\wedge\Box B\supset A),\ \Box B\to\Box(\Box A\supset B) & (\text{GL4.3}) \\ \hline
\Box(B\wedge\Box B\supset A),\ \Box B\to\Box(\Box A\supset B),\ A & (\text{thin}) \\ \hline
\Box(B\wedge\Box B\supset A),\ B\wedge\Box B\to\Box(\Box A\supset B),\ A & (\wedge\text{L}) \\ \hline
\Box(B\wedge\Box B\supset A)\to\Box(\Box A\supset B),\ B\wedge\Box B\supset A & (\supset\text{R})
\end{array}$$

Therefore

$$\begin{array}{cl}
\vdots\ \}\pi_1 \qquad \vdots\ \}\pi_2 \qquad \vdots\ \}\pi_3 & \\ \hline
\to\Box(\Box A\supset B),\ \Box(B\wedge\Box B\supset A) & (\text{GL4.3}) \\ \hline
\to\Box(\Box A\supset B)\vee\Box(B\wedge\Box B\supset A) & (\vee\text{R})
\end{array}$$

A sequent $\Gamma\to\Delta$ is called *realizable* if for some structure (S, R, s_0, D_s), $\|A\|_{s_0}=1$ for any $A\in\Gamma$ and $\|B\|_{s_0}=0$ for any $B\in\Delta$. A sequent $\Gamma\to\Delta$ which is not realizable is called *valid*, notation: $\models\Gamma\to\Delta$.

Theorem 2.1. (Soundness Theorem). *For any sequent $\Gamma\to\Delta$, if $\vdash_{\mathbf{SK4.3G}}\Gamma\to\Delta$, then $\models\Gamma\to\Delta$.*

Proof. By induction on a proof of $\Gamma\to\Delta$.

Corollary 2.2. (Consistency). *The empty sequent $\to$ is not provable in* **SK4.3G**.

Theorem 2.3. (Completeness Theorem). *For any sequent $\Gamma\to\Delta$, if $\models\Gamma\to\Delta$, then $\vdash_{\mathbf{SK4.3G}}\Gamma\to\Delta$.*

Proof. Let $\Gamma\to\Delta$ be the given sequent. We denote by Ω the set of all subformulas occurring in a formula of $\Gamma\cup\Delta$. A sequent $\Pi\to\Lambda$ is called Ω-*saturated* if it satisfies the following conditions:

(1) $\nvdash_{\mathbf{SK4.3G}}\Pi\to\Lambda$.

(2) $\Pi\cup\Lambda\subseteq\Omega$.

(3) For any $A\in\Omega-(\Pi\cup\Lambda)$, $\vdash_{\mathbf{SK4.3G}}\Pi\to\Lambda, A$ and $\vdash_{\mathbf{SK4.3G}}A, \Pi\to\Lambda$.

Assuming that $\nvdash_{\mathbf{SK4.3G}}\Gamma\to\Delta$, we shall show that $\not\models\Gamma\to\Delta$. We denote by $W(\Omega)$ the set of all Ω-saturated sequents. Since $\nvdash_{\mathbf{SK4.3G}}\Gamma\to\Delta$, the sequent $\Gamma\to\Delta$ can be extended to some $\Gamma_0\to\Delta_0\in W(\Omega)$. For any set Σ of formulas, $(\Sigma)_\Box$ denotes the set of all formulas A such that $\Box A\in\Sigma$. If $(\Delta_0)_\Box=\emptyset$, then let $S=\{\Gamma_0\to\Delta_0\}$. If $(\Delta_0)\neq\emptyset$, $\nvdash_{\mathbf{SK4.3G}}\Box(\Gamma_0)_\Box\to\Box(\Delta_0)_\Box$. Therefore, taking the rule (GL4.3) into consideration, there exist two sets Σ_1, Σ_2 of formulas such that:

(1) $\Sigma_1\neq\emptyset$.
(2) $\Sigma_1\cup\Sigma_2=(\Delta_0)_\Box$.
(3) $\Sigma_1\cap\Sigma_2=\emptyset$.
(4) $\nvdash_{\mathbf{SK4.3G}}(\Gamma_0)_\Box, \Box(\Gamma_0)_\Box, \Box\Sigma_1\to\Sigma_1, \Box\Sigma_2$.

The sequent $(\Gamma_0)_\Box, \Box(\Gamma_0)_\Box, \Box\Sigma_1\to\Sigma_1, \Box\Sigma_2$ can be extended to some $\Gamma_1\to\Delta_1\in W(\Omega)$. We notice that:

(1) $(\Gamma_0)_\Box\subset(\Gamma_1)_\Box$ (By $\subset$ we denote the proper inclusion).
(3) $(\Gamma_0)_\Box\subseteqq\Gamma_1$.
(3) $(\Delta_0)_\Box\subseteqq\Delta_1\cup(\Delta_1)_\Box$.

If $(\Delta_1)_\Box=\emptyset$, then we let $S=\{\Gamma_0\to\Delta_0, \Gamma_1\to\Delta_1\}$. If $(\Delta_1)_\Box\neq\emptyset$, we repeat the above process. In any case we finally obtain a sequence $\{\Gamma_i\to\Delta_i\}_{i=0}^{k}$ of $W(\Omega)$ such that:

(1) $(\Gamma_0)_\Box\subset(\Gamma_1)_\Box\subset\cdots\subset(\Gamma_k)_\Box$.
(2) $(\Gamma_i)_\Box\subseteqq\Gamma_{i+1}$ for any $0\leqq i<k$.
(3) $(\Delta_i)_\Box\subseteqq\Delta_{i+1}\cup(\Delta_{i+1})_\Box$ for any $0\leqq i\leqq k-1$.
(4) $(\Delta_i)_\Box\neq\emptyset$ for any $i<k$ and $(\Delta_k)_\Box=\emptyset$.

Set $S=\{\Gamma_0\to\Delta_0, \cdots, \Gamma_k\to\Delta_k\}$. We let $(\Gamma_i\to\Delta_i)R(\Gamma_j\to\Delta_j)$ iff $i<j$. Let $s_0=\Gamma_0\to\Delta_0$. We define $D_{\Gamma_i\to\Delta_i}$ as follows:

$D_{\Gamma_i\to\Delta_i}(p)=1$ iff $p\in\Gamma_i$ for any propositional variable p.

It is not difficult to show by induction on $A\in\Omega$ that for any $0\leqq i\leqq k$,

(1) $\|A\|_{\Gamma_i\to\Delta_i}=1$ if $A\in\Gamma_i$.
(2) $\|A\|_{\Gamma_i\to\Delta_i}=0$ if $A\in\Delta_i$.

In particular, we can conclude that $\Gamma_0\to\Delta_0$ is realizable and so is $\Gamma\to\Delta$. This completes the proof.

Corollary 2.4. *For any formula A, $\vdash_{\mathbf{K4.3G}}A$ iff $\vdash_{\mathbf{SK4.3G}}\to A$.*

Corollary 2.5. *The following inference rule is admissible in* **SK4.3G.**

$$\frac{\Gamma\to\Delta, A \quad A, \Pi\to\Lambda}{\Gamma, \Pi\to\Delta, \Lambda}\ \text{(cut)}$$

In the next section we will give a purely syntactical proof of Corollary 2.5.

§ 3. Cut-Elimination Theorem

The main purpose of this section is to give a proof-theoretical proof of Corollary 2.5 by amending Gentzen's original proof (for **LK** and **LJ**) such as seen in Takeuti [6].

Theorem 3.1. (Cut-elimination Theorem). *The following inference is admissible in* **SK4.3G.**

$$\frac{\Gamma_1\to\Delta_1,\ A \quad A,\ \Gamma_2\to\Delta_2}{\Gamma\to\Delta}\ \text{(cut)}$$

$$(\Gamma=\Gamma_1\cup\Gamma_2 \text{ and } \Delta=\Delta_1\cup\Delta_2)$$

For technical reasons we deal with a slightly modified version of **SK4.3G**, say **SK4.3G′**, which is obtainable from SK4.3G by restricting rules (thin) and ($\supset$L) to the following (thin)$_\Box$ and ($\supset$L)′

$$\frac{\Box(\Gamma)_\Box\to\Box(\Delta)_\Box}{\Gamma\to\Delta}\ (\text{thin})_\Box$$

$$\frac{\Gamma\to\Delta,\ A \quad B,\ \Gamma\to\Delta}{A\supset B,\ \Gamma\to\Delta}\ (\supset\text{L})'$$

and instead adopting as axioms sequents $\Gamma\to\Delta$ satisfying at least one of the following conditions:

(1) $p\in\Gamma\cap\Delta$ for some propositional variable p.

(2) $\bot\in\Gamma$.

Lemma 3.2. *The following rule* (thin L) *is admissible in* **SK4.3G′**.

$$\frac{\Gamma\to\Delta}{A,\ \Gamma\to\Delta}\ \text{(thin L)}$$

Proof. It is sufficient to deal with proof figures which contain only one (thin L) as the last inference.

$$\ddots\ \vdots\ \cdot^{\cdot^{\cdot}}$$

$$\frac{\Gamma\to\Delta}{A,\ \Gamma\to\Delta}\ \text{(thin L)}$$

The proof is carried out by double induction mainly on the formula A and secondly on the length of longest threads of the proof of $\Gamma\to\Delta$. Here we deal only with a special case of (GL4.3) being the last inference of the proof of $\Gamma\to\Delta$.

$$
\begin{array}{c}
\vdots \\
\Box\Gamma, \Gamma, B\to B \\ \hline
\Box\Gamma\to\Box B \\ \hline
A, \Box\Gamma\to\Box B
\end{array}
\quad
\begin{array}{l} \text{(GL4.3)} \\ \text{(thin L)} \end{array}
$$

If A is not of the form $\Box C$, then (thin L) degenerates into $(\text{thin})_{\Box}$ which is of course admissible. If A is of the form $\Box C$, the above proof figure is transformed into:

$$
\begin{array}{c}
\vdots \\
\Box\Gamma, \Gamma, B\to B \\ \hline
\Box C, \Box\Gamma, \Gamma, \Box B\to B \\ \hline
\Box C, C, \Box\Gamma, \Gamma, \Box B\to B \\ \hline
\Box C, \Box\Gamma\to\Box B
\end{array}
\quad
\begin{array}{l} \text{(thin L)} \\ \text{(thin L)} \\ \text{(GL4.3)} \end{array}
$$

Therefore the induction process works well.

Lemma 3.3. *The following rule* (thin R) *is admissible in* **SK4.3G′**.

$$
\frac{\Gamma\to\Delta}{\Gamma\to\Delta, A}\ \text{(thin R)}
$$

Proof. It is sufficient to deal with proof figures which contain only one (thin R) as the last inference.

$$
\begin{array}{c}
\vdots \\
\Gamma\to\Delta \\ \hline
\Gamma\to\Delta, A
\end{array}
\ \text{(thin R)}
$$

The proof is carried out by double induction mainly on the formula A and secondly on the length of longest threads of the proof of $\Gamma\to\Delta$. Here we deal only with a special case of (GL4.3) being the last inference of the proof of $\Gamma\to\Delta$.

$$
\begin{array}{c}
\vdots \\
\Box\Gamma, \Gamma, \Box B\to B \\ \hline
\Box\Gamma\to\Box B \\ \hline
\Box\Gamma\to\Box B, A
\end{array}
\quad
\begin{array}{l} \text{(GL4.3)} \\ \text{(thin R)} \end{array}
$$

If A is not of the form $\Box C$, then (thin R) degenerates into $(\text{thin})_{\Box}$, which is of course admissible. If A is of the form $\Box C$, the above proof figure is transformed into:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Box\Gamma,\Gamma,\Box B\to B}}{\dfrac{\Box\Gamma,\Gamma,\Box B,\Box C\to B}{\Box\Gamma,\Gamma,\Box B,\Box C\to B,C}\text{(thin R)}}\text{(thin L)} \quad \dfrac{\dfrac{\vdots}{\Box\Gamma,\Gamma,\Box B\to B}}{\Box\Gamma,\Gamma,\Box B\to B,\Box C}\text{(thin R)} \quad \dfrac{\dfrac{\dfrac{\vdots}{\Box\Gamma,\Gamma\to\Box B}}{\Box\Gamma,\Gamma,\Box C\to\Box B}\text{(thin L)}}{\Box\Gamma,\Gamma,\Box C\to\Box B,C}\text{(thin R)}}{\Box\Gamma\to\Box B,\ \Box C}\text{(GL4.3)}$$

Therefore the induction process works well.

Lemma 3.4. *The rule* (thin) *is admissible in* **SK4.3G′**.

Proof. Follows readily from Lemmas 3.2 and 3.3.

Lemma 3.5. *For any formula* A, $\vdash_{\mathbf{SK4.3G'}} A\to A$.

Proof. By induction on A. Here we deal only with the case of A being of the form $\Box B$.

$$\dfrac{\dfrac{\dfrac{\vdots}{B\to B}}{\Box B,\ B,\ \Box B\to B}\text{(thin)}}{\Box B\to\Box B}\text{(GL4.3)}$$

Proposition 3.6. *For any sequent* $\Gamma\to\Delta$, $\vdash_{\mathbf{SK4.3G}}\Gamma\to\Delta$ *iff* $\vdash_{\mathbf{SK4.3G'}}\Gamma\to\Delta$.

Proof. (1) If part: Trivial. (2) If only part: Use Lemmas 3.4 and 3.5.

Theorem 37. *The following inference is admissible in* **SK4.3G′**.

$$\frac{\Gamma_1\to\Delta_1,\ A \quad A,\ \Gamma_2\to\Delta_2}{\Gamma\to\Delta}\text{(cut)}$$

$(\Gamma=\Gamma_1\cup\Gamma_2$ *and* $\Delta=\Delta_1\cup\Delta_2)$

Proof. It is sufficient to deal with proof figures which contain only one (cut) as the last inference. Thus we must consider the following proof figure π.

$$\frac{\Gamma_1\to\Delta_1,\ A \quad A,\ \Gamma_2\to\Delta_2}{\Gamma\to\Delta}\text{(cut)}$$

$(\Gamma=\Gamma_1\cup\Gamma_2$ and $\Delta=\Delta_1\cup\Delta_2)$

By the *grade* of a formula B, we mean the number of logical symbols contained in B. By $\gamma(\pi)$ we denote the grade of the cut formula A. By $\delta_l^1(\pi)$ we denote the number of formulas of the form $\Box B$ that occur as subformulas of formulas in $\Gamma_1\cup\Delta_1$[1)]. We denote by $\delta_l^2(\pi)$ the number of formulas of the form $\Box B$ that occur in Γ_1[2)]. Obviously $\delta_l^1(\pi)\geqq\delta_l^2(\pi)$. We denote by $\delta_l(\pi)$ the number $\delta_l^1(\pi)-\delta_l^2(\pi)$. By $\delta_r^1(\pi)$ we denote the number of formulas of the form $\Box B$ that occur as subformulas in $\Gamma_2 \cup \Delta_2$[3]. We denote by $\delta_r^2(\pi)$ the number of formulas of the form $\Box B$ that occur in Γ_2[4]. We denote by $\delta_r(\pi)$ the number $\delta_r^1(\pi)-\delta_r^2(\pi)$. We decree that $\delta(\pi)=\delta_l(\pi)+\delta_r(\pi)$. By $\rho_l(\pi)$ we denote the number of the longest threads that end with the left upper sequent $\Gamma_1\rightarrow\Delta_1, A$ and contain the cut formula A consecutively. Similarly we denote by $\rho_r(\pi)$ the number of the longest threads that end with the right upper sequent $A, \Gamma_2\rightarrow\Delta_2$ and contain the cut formula A consecutively. By $\rho(\pi)$ we mean the number $\rho_l(\pi)+\rho_r(\pi)$. Now our proof proceeds by triple induction mainly on $\gamma(\pi)$, secondly on $\delta(\pi)$ and thirdly on $\rho(\pi)$. Since our proof is not by the usual double induction on $\gamma(\pi)$ and $\rho(\pi)$, we should be careful enough even in dealing with classical cases.

(1) $\rho(\pi)=2$: Since in rule (GL4.3) the antecedent of the lower sequent is contained in that of every upper sequent, A can be of the form $\Box B$ only when the right upper sequent $\Box B, \Gamma_2\rightarrow\Delta_2$ is an axiom sequent. In this case $\Gamma\rightarrow\Delta$ is also an axiom sequent. Therefore the only nontrivial case we must consider goes as follows:

$$\dfrac{\dfrac{\overset{\vdots}{A, \Gamma_1\rightarrow\Delta_1, B}}{\Gamma_1\rightarrow\Delta_1, A\supset B}\,(\supset\text{R}) \qquad \dfrac{\overset{\vdots}{\Gamma_2\rightarrow\Delta_2, A} \quad \overset{\vdots}{B, \Gamma_2\rightarrow\Delta_2}}{A\supset B, \Gamma_2\rightarrow\Delta_2}\,(\supset\text{L})'}{\Gamma\rightarrow\Delta}\,(\text{cut})$$

$(\Gamma=\Gamma_1\cup\Gamma_2$ and $\Delta=\Delta_1\cup\Delta_2)$

This proof figure π is transformed into:

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma_2\rightarrow\Delta_2, A} \quad \overset{\vdots}{A, \Gamma_1\rightarrow\Delta_1, B}}{\Gamma\rightarrow\Delta, B}\,(\text{cut}) \qquad \overset{\vdots}{B, \Gamma_2\rightarrow\Delta_2}}{\Gamma\rightarrow\Delta}\,(\text{cut})$$

Since the grades of A and B are smaller than that of $A\supset B$, the induction process works well.

(2) $\rho(\pi)>2$: There are several nontrivial cases, which we shall consider case by case in the following:

(2a) π is of the following form:

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma_1\rightarrow\Delta_1, B, A} \quad \overset{\vdots}{C, \Gamma_1\rightarrow\Delta_1, A}}{B\supset C, \Gamma_1\rightarrow\Delta_1, A}\,(\supset L) \qquad \overset{\vdots}{A, \Gamma_2\rightarrow\Delta_2}}{B\supset C, \Gamma\rightarrow\Delta}\,(\text{cut})$$

$(\Gamma=\Gamma_1\cup\Gamma_2$ and $\Delta=\Delta_1\cup\Delta_2)$

1)-4) Repetition is not counted.

Consider the following proof figure π_1:

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ \Gamma_1 \to \Delta_1,\ B,\ A & A,\ \Gamma_2 \to \Delta_2 \end{array}}{\Gamma \to \Delta,\ B}\ (\text{cut})$$

Since $\gamma(\pi_1)=\gamma(\pi)$, $\delta(\pi_1)\leqq\delta(\pi)$ and $\rho(\pi_1)<\rho(\pi)$, we have a cut-free proof π_1' of the sequent $\Gamma\to\Delta,\ B$ by induction hypothesis.

Consider the following proof figure π_2:

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ C,\ \Gamma_1 \to \Delta_1,\ A & A,\ \Gamma_2 \to \Delta_2 \end{array}}{C,\ \Gamma \to \Delta}\ (\text{cut})$$

Since $\gamma(\pi_2)=\gamma(\pi)$, $\delta(\pi_2)\leqq\delta(\pi)$ and $\rho(\pi_2)<\rho(\pi)$, we have a cut-free proof π_2' of the sequent $C,\ \Gamma\to\Delta$ by induction hypothesis. Therefore

$$\dfrac{\begin{array}{cc} \vdots\,\}\pi_1' & \vdots\,\}\pi_2' \\ \Gamma \to \Delta,\ B & C,\ \Gamma \to \Delta \end{array}}{B\supset C,\ \Gamma \to \Delta}\ (\supset\text{L})'$$

The following three cases are treated similarly to (2a).

(2b) $\rho_l(\pi)\geqq 2$ and the last inference of the proof of the left upper sequent of (cut) is $(\supset\text{R})$.

(2c) $\rho_r(\pi)\geqq 2$ and the last inference of the proof of the right upper sequent of (cut) is $(\supset\text{L})'$.

(2d) $\rho_r(\pi)\geqq 2$ and the last inference of the proof of the right upper sequent of (cut) is $(\supset\text{R})$.

(2e) The last inference of the proofs of both upper sequents of (cut) is (GL4.3):

We deal with the following special case, leaving the general treatment to the reader.

$$\dfrac{\dfrac{\begin{array}{ccc} \vdots & \vdots & \vdots \\ \Box,\Gamma_1,\Gamma_1,\Box A,\Box B\to A,B & \Box\Gamma_1,\Gamma_1,\Box A\to A,\Box B & \Box\Gamma_{1,1},\Box B\to A,B \end{array}}{\Box\Gamma_1\to\Box A,\ \Box B}(\text{GL4.3}) \quad \dfrac{\begin{array}{c} \vdots \\ \Box\Gamma_2,\Gamma_2,\Box A,A,\Box C\to C \end{array}}{\Box\Gamma_2,\ \Box A\to\Box C}(\text{GL4.3})}{\Box\Gamma\to\Box B,\ \Box C}(\text{cut})$$

$(\Gamma=\Gamma_1\cup\Gamma_2$ and $\Delta=\Delta_1\cup\Delta_2)$

Consider the following proof figure π_1:

$$\cfrac{\begin{matrix}\vdots & \qquad & \vdots\\ \Box\Gamma_1,\ \Gamma_1,\ \Box A,\ \Box B\to A,\ B & & \Box\Gamma_2,\ \Gamma_2,\ \Box A,\ A,\ \Box C\to C\end{matrix}}{\Box\Gamma,\ \Gamma,\ \Box A,\ \Box B,\ \Box C\to B,\ C}\ (\text{cut})$$

Since the grade of A is smaller than that of $\Box A$, there is a cut-free proof π_1' of $\Box\Gamma,\ \Gamma,\ \Box A,\ \Box B,\ \Box C\to B,\ C$ by induction hypothesis.

Consider the following proof figure π_2:

$$\cfrac{\begin{matrix}\vdots \\ \Box\Gamma_1,\ \Gamma_1\Box B\to\Box A,\ B\end{matrix}\qquad \left.\begin{matrix}\vdots \\ \Box\Gamma,\ \Gamma,\ \Box A,\ \Box B,\ \Box C\to B,\ C\end{matrix}\right\}\pi_1'}{\Box\Gamma,\ \Gamma,\ \Box B,\ \Box C\to B,\ C}\ (\text{cut})$$

Since $\gamma(\pi_2)=\gamma(\pi)$ and $\delta(\pi_2)<\delta(\pi)$, there is a cut-free proof π_2' of $\Box\Gamma,\ \Gamma,\ \Box B,\ \Box C\to B,\ C$ by induction hypothesis.

Consider the following proof figure π_3:

$$\cfrac{\begin{matrix}\vdots \\ \Box\Gamma_1,\ \Gamma_1,\ \Box B\to\Box A,\ B\end{matrix}\qquad \cfrac{\begin{matrix}\vdots \\ \Box\Gamma_2,\ \Gamma_2,\ \Box A,\ A,\ \Box C\to C\end{matrix}}{\Box\Gamma_2,\ \Box A\to\Box C}\ (\text{GL4.3})}{\Box\Gamma_1,\ \Gamma_1,\ \Box\Gamma_2,\ \Box B\to B,\ \Box C}\ (\text{cut})$$

Since $\gamma(\pi_3)=\gamma(\pi)$ and $\delta(\pi_3)<\delta(\pi)$, there is a cut-free proof π_3' of $\Box\Gamma,\ \Gamma,\ \Box B\to B,\ \Box C$ by induction hypothesis and Lemma 3.4.

Consider the following proof figure π_4:

$$\cfrac{\begin{matrix}\vdots & \qquad & \vdots\\ \Box\Gamma_1,\ \Gamma_1,\ \Box A\to A,\ \Box B & & \Box\Gamma_2,\ \Gamma_2\Box A,\ A,\ \Box C\to C\end{matrix}}{\Box\Gamma,\ \Gamma,\ \Box A,\ \Box C\to\Box B,\ C}\ (\text{cut})$$

Since $\gamma(\pi_4)<\gamma(\pi)$, there is a cut-free proof π_4' of $\Box\Gamma,\ \Gamma,\ \Box A,\ \Box C\to\Box B,\ C$ by induction hypothesis.

Consider the following proof figure π_5:

$$\cfrac{\begin{matrix}\vdots \\ \Box\Gamma_1\to\Box A,\ \Box B\end{matrix}\qquad \left.\begin{matrix}\vdots \\ \Box\Gamma,\ \Gamma,\ \Box A,\ \Box C\to\Box B,\ C\end{matrix}\right\}\pi_4'}{\Box\Gamma,\ \Gamma,\ \Box C\to\Box B,\ C}\ (\text{cut})$$

Since $\gamma(\pi_5)=\gamma(\pi)$ and $\delta(\pi_5)<\delta(\pi)$, there is a cut-free proof π_5' of $\Box\Gamma,\ \Gamma,\ \Box C\to\Box B,\ C$ by induction hypothesis. Therefore

$$\dfrac{\begin{matrix}\vdots\ \pi_2' \\ \Box\Gamma,\ \Gamma,\ \Box B,\ \Box C\rightarrow B,\ C\end{matrix} \qquad \begin{matrix}\vdots\ \pi_3' \\ \Box\Gamma,\ \Gamma,\ \Box B\rightarrow B,\ \Box C\end{matrix} \qquad \begin{matrix}\vdots\ \pi_5' \\ \Box\Gamma,\ \Gamma,\ \Box C\rightarrow \Box B,\ C\end{matrix}}{\Box\Gamma\rightarrow\Box B,\ \Box C}\ (\mathrm{GL4.3})$$

Before leaving the above proof, the reader should realize that the main reason for dealing with **SK4.3G′** instead of **SK4.3G** directly is to make the secondary induction on $\delta(\pi)$ work well. It seems that the secondary induction of Theorem 3.4 (cut-elimination theorem) of Leivant [4] indeed works well for cases like (a special case of) (2e) but fails to preserve the usual treatment of classical cases like (2c).

References

[1] Boolos, G., *The unprovability of consistency*, Cambridge University Press, 1979.

[2] Gabbay, D. M., *Investigations in modal and tense logics with applications to problems in philosophy and linguistics*, D. Reidel, 1976.

[3] Goldblatt, R., Arithmetical necessity, provability and intuitionistic logic, *Theoria*, **44** (1978), 38-46.

[4] Leivant, D., On the proof theory of the modal logic for arithmetical provability, *J. Symbolic Logic*, **46** (1981), 531-538.

[5] Sato, M., A cut-free Gentzen-type system for the modal logic S5, *J. Symbolic Logic*, **45** (1980), 67-84.

[6] Takeuti, G., *Proof theory*, North-Holland, 1975.

[7] Zeman, J. J., *Modal logic*, Oxford University Press, 1973.