Optimal Investment under Uncertainty in Household Finance

by

Toshio Kimura

Submitted to the Graduate School of Business Sciences
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

at the

University of Tsukuba

February 2009

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Abstract

The classical life cycle model has not paid much attention to the optimal investment strategy for an individual with a long-term debt such as mortgage, and presumes that a borrowing rate is equal to the deposit rate. This simplification is at odds with the reality a typical individual faces over the life cycle. In this work we investigate the effect that decisions associated with debt repayment, refinancing, and borrowing has on investment strategy under uncertainty. For this purpose we attempt to propose analytically tractable models in household finance, building on three well-established literatures: Merton's consumption and portfolio choice model, option based rational refinancing model, and a regime switching framework.

We first consider how an individual's debt repayment affects her portfolio choice over the life-cycle. By introducing the debt repayment term decision into Merton's basic consumption and portfolio choice problem, we propose a link between the debt repayment and the age-related investment in risky assets observed in households' asset allocations. Using a closed form solution, we demonstrate numerically and empirically that within Merton's framework a crucial element determining a hump-shaped age-related risky investment pattern is the individual's debt repayment term.

We next focus on the role that the changes in regime in refinancing opportunities play in the individual's refinancing strategy. The rational models of mortgage refinancing generally presume that the parameters of interest rate process do not vary over time. While the standard model under uncertainty can generate late refinancing behavior of individuals during a state with a high volatility in interest rates, the model cannot explain an early refinancing behavior. Individuals often hasten their refinancing before the optimal refinancing timing implied by standard option-based approaches. We developed a model of rational mortgage refinancing where the drift and volatility of interest rate process switches between two regimes. We find that our calibrated model can produce both late and early refinancing behavior.

Finally, we return to a consumption and portfolio choice problem where an individual has to repay her debt at a higher borrowing rate than the deposit rate. Under both investment and borrowing opportunities' regime-switching environment, we analytically examine optimal consumption and portfolio choice for an individual with direct preference for wealth (the wealth accumulation motive), as well as her future consumption (the consumption motive). The model is more general and useful in the sense that it encompasses models with the standard CRRA preference, the model with a borrowing rate equal to a risk-free rate, and the model with a single regime. We show that the optimal investment depends only on the current regime, while optimal consumption depends on both the future and the current regime. Furthermore, numerical analysis demonstrates that the extended model can potentially explain various features of household's investment behavior, such as the stable consumption to wealth ratio, time varying risky investment observed in time series data, and life-cycle risky investment profiles observed in cross-sectional data.

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Acknowledgments

I am greatly indebted to my chief thesis advisor Professor Naoki Makimoto for his support and guidance from start to the last minute of completing this thesis. His advices have always played a decisive role at various stages of my graduate study. I also thank my associate thesis advisors - Professor Hiroe Tsubaki and Professor Yuji Yamada - for their time and encouragement. Dr. Tsubaki's comments have been valuable and constructive. Dr. Yamada's suggestions have substantially improved Chapter 3. I am also grateful to the two anonymous referees, as well as the participants of the 2nd Stanford-Tsukuba Joint Workshop on Financial Engineering at Stanford, the 2007 Daiwa International Workshop on Financial Engineering, and the Asian FA-NFA 2008 International Conference, for their helpful suggestions. In addition, I wish to thank my colleagues at the Sumitomo Trust and Banking Co., for their encouragement. Mr. Richard Ginies has been a rigorous and careful reader of earlier version of this thesis. His advices have been a great support for completing this thesis in English. Finally, I would like to dedicate this work to my wife, Yuko and my son, Kazuaki, who supported and encouraged me throughout this program at the University of Tsukuba.

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Chapter 1

Introduction

Investment and liability management are important for a typical household's wealth accumulation and welfare over the life-cycle. Since most households own long-term debts such as mortgages, much attention should be directed to optimal asset allocation for households with debts under uncertainty.

This thesis analyzes optimal investment and refinancing strategy by considering different extensions to the two seminal works by Merton (1971) and Dunn and MacConnel (1981). First, we consider a model of consumption and portfolio choice for a debt-holding individual investor when her debt repayment term decision is introduced into the classic work by Merton. Next, we focus on the role of regime switches in interest rate process in a model of rational mortgage refinancing decision. Finally, we consider the optimal consumption and portfolio strategies when a borrowing rate is higher than the deposit rate and the investment opportunities are regime switching.

We begin in chapter 2 by surveying recent aspects and developments in major dynamic investment frameworks in household finance. In particular, we discuss two popular models, Merton's lifetime consumption-portfolio choice problem and the option-based approach to refinancing behavior. To motivate extensions of these basic models, we pay attention to two empirical findings: a hump-shaped life-cycle investment pattern in risky assets and an early mortgage refinancing behavior. We also review studies applying a regime-switching framework, which has been recognized as an attractive tool to describe the stochastic behavior of assets return.

In Chapter 3 we examine how an individual's debt repayment affects her portfolio choice over the life-cycle. By introducing the debt repayment term decision into Merton's basic consumption and portfolio choice problem, we propose a link between the debt repayment and the age-related investment in risky assets observed in households' asset allocations. With our analytical solution at hand, we proceed to investigate the effects of debt repayment term on life-cycle portfolio choice. We find that the individual's debt repayment affects optimal consumption-portfolio choice through the individual's future net human wealth and current financial accumulation, both of which reflect the individual's debt repayment. The debt repayment dampens the individual's financial wealth accumulation, which results in a larger equity share in total financial wealth during the individual's debt-repayment term. Moreover, the model with debt repayment can generate a hump-shaped pattern of risky asset holdings over the individual investor's life-cycle.

In Chapter 4, we consider an optimal debt refinancing strategy where the individual's investment opportunities switch among regimes. The rational models of mortgage refinancing generally presume that the parameters of interest rate process do not vary over time. The standard option-based models under uncertainty can generate late refinancing behavior by demonstrating that the mortgage holder will delay her refinancing when the volatility of interest rates is high. The empirical evidence on mortgage refinancing, however, has revealed that households occasionally not only delay, but also hasten their refinancing, even when it appears not optimal to refinance under the standard models. Motivated by the empirical puzzle, we develop a model of rational mortgage refinancing where the drift and volatility of interest rate process switch between two regimes. The rational model we consider is an extension of the analytical work by Agarwal, Driscoll, and Laibson (2002). We evaluate the predictions of our model, based on the estimated parameters for a two-regime interest rate process to capture the evolution of the mortgage rates in the U.S. Our model can produce both early and late refinancing, which is consistent with the findings documented empirically.

In Chapter 5, we return to the individual's consumption and portfolio choice problem. By explicitly considering both investment and borrowing opportunities under a regime-switching environment, we develop a model of consumption and portfolio choice for an individual with direct preference for wealth, future consumption and social status. The individual has to repay her debt at a higher borrowing rate than the deposit rate under stochastic investment opportunities with regime switches. We derive explicit formulas for the optimal consumption and investment strategies. Once we account for changes in regime, we find that the optimal investment in risky assets is the current regime dependent while the optimal consumption depends on both current and future regimes. Numerical experiments demonstrate that in each regime, the individual's optimal consumption and portfolio strategy recognizes the possibility of a regime shift. The model is more general and useful in the sense that it encompasses the model with a standard CRRA preference for consumption, the model with a borrowing rate equal to the deposit rate, and the model with a single regime.

Our work contributes to three areas of the financial literature. First, Chapter 3 and Chapter 5 contribute to the literature on the classical consumption and portfolio choice problem under uncertainty. By introducing realistic factors related to an individual's borrowing behavior into the standard Merton model, we derive properties of optimal policies with debt under stochastic investment opportunities. We show how debt repayment term can be determined, and show that within the standard model, the debt repayment term decision is independent of the optimal consumption and investment policies. In terms of stochastic investment opportunities, we combine a model whose borrowing rate is higher than the deposit rate [e.g. Fleming and Zariphopoulou (1991), Xu and Chen (1998)] with the model where the individual investor has direct preference for wealth status [e.g. Bakshi and Chen (1998)]. Moreover, we incorporate regime switches in the investment opportunities and wealth status of the individual into the problems examined by Xu and Chen (1998) and Bakshi and Chen (1998) . The analytically tractable model has favorable features to explain a household's investment behavior, such as the stable consumption to wealth ratio and time varying risky investment observed in time series data.

Second, we add to the literature on option-based refinancing models by providing a new explanation for why the timing of refinancing ought to vary. In particular, in Chapter 4 we rationally demonstrate why early or late refinancing arises, countering to the optimal refinancing timing implied by standard option-based approaches. The optimal refinancing strategy is affected by the future regime in interest rate process, which produces early or late refinancing. This explanation stands in contrast to those of the heterogeneous transaction costs [e.g. Dunn and MacConnel (1981)], the discontinuous decision [e.g. Stanton (1995), Agarwal, Driscoll, and Laibson (2004)], and housing price dynamics [e.g. Downing, Stanton, and Wallace (2005)].

Finally, our work contributes to the growing literature on optimal investment strategies under investment opportunities switching in regimes. In Chapter 4 and Chapter 5 we obtain the semi-analytical solutions to the refinancing problem and consumption-portfolio choice problem. Our results add to the literature on analytical models, such as option pricing [Guo and Zhang (2004)], investment decision of a firm [Guo, Miao, and Morellec (2005), Makimoto (2008)], and an option stock selling rule [Zhang (2001)].

Chapter 2

Literature Review

The main objective of research in household finance is to develop theoretically well-founded models to take into account serious aspects of an individual's decisions over the entire life-cycle. Better models will improve both investment decisions and our understanding of the pricing mechanisms in financial markets. Merton's lifetime consumption-portfolio choice problem and the option based refinancing model are two of the most influential dynamic decision frameworks in household finance. The former model is a foundation to studies on optimal investment and consumption strategy for individuals. It also constructs a basic framework for understanding of asset pricing. The latter model is applied to consider the refinancing behavior of individuals who hold mortgage debts. The model is often used to evaluate the price of Mortgage Backed Securities (MBS).

Because decision problems under uncertainty generally presume that an individual's investment opportunities obey specific stochastic processes, the structure of stochastic processes plays a crucial role in their optimal policy. Recent evidence on both states of the economy and behavior of financial markets supports the notion that the investment opportunities shift among different states. For instance, economic variables such as the growth rates of an economy mostly tend to behave quite differently during an economic downturn. Abrupt changes are also a prevalent feature of financial data. After the seminal work by Hamilton (1989), applying an idea of regime-switching to characterize observations in financial markets has become one of the most active research areas in finance literature.

Motivated by these theoretical and empirical literatures, this chapter surveys the progress and limitations to date in three related research areas: optimal consumption and portfolio choice, rational mortgage refinancing, and regime switching. We particularly pay attention to analytical studies on the optimal investment strategy for an individual investor who holds debt and has borrowing opportunities. While analytical works cannot incorporate all of the individual's facing realistic factors into the rational models, optimal policies derived as closed-form solutions make the roots and causes of the effects clearer and thus give us useful insights into the understanding of complicated decision problems.

This chapter starts by reviewing the basic analytical studies on consumption and portfolio choices in Section 2.1. Section 2.2 covers the developments of rational refinancing models. Section 2.3 summarizes the empirical and analytical works on investment decision under regime switching. Section 2.4 provides suggestions for future research.

2.1 Optimal Consumption and Investment

2.1.1 Basic Merton's Model

The consumption and portfolio choice problem was formulated and solved in two seminal papers by Merton (1969,1971). A series of his works provides a conceptual framework for the long-term financial planning under both constant and stochastic investment opportunities [e.g. Merton (1973)]. We outline the assumptions and results for the dynamic consumption and portfolio choice problem in a continuous-time environment.

The basic setup in the problem is that an individual maximizes her expected utility function that depends on the rate of consumption at all future dates. The individual investor is assumed to have two assets available for investment, a risk-free asset such as the money market account whose price is denoted by X_t and risky assets such as stocks or equity. Return on the risk-free asset is assumed to be r and the price of the equity S_t follows a geometric Brownian motion:

$$\begin{aligned} dX_t &= rX_t dt, && (2.1) \\ dS_t &= \mu S_t dt + \sigma S_t dz_t && (2.2) \end{aligned}$$

where $\{z_t\}$ is a standard Brownian motion. When the μ, σ, and r are time-varying, the investor is said to face stochastic investment opportunities. As a benchmark for later discussion, we first consider a model under constant investment opportunities where these parameters are assumed to be constant. Given the investor's wealth W_t at time t, she is assumed to invest π_t into equity and save the rest $W_t - \pi_t$ in a money market account, and consume the rate of c_t. These assumptions on investment opportunities imply that the wealth dynamics for the investor are given by

$$dW_t = \{rW_t - c_t + (\mu - r)\pi_t\}dt + \sigma \pi_t dz_t. \qquad (2.3)$$

The individual's objective is to maximize her expected accumulated utility

$$E\left[\int_t^T e^{-\delta s}u(c_s)ds + e^{-\delta(T-t)}U(W_T)\;\middle|\; W_t = w\right]$$

by choosing an optimal investment into the equity π_t and instantaneous consumption rate c_t, subject to the wealth dynamics in (2.3). The date s utility function $u(c_s)$ is assumed to be strictly increasing and concave in c_s and the bequest function $U(W_T)$ is assumed to be strictly increasing and concave in terminal wealth W_T. We denote the individual's value function by

$$J(w,t) = \max_{\{c_s\},\{\pi_s\}} E\left[\int_t^T e^{-\delta s}u(c_s)ds + e^{-\delta(T-t)}U(W_T)\;\middle|\; W_t = w\right]. \tag{2.4}$$

Merton (1971) solved the above problem for a more general class of hyperbolic absolute risk aversion (HARA) utility functions. A well-known special case in HARA family is the utility with a constant relative risk aversion (CRRA):

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{2.5}$$

where γ is a parameter of constant relative risk aversion.

With CRRA utility and no bequest function $U(W_T) \equiv 0$ for simplicity, the optimization problem can be analytically solved and the individual's value function is expressed as

$$J(w,t) = e^{-\rho t}\left[\frac{1-e^{-K(T-t)}}{K}\right]^{\gamma}\frac{w^{1-\gamma}}{1-\gamma} \tag{2.6}$$

where

$$K \equiv \frac{1}{\gamma}\left[\rho - (1-\gamma)r - \frac{(1-\gamma)(\mu-r)^2}{2\gamma\sigma^2}\right]. \tag{2.7}$$

The resulting optimal policies for equity investment $\hat{\pi}_t$ and consumption $\hat{c}_t$ are solved as the closed form,

$$\hat{\pi}_t = \frac{1}{\gamma}\frac{\mu-r}{\sigma^2}w_t, \tag{2.8}$$

$$\hat{c}_t = \frac{K}{1-e^{-K(T-t)}}w_t. \tag{2.9}$$

The analytical solution for the very basic model reveals clear messages for the optimal investment strategy for an individual who faces the investment opportunities over the life-cycle, and the empirical validation. First, the individual's value function $J(w,t)$ depends only on wealth w and time t. As a result, both optimal consumption $\hat{c}_t$ and equity investment $\hat{\pi}_t$ are linear functions of wealth w. Second, the optimal 'share' of equity on total financial wealth $\hat{\pi}_t/w_t$ decreases in risk aversion γ. This is consistent with the popular advice that more risk-averse investors should reduce their portfolio allocation in risky assets. Third, given a specific risk attitude, the allocation between risk-free and risky assets is independent of time horizon T. In other words, a constant share of equity investment strategy is optimal for households. The strategy clearly runs counter to the popular advice that investors should reduce their allocation in risky asset as they age.

These policy implications rely heavily on the assumptions of the model: an investor (a) faces constant investment opportunities, (b) receives no labor income, and (c) has a time-separable utility function. Models with these restricted assumptions predict that the individual investor will allocate a constant fraction of her wealth to risky assets over her lifetime. In the next section, we discuss the empirical puzzles and review quickly on the developments of analytical studies.

2.1.2 Empirical Puzzles and Development of Analytical Study

Despite the sound analytical foundations in the classical consumption-portfolio choice problem, the observed households' behaviors do not support the prediction of the model. Moreover, there are several discrepancies among analytically solved optimal investment strategy, popular advice and observed behavior. For instance, the traditional portfolio strategy under constant investment opportunities suggests that the share invested in risky assets should be constant over the life-cycle. On the other hand, financial planner's typical advice is that long-term investors like young households with their long-horizon, should have a much greater share of risky financial assets to take advantage of the equity premium. Empirical studies, such as Ameriks and Zeldes (2004), however, have revealed that portfolio choices observed in households in developed countries have the following age-related patterns:

1. Investment in risky assets has a hump-shaped life-cycle pattern.
2. The proportion of a household's holding equity is hump-shaped with age.
3. Substantial non-participation in equity market is observed.

Furthermore, Iwaisako (2003) argues that in Japanese households' data, the age-related pattern in equity holding is mostly observed in households that own their houses, but not in those that do not. Such age-related profile is substantially contrasted in both theory and the popular advice.

To fill these gaps, burgeoning studies have been motivated to attempt to relax the restricted assumptions in the basic Merton's framework. Most studies consider the effects of various realistic factors, such as stochastic investment opportunities, an individual's labor income, a class of time-inseparable preference, and borrowing opportunities.

Stochastic Environment

Merton (1971,1973) first introduces a conceptual framework for long-term investment planning given time-varying investment opportunities. When the mean and variance of assets returns are time-varying, so that investment opportunities are stochastically changing, the optimal portfolio policy $\hat{\pi}$ no longer satisfies the constant risky asset share over the life-cycle, such as the policy $\hat{\pi}_t/w_t$ in (2.8). From a technical viewpoint, the optimal portfolio policy includes an additional component that reflects the need to hedge against unfavorable shifts in investment opportunities, as we will discuss later.

In regard to the age-related profile, the additional hedging component needs to be positive and increasing as the investment horizon. Among analytical models under time-varying investment opportunities, one candidate to satisfy such property is a model with a stochastic equity premium. The most popular setting is to assume that the excess return follows a mean-reverting process. Both Kim and Omberg (1996) and Wachter (2002) obtain a closed-form solution for the optimal investment policy. In their setup with a constant risk-free rate r, the process of the price of risky asset S_t in (2.2) is extended to

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma S_t dz_t \\ &= (r + \sigma\lambda_t) S_t dt + \sigma S_t dz_t, \end{aligned} \tag{2.10}$$

$$d\lambda_t = \kappa[\bar{\lambda} - \lambda_t]dt + \rho\sigma_\lambda dz_t + \sqrt{1-\rho^2\sigma_\lambda}d\tilde{z}_t \tag{2.11}$$

where the market price of risk $\lambda_t = (\mu_t - r)/\sigma$ follows an Ornstein-Uhlembeck process with a long-term average $\bar{\lambda}$, a mean-reversion speed κ, and a volatility σ_λ. Here $\{\tilde{z}_t\}$ is a standard Brownian motion independent of $\{z_t\}$. If the market price of risk λ_t is constant, then the process of S_t in (2.10) reduces to the process in (2.2). All constants are assumed to be positive, except the correlation parameter ρ. In Kim and Omberg (1996) where investor's problem is to maximize the CRRA utility from terminal wealth only, the optimal equity investment $\pi^s(w, \lambda, t)$ under the stochastic environment with finite time horizon T is explicitly solved as

$$\begin{aligned}\pi^s(w_t, \lambda, t) &= \frac{1}{\gamma}\frac{\mu_t - r}{\sigma^2}w_t + \frac{\rho\sigma_\lambda}{\sigma}\left[A_b(T-t) + A_c(T-t)\frac{\mu_t - r}{\sigma}\right]w_t \\ &= \hat{\pi}_t + \frac{\rho\sigma_\lambda}{\sigma}\left[A_b(T-t) + A_c(T-t)\frac{\mu_t - r}{\sigma}\right]w_t \qquad (2.12)\end{aligned}$$

where $\hat{\pi}_t$ is optimal investment policy (2.8) in Merton's basic model, and

$$\begin{aligned}A_b(\tau) &= \frac{1-\gamma}{\gamma^2}\frac{\kappa\bar{\lambda}(1-e^{-\sqrt{q}\tau})^2}{\sqrt{q}[2\sqrt{q} - (\sqrt{q} - \bar{\kappa})(1 - e^{-2\sqrt{q}\tau})]}, \\ A_c(\tau) &= \frac{1-\gamma}{\gamma^2}\frac{1 - e^{-\sqrt{q}\tau})^2}{2\sqrt{q} - (\sqrt{q} - \bar{\kappa})(1 - e^{-2\sqrt{q}\tau})}\end{aligned}$$

with

$$\begin{aligned}\bar{\kappa} &= \kappa\frac{1-\gamma}{\gamma}\rho\sigma_\lambda, \\ q &= \bar{\kappa}^2 - \sigma_\lambda^2(\rho^2 + (1-\gamma)(1-\rho^2))\frac{1-\gamma}{\gamma^2}.\end{aligned}$$

Kim and Omberg (1996) and Wachter (2002) showed that for $\gamma < 0$, both function of $A_b(\tau)$ and $A_c(\tau)$ are negative and decreasing. If the current value of the equity premium or the market price of risk $\lambda_t = (\mu_t - r)/\sigma$ is positive and the correlation ρ is negative, the second term in (2.12), namely, the hedge term of optimal portfolio is positive and increasing with the investor's time horizon. On empirical ground, the assumption that the equity premium λ_t and the price of stock S_t is negatively correlated ($\rho < 0$) is not unrealistic. Thus, an individual investor with a long horizon tends to invest a larger proportion of financial wealth into risky assets than one with the same preference but a shorter horizon. The result generates a decreasing age-related profile in risky investment and therefore is consistent with the financial planner's advice.

Another direction of studies on stochastic investment opportunities is to construct a model where the interest rate r_t obeys a stochastic process such as Vasicek (1977), Cox, Ingersoll, and Ross (1985). Under a stochastic interest rate process, bonds are a natural instrument that hedge against stochastic interest rates. Analytical studies in Liu (2007) and the related work by Munk and Sorensen (2004) show that the hedge component is not involved in the optimal policy for equity, but involved in that for bonds. This indicates that the model under stochastic interest rate with constant price of risk cannot induce the investment strategy for stocks to be time-varying. As a result, the portfolio share of equity is independent of the investor's time horizon. Instead, the model demonstrates that the demand for bonds increases while the demand for equity decreases as the risk aversion increases. Because this strategy implies that the bond-stock mix increases with risk aversion, the optimal policy is consistent with the popular advice by a typical financial planner.

In sum, the model with stochastic interest rate cannot resolve the age-related profile, but can resolve "an asset allocation puzzle" which is first documented by Canner, Mankiw, and Weil (1997). They point out that financial planners typically advise conservative investors to hold more bonds relative to equities in their risky portfolios.

Labor Income

The second candidate to explain the age-related profile is a model with investor's labor income. With the labor income denoted by y_t, the wealth process in (2.3) evolves as

$$dW_t = \{rW_t - c_t + (\mu - r)\pi_t + y_t\}dt + \sigma\pi_t dz_t. \tag{2.13}$$

Merton (1971) demonstrates that the optimal investment policy under a constant labor income $y_t = y$ and a constant interest rate r can be explicitly expressed as

$$\pi_t^L = \frac{1}{\gamma}\frac{\mu - r}{\sigma^2} w_t \left[1 + \frac{y(1 - e^{r(t-T)})}{r}\frac{1}{w_t}\right] = \hat{\pi}_t \left[1 + \frac{H(y,r)}{w_t}\right] \tag{2.14}$$

where

$$H(y,r) = \frac{y(1 - e^{r(t-T)})}{r}. \tag{2.15}$$

Note that $H(y,r)$ is the present value of future labor income y discounted by constant risk-free rate r, which is generally called as the individual's "human wealth". If the labor income $y_t = 0$, the optimal equity investment π_t^L reduces to the original optimal policy $\hat{\pi}_t$ under constant investment opportunities.

Because human wealth is increasing in the investor's horizon T and decreasing as time t goes, the optimal equity share π_t^L/w_t shows that younger households have a larger proportion invested in equity than older investors. This is consistent with a financial planner's advice but does not match the hump-shaped profile observed in data. However, Lynch and Tan (2004) argue that calibrating the business-cycle variations in the first two moments of labor income growth to U.S. data leads to large reduction in equity holding by young households with low wealth-income ratios.

Habit Formation

The third approach to produce a hump-shaped profile is a model with habit formation of past consumption in utility. Because individuals mostly are accustomed to their past standard of living, the idea that an individual's consumption choice today affects her utility in the future is a natural extension as an alternative utility form. Then we replace the time-additive standard CRRA utility function in (2.5) by

$$u(c, x_t) = \frac{(c - bx_t)^{1-\gamma}}{1-\gamma} \tag{2.16}$$

where

$$x_t = e^{-at}x_0 + b\int_0^t e^{a(s-t)}c_s ds. \tag{2.17}$$

The habit stock x_t depends on an exponentially weighted average of the entire history of past consumption. The individual investor's utility depends on the level of excess consumption to the habit or past standard of living level. Note that if $b = 0$, utility is of the standard time-separable CRRA form with a coefficient of relative risk aversion equal to γ. When $b > 0$, bx_t can be interpreted as a "subsistence" or "habit" consumption standard. Alternatively, if $b < 0$ so that past consumption adds to rather than subtracts from current utility, then the model displays "durability" in consumption rather than habit persistence.

Sundaresan (1989) and Constantinides (1990) solve Merton's basic problem with habit formation for infinite time horizon. For finite time horizon problems, Schroder and Skiadas (2000) develop a method to transform a problem without habit into the dual problem with habit persistence. By applying their method, we can obtain the closed form solution under habit formation from the solution to a dual problem where utility is of the same power form but no habit ($b = 0$, $x_0 = 0$). The optimal equity investment π_t^H under the habit formation in (2.16) and (2.17) is derived as

$$\begin{aligned} \pi_t^H &= \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}\left[w_t - \frac{bx_t}{r+a-b}\left\{1 - e^{-(r+a-b)(T-t)}\right\}\right] \\ &= \hat{\pi}_t\left[1 - \theta_t\frac{bx_t}{w_t}\right] \end{aligned} \tag{2.18}$$

where

$$\theta_t = \frac{1}{r+a-b}\left\{1 - e^{(b-a-r)(T-t)}\right\}. \tag{2.19}$$

Here $\hat{\pi}_t$ is again the optimal investment policy given in (2.8). At time t, the individual investor saves an amount equal to $\theta_t b x_t$ into a risk-free asset, and allocates the remainder of his wealth between equity and the risk-free asset in the same manner the basic Merton-type investor does.

It is worthwhile to consider the meaning of $\theta_t b x_t$. As Lax (2001) pointed out, $\theta_t b x_t$ implies the amount of a risk-free asset (e.g. bank account) ensuring that the future consumption will not fall below the habit level. If the individual investor chooses to consume the minimum amount x_t necessary to survive now and at every time until T, then she needs to save an amount equal to all her future habit levels discounted at the risk-free rate r into the bank account. To do so, she assures survival level until T. This implies that given $c_s = b x_s$ for $t \leq s \leq T$, the habit stock to be ensured evolves as

$$b x_s = b x_t e^{(b-a)(s-t)}, \quad t \leq s \leq T. \tag{2.20}$$

The amount she needs to place in the bank account defined by $\hat{x}_t$ is given by

$$\hat{x}_t \equiv \int_t^T e^{-r(s-t)} b x_s ds = \frac{b x_t}{r+a-b} \left\{ 1 - e^{(b-a-r)(T-t)} \right\} = \theta_t b x_t. \tag{2.21}$$

This is exactly the implication of $\theta_t b x_t$. This amount is set aside in the riskless asset, and the remainder of invested wealth is allocated using the standard Merton formula for CRRA utility. In regard to the effect of age on the proportion invested in equity, we also notice that

$$\frac{\partial(\pi_t^H / w_t)}{\partial t} = e^{(b-a-r)(T-t)} \frac{b x_t}{w_t} > 0.$$

Therefore, the equity share to total financial wealth is an increasing function of time, which results in a hump-shaped equity share profile. Munk (2007) also investigates the basic Merton-type model with habit formation in preference under stochastic investment opportunities. He obtains a closed-form solution for mean-reverting equity premium and shows that the solution is a partial differential equation for a model with Cox-Ingersoll Ross (CIR) interest rate process.

2.1.3 Model with Borrowing

Debt contracts have potentially significant impacts on an individual's consumption and investment choices over the life-cycle. Because debt repayments might change the individual's disposable income and her "home equity", which is defined by the net home values subtracted by mortgage debt, mortgage debt repayment affects her consumption and asset accumulation in a persistent manner. To our best knowledge, there are three approaches towards consumption-portfolio problem with borrowing: mortgage choice, borrowing constraints, and higher borrowing rate than interest rate.

Mortgage Choice and Borrowing Constraints

Campbell and Cocco (2003) numerically solve a model of household mortgage choice between an adjustable-rate mortgage (ARM) and a fixed-rate mortgage (FRM). They show that ARM should be attractive to non-constrained households while FRM should be attractive to risk-averse borrowing-constrained households, particularly those have high mortgage debt relative to their income.

Zariphopoulou (1994) and Vila and Zariphopoulou (1997) consider a portfolio problem with borrowing constraint such as,

$$\pi_t \leq w_t + L \tag{2.22}$$

where $L \geq 0$. Notice that when $L = 0$, $\pi_t \leq w_t$ which implies no borrowing is allowed. When $L = \infty$, the problem is exactly Merton's model in which the investor is allowed to have unlimited borrowing. They show that with borrowing constraint L, the optimal risky asset share π_t^c is

$$\pi_t^c = \min\left\{\frac{1}{\gamma}\frac{\mu - r}{\sigma^2} w_t, w_t + L\right\} = \min\left\{\hat{\pi}_t, w_t + L\right\}. \tag{2.23}$$

Model with Borrowing Rate Higher than Deposit Rate

Fleming and Zariphopoulou (1991) and Xu and Chen (1998) derived a closed form solution for the standard consumption and portfolio choice problem with higher borrowing rate (R) than interest rate (r) under both finite and infinite time horizon. The return on the risk-free asset and the wealth process are now modified to

$$dX_t = \begin{cases} rX_t dt, & X_t \geq 0 \\ RX_t dt, & X_t < 0, \end{cases} \tag{2.24}$$

$$dW_t = \left\{rW_t - c_t + (\mu - r)\pi_t - (R - r)(W_t - \pi_t)^-\right\}dt + \sigma\pi_t dz_t. \tag{2.25}$$

where a function $(a)^- = -a$ if $a < 0$ and 0 otherwise.

It is worthwhile to mention that the structure of the problem is different from those we have discussed in the sense that the investment opportunities themselves are constant but endogenously switched by the individual investor's choice variable π_t. By solving the associated Hamilton-Jacobi-Bellman (HJB) equation, the optimal investment for risky assets with higher borrowing rate π_t^{HB} is derived as

$$\pi_t^{HB} = \begin{cases} \frac{1}{\gamma}\frac{\mu-R}{\sigma^2}w_t, & \gamma\sigma^2 < \mu - R \\ w_t, & \mu - R \leq \gamma\sigma^2 \leq \mu - r \\ \frac{1}{\gamma}\frac{\mu-r}{\sigma^2}w_t, & \mu - r < \gamma\sigma^2. \end{cases} \tag{2.26}$$

As (2.26) shows, the optimal investment policy is characterized as three investment strategies, depending on the parameter of constant relative risk aversion γ, the equity premium $(\mu - R)/\sigma^2$, and $(\mu - r)/\sigma^2$:

1. Borrowing: Borrow money $(\frac{1}{\gamma}\frac{\mu-R}{\sigma^2}w_t - w_t)$ to invest $\frac{1}{\gamma}\frac{\mu-R}{\sigma^2}w_t$ into stock if the premium $\mu - R$ is higher enough to produce that $\gamma\sigma^2 < \mu - R$.
2. No borrowing but all in stocks: Invest all her present wealth w_t into stocks without borrowing if the borrowing rate R is higher and the deposit rate r is lower in the sense that $r < \mu - \gamma\sigma^2 < R$.
3. No Borrowing and partly in stocks: Save in bank deposit $(w_t - \frac{\mu-r}{\gamma\sigma^2})$ if the interest rate r become higher $\mu - \gamma\sigma^2 \leq r$.

One of the attractive features of the model is that the optimal policy has rich insights into the leverage effect on risky asset investment. As the first strategy shows, equity demand increases drastically with higher leverage, due to the combination of lower volatility and lower borrowing rate. The analytical solution supports numerical results conducted by Davis, Kubler and Willen (2005). They study the consumption and portfolio behavior in a life-cycle model with constant borrowing costs that exceed a constant risk-free investment return. Based on the numerical method and calibration to match U.S. data, they show that a modest wedge between borrowing costs and the risk-free return dramatically changes the demand for equity.

We have reviewed developments of analytical studies on a consumption and portfolio choice model. While the early literature on dynamic consumption and investment decision ignored realistic factors, progress has been made with respect to incorporating stochastic investment opportunities, labor income, and nonstandard preference into Merton's basic framework. Table 2.1 summarizes and compares the basic structure of major analytical frameworks. Although these models in part do a better job of explaining stylized facts among households' behavior, there are still open topics to be conducted, such as explaining non-participation in the equity market and clarifying the effect of mortgage debt and housing on portfolio choices.

Table 2.1: Analytical Consumption-Portfolio Choice Models

State var.	Investment Opportunities	Wealth Process	Utility	References
w	(constant) $dX_t = rX_t dt$ $dS_t = \mu S_t dt + \sigma S_t dz_t$	$dW_t = [rW_t - c_t + (\mu - r)\pi_t]dt + \sigma\pi_t dz_t$	c^γ/γ	Merton (1969)
w,λ	(stochastic) $dX_t = rX_t dt$ $dS_t = (r + \sigma\lambda_t)S_t dt + \sigma S_t dz_t$ $d\lambda = \kappa[\bar{\lambda} - \lambda_t]dt - \sigma_\lambda d\tilde{z}_t$	$dW_t = [rW_t - c_t + \sigma\lambda_t\pi_t]dt + \sigma\pi_t dz_t$	c^γ/γ	Wachter (2002)
w,λ,x	(stochastic) $dX_t = rX_t dt$ $dS_t = (r + \sigma\lambda_t)S_t dt + \sigma S_t dz_t$ $d\lambda = \kappa[\bar{\lambda} - \lambda_t]dt - \sigma_\lambda d\tilde{z}_t$	$dW_t = [rW_t - c_t + \sigma\lambda_t\pi_t]dt + \sigma\pi_t dz_t$ $x_t = e^{-at}x_0 + \int_0^t e^{-a(t-s)}c_s ds$	$(c - x_t)^\gamma/\gamma$	Lui (2007) Munk and Sorensen (2004)
w,r	(stochastic) $dB_t = [r_t + \sigma_B(r_t)\lambda_B]B_t dt + \sigma_B(r_t)B_t d\tilde{z}_t$ $dS_t = (r + \sigma\lambda_S)S_t dt + \sigma_S S_t dz_t$ $dr_t = \kappa(\bar{r} - r_t)dt - \sigma_r\sqrt{r_t}d\tilde{z}_t$	$dW_t = [rW_t - c_t + \pi_t^{'}\sigma_t\lambda_t]dt + \pi_t^{'}\sigma_t d\boldsymbol{z}_t$	c^γ/γ	Liu (2007)
w,r,y	(stochastic) $dB_t = [r_t + \sigma_B(r_t)\lambda_B]B_t dt + \sigma_B(r_t)B_t d\tilde{z}_t$ $dS_t = (r + \sigma\lambda_S)S_t dt + \sigma_S S_t dz_t$ $dr_t = \kappa(\bar{r} - r_t)dt - \sigma_r\sqrt{r_t}d\tilde{z}_t$ $dy_t = y_t[(\zeta_0 + \zeta_1 r_t dz_{yt})dt + \sigma_y]$	$dW_t = [rW_t - c_t + \pi_t^{'}\sigma_t\lambda_t + y_t]dt + \pi_t^{'}\sigma_t d\boldsymbol{z}_t$	c^γ/γ	Munk and Sorensen (2006)
w, r, R	(constant but endogenous) $dX_t = rX_t dt, \quad X_t \geq 0$ $dX_t = RX_t dt \quad X_t < 0$ $dS_t = \mu S_t dt + \sigma S_t dz_t$	$dW_t = [rW_t - c_t + (\mu - r)\pi_t$ $-(R - r)(W_t - \pi_t)^-]dt + \sigma\pi_t dz_t$	c^γ/γ	Xu and Chen (1998)

2.2 Rational Mortgage Refinancing

2.2.1 Basic Option Based Refinancing Model

The option-based prepayment or refinancing model has been developed mostly for the purpose of the valuation of mortgaged-backed securities (MBS). Because the cash flow of MBS depends on the prepayment that borrowers make on the underlying mortgages, modeling the borrowers' prepayment behavior adequately is critically important for the valuation of MBS. The option-based approach focuses on determining the rational prepayment behavior where the optimal prepayment strategy is characterized exactly as the optimal exercise strategy for an American call option.

Under the basic option-based prepayment model, we generally assume that an individual who holds a mortgage debt faces a specific diffusion interest rate process r_t,

$$dr_t = \mu(r_t)dt + \sigma(r_t)d\tilde{z}_t \tag{2.27}$$

where $\{\tilde{z}_t\}$ is a standard Brownian motion. The individual's objective is to minimize the expected value of future discounted interest payments ψ

$$E\left\{\int_0^{\tau\wedge T} e^{-\int_t^s r_u du}\psi ds + 1_{\{\tau<T\}}e^{-\int_t^\tau r_u du}D_\tau \;\middle|\; r_0 = r\right\} \tag{2.28}$$

by choosing her prepayment time τ. Here $1_A = 1$ if A is true and 0 otherwise, and the individual's scheduled payment ψ is denoted by

$$\psi = D_0\left[\int_0^T e^{-r_0 u}du\right]^{-1} \tag{2.29}$$

where D_0 is initial mortgage debt and r_0 is an initial fixed borrowing rate. The value of the optimally prepaid mortgage $V(r,t)$ is then obtained as

$$V(r,0) = \min_{t<\tau<T}\left[E\left\{\int_0^{\tau\wedge T} e^{-\int_t^s r_u du}\psi ds + 1_{\{\tau<T\}}e^{-\int_t^\tau r_u du}D_\tau \;\middle|\; r_0 = r\right\}\right] \tag{2.30}$$

where D_τ is remaining mortgage debt, which is given by

$$D_\tau = \int_\tau^T e^{-r_0(s-\tau)}\psi ds. \tag{2.31}$$

Note that the first term in (2.30) represents the net present value of the scheduled payment until prepayment τ or end of the contract T. The second term represents the present value of the lump-sum prepayment D_τ. For a given interest rate level, it is optimal for a mortgagor to prepay at time t if that leads to a lower mortgage value. Therefore, the optimal prepayment strategy is generally characterized as a threshold type strategy such that the individual should prepay when the market interest rate first hit some threshold value of r^*, which in turn determines the mortgagor's optimal prepayment time τ^*.

2.2.2 Empirical Puzzles and Development of Model

The option-based approach in general induces rational prepayments where prepayments are caused by the fact that the prepayment option is "in the money". However, several empirical evidences on refinancing have revealed that the actual behavior of mortgagors differs in significant ways from the prediction of the pure option-based models. Agarwal, Driscoll, and Laibson (2004) briefly summarize these discrepancies as three types of empirical findings.

1. Late Refinancing: Some mortgagors will not do so even if market interest rates have fallen substantially.

2. Early Refinancing: Indispensable part of mortgagors refinance even if opportunity to refinance is a little premature based on the standard model.

3. Change in Frequency of Late and Early Refinancing: Late refinancing was more likely observed in the 1980s, but early refinancing was common in the 1990s.

Modifications with Realistic Factors

The development of the rational refinancing model can be traced back to Dunn and McConnel (1981a, 1981b). They assume that markets are frictionless and that mortgagors exercise their call option as soon as the value of mortgage would exceed the face value of the loan. After their seminal work, there have been several approaches to attempt to explain the observed refinancing behavior. Given the empirical observation that the actual refinancing is often delayed, Dunn and Spatt (2005) incorporate the transaction costs associated with mortgage refinancing into the rational valuation framework. Dunn and McConnel (1981a, 1981b) have introduced the prepayment due to exogenous reasons, which is described by a hazard rate η. The hazard rate η can be modeled to depend on time and the current interest rate

r_t such as $\eta_t = \eta(r_t, t)$ to capture time-varying state-dependent suboptimal prepayment.

Archer and Ling (1993) and Stanton (1995) add heterogeneity in transaction costs to the standard model. Stanton (1995) also attempts to combine the non-continuous decision-making by the borrowers with the basic model, where prepayment decision is considered according to another hazard rate $\tilde{\eta}_t = \tilde{\eta}(r_t, t)$. To account for "delayed" prepayments when interest rates decline, he formulates that mortgagors evaluate their prepayment options only at discrete intervals rather than continuously. Thus the model of Stanton (1995) incorporates heterogeneity (transaction cost), suboptimal prepayment (exogenous prepayment), and non-continuous decision, into the basic option-based model. Although the average rethinking interval $(1/\tilde{\eta})$ implied by the estimate for $\tilde{\eta}$ is eight months and seems to be too long, Stanton (1995) demonstrates that the predictions of the model are more accurate than a popular reduced-form model by Schwartz and Torous (1989).

Another strand of extension of the model is to introduce additional state variables to play in refinancing behavior. Deng, Quigley, and Van Order (2000) argue that it is important to consider the prepayment option and default option simultaneously. Motivated by the idea that default option depends on the associated asset value, Downig, Stanton and Wallace (2003) introduce the effect of variations of housing price into prepayment behavior. They develop a two-factor structural mortgage pricing model where rational mortgagors choose when to prepay and default in response to changes in both interest rates and house prices. To explain early refinancing, Hurst and Stafford (2002) consider the model in which an individual uses her housing wealth to smooth her consumption profile.

Study on Analytical Model

Despite a large body of literature on rational refinancing models attempting to explain the pricing mortgage-backed securities, surprisingly little work has been done in regard to the analytical modeling for refinancing decisions from a household's point of view.

Agarwal, Driscoll, and Laibson (2002) first develop an analytically tractable model of refinancing where the interest rate r_t obeys the Ito process. They modify a more general refinancing behavior to be a simple problem where a mortgagor minimizes the net present value of her interest payments with her discount rate δ. Mortgage of size M is issued at interest rate r_0 at initial sate $t = 0$. Each period the mortgagor pays $r_0 M$ to the bank, until the mortgage is repaid. Each period she can refinance at a fixed cost C.

With a constant hazard rate η, the mortgagor sells her home for exogenous reasons and repays her mortgage. If the interest rate process is assumed to follow an Ito process:

$$dr_t = \sigma d\tilde{z}_t, \tag{2.32}$$

the mortgage holder's objective is to minimize the expected net present value of future interest payments and associated refinancing costs discounted by her personal discount rate δ. When the mortgage holder refinances at $\tau_1, \tau_2, \ldots$ according to a certain refinancing policy, the present value of total payments starting with initial state (r_0, r_t) becomes

$$\begin{aligned} U(r_0, r_t) &= r_0 M \int_0^{\tau_1} e^{-\delta t} dt + \sum_{k=1}^{N-1} r_{\tau_k} M \int_{\tau_k}^{\tau_{k+1}} e^{-\delta t} dt \\ &\quad + r_{\tau_N} M \int_{\tau_N}^{\kappa} e^{-\delta t} dt + \sum_{k=1}^{N} e^{-\delta \tau_k} C + e^{-\delta \kappa} M \end{aligned} \tag{2.33}$$

where N is the number of refinances before κ. The first three terms in (2.33) represent instantaneous payments while the fourth and fifth terms respectively indicate refinancing costs and debt payment. Optimal refinancing policy to minimize $\mathrm{E}\left(U(r_0, r_t)\right)$ is described such that the mortgage holder attempts to find a sequence $\boldsymbol{\tau} = \{\tau_1, \tau_2, \ldots\}$ of refinancing epochs where τ_i's are stopping times with respect to the process of $\{r_t\}$.

Agarwal, Driscoll, and Laibson (2002) have solved the above problem and found that the optimal refinancing policy takes the form of a trigger policy for a specific interest rate differential between market rate and borrowing rate $x_t = r_0 - r_t$. The trigger is of a threshold type in the sense that under the optimal policy the mortgage holder should start refinancing for the first time when the interest rate differential x_t reaches a threshold x^*. The optimal threshold x^* can be explicitly found by solving the following system of equations

$$\begin{aligned} K e^{\varphi x^*} &= K + \frac{x^* M}{\delta + \eta} + F + fM \\ K \varphi e^{\varphi x^*} &= \frac{M}{\delta + \eta}. \end{aligned} \tag{2.34}$$

The presumptions that the mortgage is an infinite-horizon mortgage that pays only interests continuously and the borrower is risk neutral seem at first glance to be over simplistic. However, by adequately choosing η so that the expected time until future full repayment, $1/\eta$, is between twenty years

($\eta = .05$) and ten years ($\eta = .01$), the mortgage contract in this analytical model can approximate reality. Agarwal, Driscoll, and Laibson (2004) use their closed form solution to consider the effect of a distracted consumer, who only reconsiders her refinancing decision from time to time. Introducing such infrequent behavior of households into their continuous-time analytical model, they demonstrate that the distracted refinancing decisions induce both late and early refinancing. One shortcoming of their model, however, is that the model solves the optimal policy under the simple Ito process, which has no trend and no changes in states.

To conclude this subsection, Table 2.2 summarizes and compares major developments of option-based rational refinancing models. As Table 2.2 shows, there has been quite a few analytical studies to date. Most option-based refinancing models apply numerical solution techniques based on a discretization of time and the state space. The development of analytical models taking into account realistic factors is a demanding research area.

2.3 Regime Switching Model

2.3.1 Regime Switching

The notion that the stochastic behavior of asset prices in financial markets varies over time has been widely recognized among academic researchers and practitioners. After Hamilton (1989) first applies a Markov switching process to analyze economic dynamics, the regime switching approach has become an active research area in macroeconomics and finance.

The most attractive characteristic in the regime switching framework is that the model with regime switches can capture a number of stylized facts of asset returns documented by empirical finance literature: Stock and bond returns are time-varying and partly predictable [Campbell (1987) and Fama and French (1988,1989)], their volatility changes over time [Bollerslev, Chou, and Kroner (1992)], and correlations behave quite differently during a bear market [Ang and Chen (2002)]. One important property in regime switching framework is that mixture of standard distributions in different regime results in non-standard distributions, which has been typically observed in the distribution of asset returns.

Regime switching models can be applied to identify bull and bear regimes with different means, volatility, and correlations across assets. Suppose x_t represents an investment opportunity, such as return on equity and interest rate. In a continuous time setting with regime shifts, the dynamics of asset

Table 2.2: Development of Option-based Prepayment and Refinancing Models

State Variables	Interest Rate Process $dr = \mu(r)dt + \sigma(r)d\tilde{z}$	Other Process	Transaction Cost	Default Option	Analytical Solution	References
r	$\mu(r) = k(m-r)$ $\sigma(r) = \sigma\sqrt{r}$		No	No	No	Dun and MacConnel (1981)
r, H	$\mu(r) = k(m-r)$ $\sigma(r) = \sigma\sqrt{r}$	$dH = (\alpha - s)Hdt + \sigma_H dz_H$	No	Yes	No	Kau, Keenan, Muller and Epperson (1992)
r	$\mu(r) = k(m-r)$ $\sigma(r) = \sigma\sqrt{r}$		No	No	No	Stanton (1995)
r, H	$\mu(r) = k(m-r) - \eta r$ $\sigma(r) = \sigma\sqrt{r}$	$dH = (r - q)Hdt + \sigma_H dz_H$	Yes	Yes	No	Downing, Stanton and Wallace (1995)
r	$\mu(r) = 0$ $\sigma(r) = \sigma$		Yes	No	Yes	Agarwal, Driscoll, and Laibson (2004)
r,I	$\mu(r) = k(I_t)$ $\sigma(r) = \sigma(I_t)$	$\{I_t\}$ is a Markov Process	Yes	No	Yes	Kimura and Makimoto (2008)

variables x_t are generally assumed as the following diffusion process:

$$dx_t = \mu(I_t)x_t dt + \sigma(I_t)x_t dz_t \tag{2.35}$$

where $\{z_t\}$ is a standard Brownian motion and $\{I_t\}$ is a Markov process independent of $\{z_t\}$. The pair $\{\mu(I_t), \sigma(I_t)\}$ takes different values when the process $\{I_t\}$ is in different states. It is usually assumed that for each state i, there is a known drift parameter μ_i and a known volatility parameter $\sigma_i > 0$. In the typical case with two states (e.g. "Bull" and "Bear"), the transition rate matrix is generally described by

$$\boldsymbol{Q} = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}. \tag{2.36}$$

Here q_i denotes the transition rate of the Markov chain that governs the transition across regimes. The probability that a transition occurs from state i to another state in a small time interval $(t, t+dt)$ is equal to $q_i dt + o(dt)$ where $o(dt)$ is a small order of dt such that $o(dt)/dt \to 0$ as $dt \to 0$. Thus $1 - q_i dt + o(dt)$ is the probability that the process remains in state i. It is also noted that $1/q_i$ is the average interval during which the regime process stays in regime i. The smaller positive value q_i is, the longer the regime i persists. The transition probability matrix in a small interval $(t, t+dt)$ can be approximated by $\boldsymbol{I} + \boldsymbol{Q}dt$ where $\boldsymbol{I}$ is an identity matrix.

2.3.2 Applications in Regime Switching Model

The evidence on the behavior of macroeconomic variables as well as asset prices is consistent with the notion that investment opportunities shift between different states. A number of papers have applied the regime-switching framework to capture the behavior of these variables. Hamilton (1989) applies an autoregressive process with a two-state Markov chain to analyze the growth rate in the U.S. He showed that the average durations and the volatility of growth rates are different between contraction and expansion. In his model, the growth rate of the economy is given by a discrete-time version of process dx_t in (2.35). Ang and Bekaert (2002) demonstrate that the regime switching models forecast better out-sample movements of interest rates than single regime models. They also indicate that the regimes correspond reasonably well with the business cycles in the US. Extending the univariate short interest rate process with regime switches, Bansal and Zhou (2002) develop a term structure model where the short interest rate and market price of risks are subject to discrete regime shifts. They demonstrate that their two-factor regime switching model with regime-dependent market price of risk is supported by the data, while both the CIR and affine model with three factors and the two-factor regime switching model with constant price of risk are rejected. Dai, Singleton, and Yang (2007) also develop an arbitrage-free, dynamic term structure model with a regime-switching price factor, which gives a closed-form solution for a zero-coupon bond price. Based on the empirical test statistics, they argue that the regime-switching term structure with constant regime transition probabilities is rejected.

Motivated by these empirical studies, research on asset allocation models with regime shifts attracts attention in finance literature. Since correlations between cross-country equity returns are widely recognized as higher during bear markets than during bull markets, work on international asset allocation with regime shifts is a natural direction for research. Ang and Bekaert (2002) are the first to numerically solve and develop intuition on the dynamic asset allocation problem in the presence of regime switches for investors with CRRA preference. In their setup, the stock returns in different countries are assumed to follow the process dx_t with a two-state Markov chain I_t in (2.35). They demonstrate that regime switching model can reproduce Longin-Solnik's (2001) exceedance correlations, which is a correlation in extreme environments. They also show that international diversification is still valuable with regime shifts. From an individual investor's point of view, Guidolin and Timmermann (2005) study dynamic portfolio and consumption choice in the presence of regime switching in asset returns. They consider a US investor whose problem is asset allocation among stocks, bonds, and Treasury bills. They demonstrate that regime shifts in asset returns induce a non-monotonic relationship between the investment horizon and demand for stocks.

2.3.3 Analytical Study under Regime Switching

Despite a number of researches on the application of regime switching approach to analyze abrupt changes in financial markets, there are quite few analytical studies on optimal decision under regime shifts. In the context of option pricing, Guo and Zhang (2001) derive a closed-form solution to the optimal stopping problem for pricing perpetual American put options in a regime switching process (2.35). Their optimal stopping problem is to obtain the optimal execution time τ so as to maximize the expected discounted payoff

$$E\left[e^{-r\tau}(K - S_\tau)^+ \,\middle|\, S_0 = x, I_0 = i\right] \tag{2.37}$$

where a function $(a)^+ = a$ if $a > 0$ and 0 otherwise.

Real option analysis is another area with which regime switching approach is combined. Guo, Miao, and Morellec (2005) solve a model of firm's investment decision in which the growth rate and volatility of demand x_t shift between different states in such a process in (2.35). They describe the firm's objective value function as

$$V(x_t, k_t, i) \equiv \max_{F_{t+u} \geq 0} E\left[\int_0^{\infty} e^{-\delta u}\left[\frac{1}{\gamma} x_{t+u}^{1-\gamma} k_{t+u}^{\gamma} du - F_{t+u}\right] \middle| x_0 = x, k_0 = k, I_0 = i\right] \tag{2.38}$$

where

$$dk_t = F_t - \nu k_t dt.$$

Here x_t is the price of a production good, which is made through a production function $k_t^{1-\gamma}$ where k_t is the capital stock and γ is a parameter of the production function. F_t is an investment for the capital stock and ν is the depreciation rate of the capital stock. The firm's objective is to determine the investment policy $\{F_t\}$ that maximize the expected present value of profits net of investment costs: $1/\gamma(x^{1-\gamma}k^{\gamma}) - F$.

They show that the optimal policy is characterized by a different trigger threshold for each regime i. Moreover, because of the possibility of a regime shift of future demand, the investment policy in each regime reflects the possibility for the firm to invest in the other regime. Under a simpler setup, Makimoto (2008) derives a closed form solution for a real option problem where a multiplicative shock to the output price shifts at a random time.

In the research area on asset allocation decision, a closed-form solution with regime switching is derived in static mean-variance framework. Zhou and Yin (2006) analyze a continuous-time version of the Markowitz mean-variance portfolio selection model for a market consisting of one bank account and multiple stocks. Given the drift and volatility of the stocks switching among a finite number of states, they derived explicitly mean-variance efficient portfolios and efficient frontiers in closed forms.

2.4 Discussion on Future Issues

This chapter has reviewed the two influential dynamic models in household finance: the optimal consumption-portfolio choice and the option-based mortgage refinancing model. The simple version of the models fails to explain some key features of the actual portfolio choices and mortgage refinancing observed in data, such as hump-shaped risky investment with age and the changes in frequency of late and early refinancing. However, recent theoretical extensions of the basic model have succeeded in explaining some of these features and reducing the magnitude of empirical puzzles. These extensions include stochastic investment opportunities, labor income, habit formation in consumption and portfolio choice model. The others include incorporating heterogeneous transaction cost, housing price dynamics, infrequent decision timing, and default option into the basic option-based refinancing model. We also have made a quick review on applications of regime switching framework to optimal decision models.

Issues for future research on dynamic decision models in household finance are summarized in the following three directions. First, incorporating decision associated with borrowing into the basic consumption and portfolio choice problem is one of important research directions. From a practical view point, debt management as well as investment strategy is crucial in financial decision for typical middle-class households. Campbell and Cocco (2003) is one of the serious attempts to take into account debt management for households. To combine higher borrowing rate with developments of basic models is also an interesting topic. Life-cycle portfolio choices with housing investment and mortgage debt are another candidate to explain heterogeneous features observed among households. Committed repayment associated mortgage debt may play a significant role in determining investment patterns of individuals over their life-cycle. Fratantoni (1998, 2001), Cocco (2005), Yao and Zhang (2005) focus on this effect. Almost all of these researches resort to rather coarse and computationally very intensive numerical solutions.

Second, to develop analytical models associated with realistic factors is quite demanding to fully understand the mechanism and factors affecting an individual's investment decision. Due to the complexity of reality, most studies apply numerical solution techniques with an unknown precision. The analytical studies are much easier to analyze, interpret and implement and thus clarify an evaluation of the economic forces at play.

Finally, applying regime switching to optimal dynamic decision models in household finance is a challenging research area. Simple diffusion processes themselves cannot describe abrupt changes in investment opportunities. The regime switching framework is an attractive tool to consider realistic problems many investors face.

Chapter 3

Optimal Investment with Mortgage Debt Repayment

3.1 Introduction

The idea that the mortgage debt held by an individual influences her consumption and portfolio choice is naturally accepted. Because committed debt repayments depress a household's disposable income profile in the long-term, the mortgage debt affects the individual's optimal consumption and asset accumulation. After a dynamic consumption-portfolio choice framework was developed by Merton (1969,1971), considerable studies have attempted to incorporate various realistic factors into the basic model, such as stochastic investment opportunities, stochastic income, and borrowing constraints. Yet, despite the potential importance of the effect of debt repayment, there is little focus on an explicit link between optimal debt repayment and investment policies. As a result, the impact and mechanism of mortgage debt on optimal consumption and portfolio choice has not been settled.

On empirical grounds, recent studies such as Amerikis and Zeldes (2004) reveal several common investment patterns in households: First, equity shares in financial assets have a hump-shaped pattern over their life-cycle, peaking in the late forties and fifties. Second, the proportion of population holding equity displays a hump-shaped pattern with age. Third, equity shares in financial assets conditional on ownership are mostly constant with age. In addition to these stylized facts, Iwaisako (2003) uses Japanese households' data to find an additional empirical fact: the age-related pattern in equity holding is mostly observed in households that own their houses, but not in those that do not. Because most households owning house finish paying off their mortgage debt in their fifties or sixties, these empirical findings suggest that their debt repayments are one of important sources to cause such investment patterns.

In this chapter, we study the properties of optimal consumption and portfolio choices with long-term debt repayment under a stochastic interest rate process, to explain the variation in the ratio of equity to financial wealth observed among households. Emphasizing the impact of debt repayment on asset allocation, we construct a framework to incorporate mortgage debt repayments into the basic consumption and portfolio choice problem developed by Merton (1969). Specifically, we consider a finite lived individual investor who holds a fixed-rate mortgage debt and maximizes power utility defined over consumption and terminal wealth.

The notion that an optimal repayment depends on current market interest rates relative to the contract mortgage rate is well established in the mortgage refinancing literature. However, there is little focus on an explicit link between the optimal debt repayment and consumption-investment policies. We consider the problem within the standard consumption and portfolio choice framework. After clarifying the effects of the debt repayment term on portfolio choice in a continuous-time setting, we empirically examine how debt repayments actually influence a household's equity holding behavior. We use a survey data on Japanese households from 1993 to 2000 in order to estimate an equity share demand function among households. The observed differences in equity investment patterns between households with and without debt mostly match the implications suggested by the optimal consumption-investment model with debt repayments.

The key feature of our model is the role that the debt repayment term plays in the individual's optimal investment policy over the life-cycle. Under a deterministic income and a stochastic interest rate process, the model can express the optimal consumption and investment policies as functions of the debt repayment term which the individual choose. The debt repayment decision, on the other hand, does not depend on the optimal consumption and investment policies. Moreover, the debt repayment dampens the wealth accumulation for the individual with long-term debt. Thus, given the stochastic interest rate process and even a constant labor income for an individual, the smaller financial wealth dampened by debt repayments in turn induces much larger risky assets share of the individual's total financial wealth. Such a counter-intuitive implication reflects the optimal investment policy, which is determined by the sum of total financial wealth and human wealth. Since human wealth, defined by the net present discounted value of labor income, has a much larger effect on investment policy, smaller total financial wealth results in a larger equity share in total financial wealth.

Depressed financial wealth, however, is likely to deter the individual from participating in the equity market. As Vissing-Jorgensen (2003) demonstrates, transaction costs, such as a minimum amount requirement for investment or fixed transaction costs cause zero-holding of equity for individuals who do not have enough financial wealth to enter the market. These opposing effects (i.e. relatively higher equity share and non-participation) associated with the evolution of the financial wealth profile for individuals with debt could cause a hump-shaped pattern in equity investment. Our empirical study indicates that households with mortgage debt generally tend to have a smaller financial wealth, which deters them from participating in the equity market. On the other hand, a smaller financial wealth produces much a larger equity share in financial wealth, once households start to hold equity. Moreover, such a dampened wealth effect disappears after the individual finishes paying back all the mortgage debt in the fifties and sixties. This could cause a positive effect on her holding equity but a negative effect on equity share in her total financial wealth.

Interestingly, our empirical analysis also reveals that the equity share in total net worth, defined by total wealth minus the value of outstanding debt, decreases with age, while the equity share in financial wealth is mostly constant. Another additional interesting result shows that the average equity share in net worth, conditional on equity holding is almost same among households with and without mortgage debt. These results, therefore, imply that any serious attempt to explain household dynamic portfolio choice should consider the variations of asset share in net worth.

This study stands on the seminal work by Merton (1969,1971) and Munk and Sorensen (2007), the latter of which developed an optimal consumption-investment problem with stochastic income under a stochastic interest rate process. In addition, our work relates to several other papers in the portfolio choice literature. Campbell and Cocco (2003) numerically compute the dynamic consumption and mortgage strategies for an investor with non-tradable risky human wealth. They discuss the choice between adjustable-rate and fixed-rate mortgages, but do not focus on portfolio choice. Fleming and Zariphopoulou (1991) and Xu and Chen (1998) explicitly solve the optimal consumption and portfolio choice problem with a borrowing rate higher than the risk-less interest rate, but do not take into account the long-term debt, such as fixed-rate mortgages. The closest to our study is Fratantoni (2001) where he applies numerical simulation to examine the effects of committed payment risks associated with home mortgage on stock holding behavior.

Despite a series of theoretical studies on portfolio choice for an individual, the effect of home ownership and mortgage debt on dynamic asset allocation has not yet been empirically settled. Fratantoni (1998) finds that households with higher mortgage payment to income ratios have lower equity holding.

Flavin and Yamashita (2002) show a life-cycle pattern in holding stocks and bonds, with households holding more stocks as they age and reduce their amount of mortgage debt. Contrary to these results, Yao and Zhang (2005) point out that in the presence of labor income risk, the homeowner increases her equity share to liquid assets because of the diversification of benefits.

The main contribution of this chapter is to introduce the debt repayment term into the basic consumption-portfolio choice problem. From an analytical viewpoint, we provide an explicit link between the debt repayment and the consumption-portfolio choice. Most studies on portfolio choice problems with debt have not considered debt repayment in explicit forms. The lack of explicit solution may make unclear implications from their empirical results on asset allocation. By using explicit formula, we can investigate the effect that debt repayment has on life cycle portfolio allocation. From an empirical viewpoint, we assess the effect of debt repayment on portfolio choice by applying a bivariate probit sample selection approach. One difficulty in the empirical work on the relationship between debt and portfolio choice lies in the simultaneous features in both debt repayment and financial wealth. Debt repayment has a direct effect on wealth accumulation over the repayment term. Therefore, a sample selection bias arises in the different wealth accumulation process between households with debt and those without debt. By applying the bivariate probit sample selection technique, we to estimate the evolution of equity share profile, conditional on equity and debt holding.

This chapter is organized as follows. In Section 3.2, after presenting our problem in detail, we derive the value function and optimal consumption-investment policy for a household with debt. Implications of the optimal policy are discussed with some numerical examples in Section 3.3. In Section 3.4, we describe empirical methodology, the data we use, and the results. Finally, we conclude the study in Section 3.5.

3.2 A Household's Problem with Debt

The framework for asset allocation in this chapter builds on the classic work of Merton (1969) and the model under stochastic interest rate process developed by Munk and Sorensen (2007). In contrast to these papers, we are interested in the effect that debt repayment has on the optimal portfolio policy of an individual investor who holds mortgage debt.

We consider an individual investor with mortgage debt who has three assets available for investment: a risk-less bank account, bonds, and equity. At each period, the investor receives deterministic labor income, repays the committed debt, and chooses the amount to be invested. We assume that the mortgage debt is exogenously given. The committed debt repayments are determined by the debt repayment term, which is chosen by the individual at the initial period. While Munk and Sorensen (2007) consider the consumption and investment problem with stochastic income under a stochastic interest rate process, our model considers the problem with deterministic income under stochastic interest rate process. Instead, we allow an individual to choose the debt repayment schedule as well as consumption and portfolio choice. In the proceeding subsections, we will specify the main elements of our modeling framework.

3.2.1 Investment Opportunities

The individual can invest in three financial assets: a money market account, bonds, and stocks. The return on the money market account equals the continuously compounded short-term interest rate r_t, which is assumed to follow an Ornstein-Uhlenbeck process:

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma_r d\tilde{z}_t \tag{3.1}$$

where κ, $\bar{r}$, and σ_r are positive constants, and $\{\tilde{z}_t\}$ is a standard Brownian motion. The model is first introduced by Vasicek (1977). The amount of money market account X_t satisfies

$$\frac{dX_t}{X_t} = r_t dt. \tag{3.2}$$

The bond price at time t with maturity s is denoted by $B^s(r,t)$ when $r_t = r$. If we write $B_t^s \equiv B^s(r,t)$, then in absent of arbitrage the dynamics of B_t^s follows

$$dB_t^s = B_t^s \Big[\big(r_t + \lambda_r \sigma_B(r,t) \big) dt + \sigma_B(r,t) d\tilde{z}_t \Big] \tag{3.3}$$

where $\sigma_B(r,t) = -\sigma_r(\partial B^s(r,t)/\partial r)/B^s(r,t)$ is the volatility of the bond price and λ_r is the market price of interest rate risk, which we assume to be constant. As Vasicek (1977) shows, the price of a zero-coupon bond paying one unit of account at some maturity s is given by

$$B^s(r,t) = e^{-a(s-t)-b(s-t)r} \tag{3.4}$$

where

$$\begin{aligned} b(u) &= \frac{1}{\kappa}(1 - e^{-\kappa u}), \\ a(u) &= \bar{R}[u - b(u)] + \frac{\sigma_r^2}{4\kappa} b(u)^2, \\ \bar{R} &= \bar{r} - \frac{\sigma_r}{\kappa}\lambda_r - \frac{\sigma_r^2}{2\kappa^2}. \end{aligned} \tag{3.5}$$

Here $\bar{R}$ is the asymptotic zero-coupon yield as the time to maturity goes to infinity. From the formula for a zero-coupon bond price in (3.4), the volatility $\sigma_B(r,t)$ can be rewritten by

$$\sigma_B(r,t) = \sigma_r b(s-t). \tag{3.6}$$

The volatility $\sigma_B(r,t)$ depends only on the time-to maturity s and not on the spot rate r. Thus we denote $\sigma_B(r,t)$ in (3.6) as σ_B. The stock price S_t is assumed to follow

$$dS_t = S_t\left[(r_t + \varphi\sigma_S)dt + \rho\sigma_S d\tilde{z}_t + \sqrt{1-\rho^2}\sigma_S dz_t\right] \tag{3.7}$$

where $\{z_t\}$ is a standard Brownian motion independent of $\{\tilde{z}_t\}$. The parameter ρ is the correlation between bond return and stock return, and σ_S is the instantaneous volatility of the stock return. The Sharp ratio of the stock return is denoted by $\varphi = (\mu_t - r_t)/\sigma_S$. The ρ, σ_S, and φ are assumed to be constant. We rewrite the price dynamics in (3.3) and (3.7) as

$$\begin{aligned}\begin{pmatrix} dB_t^s \\ dS_t \end{pmatrix} &= \begin{pmatrix} B_t^s & 0 \\ 0 & S_t \end{pmatrix}\left[\left(r_t\mathbf{1} + \begin{pmatrix} \sigma_B & 0 \\ \rho\sigma_S & \sqrt{1-\rho^2}\sigma_S \end{pmatrix}\begin{pmatrix} \lambda_r \\ \lambda_S \end{pmatrix}\right)dt\right. \\ &\qquad \left. + \begin{pmatrix} \sigma_B & 0 \\ \rho\sigma_S & \sqrt{1-\rho^2}\sigma_S \end{pmatrix}\begin{pmatrix} d\tilde{z}_t \\ dz_t \end{pmatrix}\right] \end{aligned} \tag{3.8}$$

where $\mathbf{1} = (1,1)^\top$, $\top$ implies the transpose of matrix, and $\lambda_S = (\varphi - \rho\lambda_r)/\sqrt{1-\rho^2}$. To simplify the following expressions, we introduce the vector $\boldsymbol{P}_t = (B_t^s, S_t)^\top$. We rewrite equation (3.8) as

$$d\boldsymbol{P}_t = \mathrm{diag}(\boldsymbol{P}_t)[(r_t\mathbf{1} + \Sigma\Lambda)dt + \Sigma d\boldsymbol{z}_t] \tag{3.9}$$

where $\mathrm{diag}(\boldsymbol{P}_t)$ is the diagonal matrix given as

$$\mathrm{diag}(\boldsymbol{P}_t) = \begin{pmatrix} B_t^s & 0 \\ 0 & S_t \end{pmatrix},$$

$\boldsymbol{z}_t = (\tilde{z}_t, z_t)^\top$, $\Lambda = (\lambda_r, \lambda_S)^\top$, and

$$\Sigma = \begin{pmatrix} \sigma_B & 0 \\ \rho\sigma_S & \sqrt{1-\rho^2}\sigma_S \end{pmatrix}.$$

3.2.2 Mortgage Debt Repayment and Labor Income

In the Vasicek's model, the yield to maturity, denoted by $I^s(r,t)$ has an affine form,

$$I^s(r,t) = \frac{1}{s-t}\{a(s-t) + b(s-t)r_t\}. \tag{3.10}$$

We assume that the mortgage rate at time $t = 0$ is set to be the sum of the yield to debt repayment term τ and a positive lending premium η_τ, which generally reflects the default risk depending on the lending term τ:

$$\theta(\tau) = I^\tau(r,0) + \eta_\tau. \tag{3.11}$$

Here we assume that lending premium η_τ is determined to satisfy no arbitrage condition in the sense that nobody can obtain profits certainly by borrowing money at rate $\theta(\tau)$ and investing them into bonds.

Given the individual's mortgage debt D_0 and interest rate r_0 at the initial borrowing time $t = 0$, the scheduled payments $\psi(r_0,\tau)$ associated with a debt repayment term τ and the mortgage rate $\theta(\tau)$, can be written as

$$\psi(r_0,\tau) = \frac{D_0}{f_\tau}, \qquad f_\tau = \int_0^\tau e^{-\theta(\tau)u}du. \tag{3.12}$$

It is worthwhile to mention that $\psi(r_0,\tau)$ depends on the debt repayment term τ, the spot rate at the initial period r_0, and the yield curve.

In the Vasicek's model, the zero-coupon yield curve $s \longmapsto I^s(r_0,t)$ will have one of three shapes depending on the parameter values and the current spot rate r_0: (i) increasing if $r_0 < \bar{R} - \sigma_r^2/4\kappa^2$, (ii) decreasing if $r_0 > \bar{R} + \sigma_r^2/2\kappa^2$, and (iii) hump-shaped for the intermediate values of r_0. Figure 3.1 in Section 3.3 shows examples of a zero-coupon yield curve.

The individual is assumed to receive her exogenous constant income flow y in each period. Her net income after debt repayments, denoted by α_t, is then described as

$$\alpha_t = \begin{cases} y - \psi(r_0,\tau), & 0 \le t \le \tau, \\ y, & \tau < t \le T. \end{cases} \tag{3.13}$$

3.2.3 Individual's Optimization Problem

With the financial wealth of the individual investor W_t and her net income α_t at time t, she chooses consumption c_t and the amounts invested in bonds and stocks, denoted by a vector $\Pi_t = (\pi_{B,t}, \pi_{S,t})^\top$. The amount invested in the bank account is determined as the residuals, $W_t - \pi_{B,t} - \pi_{S,t}$. Given a consumption strategy c_t, an investment strategy Π_t, and net income α_t, the financial wealth of the individual W_t evolves as

$$dW_t = (r_t W_t + \Pi_t^\top \Sigma\Lambda - c_t + \alpha_t)dt + \Pi_t^\top \Sigma d\boldsymbol{z}_t. \tag{3.14}$$

At the initial period $t = 0$, the individual first chooses an appropriate debt-repayment term τ. For $0 \leq t \leq T$, she determines the optimal instantaneous consumption c_t and investment policy $\Pi_t = (\pi_{B,t}, \pi_{S,t})^\top$ to maximize her expected utility

$$E\left[\int_0^T e^{-\delta s} u(c_s) ds + e^{-\delta T} U(W_T) \middle| W_0 = w, r_0 = r\right]. \tag{3.15}$$

We define the household's value function conditional on τ as

$$J^{(\tau)}(w, r, t) = \max_{\{c_s\},\{\Pi_s\}} E\left[\int_t^T e^{-\delta s} u(c_s) ds + e^{-\delta(T-t)} U(W_T) \middle| W_t = w, r_t = r\right]. \tag{3.16}$$

We will solve the individual investor's problem in two steps: (1) given the debt repayment term τ, the individual chooses optimal process of instantaneous consumption $\{c_t\}$ and the amounts to be invested $\{\Pi_t\}$ to maximize her utility over the life cycle; (2) given the optimal values of c_t and Π_t conditional on τ, the individual chooses τ to maximize the value function

$$\max_\tau J^{(\tau)}(w, r, 0). \tag{3.17}$$

We assume that the individual's utility of consumption and terminal wealth have the power utility function with a positive risk-aversion coefficient γ:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \tag{3.18}$$

$$U(w) = \frac{w^{1-\gamma}}{1-\gamma}. \tag{3.19}$$

3.2.4 Individual's Value Function

In what follows, we may write J instead of $J^{(\tau)}$ to simplify the notation, as long as no confution will occur. Applying the principle of optimality and Ito's lemma to (3.16) yield the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}
\delta J \;=\; & J_t + (rw + \alpha_t)J_w + \max_{c_t}\Big\{u(c_t) - J_w c_t\Big\} \\
& + \max_{\Pi_t}\Big\{J_w \Pi_t^\top \Sigma\Lambda - J_{wr}\Pi_t^\top \Sigma \boldsymbol{e}_1 \sigma_r + \frac{1}{2} J_{ww} \Pi_t^\top \Sigma\Sigma^\top \Pi_t\Big\} \\
& + J_r \kappa[\bar{r} - r] + \frac{1}{2} J_{rr}\sigma_r^2
\end{aligned} \tag{3.20}$$

where $\boldsymbol{e}_1 = (1,0)^\top$. Here J_t, J_w, J_r, J_{ww}, J_{wr}, and J_{rr} are defined by $J_t \equiv \partial J/\partial t$, $J_w \equiv \partial J/\partial w$, $J_r \equiv \partial J/\partial r$, $J_{ww} \equiv \partial^2 J/\partial w^2$, $J_{wr} \equiv \partial^2 J/\partial w \partial r$, and $J_{rr} \equiv \partial^2 J/\partial r^2$. The derivation of (3.20) is provided in Appendix A.1.

Maximizing the right-hand side of (3.20) with respect to c_t gives the first order condition

$$u'(\hat{c}_t) = J_w \tag{3.21}$$

where $\hat{c}_t$ denotes the optimal consumption. Noting the marginal utility for consumption is $u'(c) = c^{-\gamma}$, we obtain

$$\hat{c}_t = \Big[J_w\Big]^{-1/\gamma}. \tag{3.22}$$

Maximizing (3.20) with respect to Π_t gives the first order condition

$$J_w \Sigma\Lambda - J_{wr}\Sigma \boldsymbol{e}_1 \sigma_r + J_{ww}\Sigma\Sigma^\top \hat{\Pi}_t = 0, \tag{3.23}$$

which yields

$$\hat{\Pi}_t = -\frac{J_w}{J_{ww}}(\Sigma^\top)^{-1}\Lambda + \frac{J_{wr}}{J_{ww}}(\Sigma^\top)^{-1}\boldsymbol{e}_1\sigma_r. \tag{3.24}$$

The first term of (3.24) represents the standard mean-variance optimal portfolios and the second term represents a hedge against the changes in the interest rate. The inverse of the transposed volatility matrix is

$$(\Sigma^\top)^{-1} = \begin{pmatrix} \sigma_B & \rho\sigma_S \\ 0 & \sqrt{1-\rho^2}\sigma_S \end{pmatrix}^{-1} = \frac{1}{\sqrt{1-\rho^2}\sigma_B\sigma_S}\begin{pmatrix} \sqrt{1-\rho^2}\sigma_S & -\rho\sigma_S \\ 0 & \sigma_B \end{pmatrix}$$

so that we can write

$$\hat{\Pi}_t = -\frac{J_w}{J_{ww}}(\Sigma^\top)^{-1}\Lambda + \frac{J_{wr}}{J_{ww}}\frac{\sigma_r}{\sigma_B}\boldsymbol{e}_1. \tag{3.25}$$

Following Munk and Sorensen (2007), the value function is given by

$$J(w,r,t) = \frac{1}{1-\gamma}\ell(r,t)^\gamma\{w + h(r,t)\}^{1-\gamma} \tag{3.26}$$

where

$$\ell(r,t) = k(T)(B^T(r,t))^{\frac{\gamma-1}{\gamma}} + \int_t^T k(s)(B^s(r,t))^{\frac{\gamma-1}{\gamma}}ds \tag{3.27}$$

with $k(u)$ defined by

$$\begin{aligned} k(u) \;=\; \exp\Bigg\{ & \left(-\frac{\delta}{\gamma} + \frac{1-\gamma}{2\gamma^2}(\lambda_S^2 + \lambda_r^2)\right)u \\ & + \frac{1-\gamma}{\gamma^2}\left((\bar{r} - \bar{R})(u - b(u)) - \frac{\sigma_r^2}{4\kappa}b(u)^2\right)\Bigg\} \end{aligned} \tag{3.28}$$

and

$$h(r,t) = \int_t^T \alpha_s B^s(r,t)ds. \tag{3.29}$$

The result can be verified by substitution of (3.26) into the HJB equation (3.20), which is provided in Appendix A.2.

3.2.5 Optimal Consumption and Investment Policy

Once debt repayment term τ is given, the optimal policy for the individual is derived from her value function. From the first-order condition, the optimal consumption is given by

$$\hat{c}_t = \ell(r_t,t)^{-1}\{w_t + h(r_t,t)\}. \tag{3.30}$$

The optimal investment mix of bonds and equity holdings is

$$\begin{aligned}\hat{\Pi}_t &= \frac{1}{\gamma}(\Sigma^\top)^{-1}\Lambda(W_t + h(r_t,t)) \\ &\quad + \left(h_r(r_t,t) - \frac{\ell_r(r_t,t)}{\ell(r_t,t)}(W_t + h(r_t,t))\right)\frac{\sigma_r}{\sigma_B}\boldsymbol{e}_1\end{aligned} \tag{3.31}$$

where

$$h_r(r,t) = \int_t^T b(s-t)\alpha_s B^s(r,t)ds \tag{3.32}$$

and

$$\begin{aligned}\ell_r(r,t) &= \frac{\gamma-1}{\gamma}\Bigg[b(T-t)k(T-t)(B^T(r,t))^{\frac{\gamma-1}{\gamma}} \\ &\quad + \int_t^T b(s-t)k(s-t)(B^s(r,t))^{\frac{\gamma-1}{\gamma}}ds\Bigg].\end{aligned} \tag{3.33}$$

Note that optimal equity to be invested $\hat{\pi}_{S,t}$ is expressed as

$$\hat{\pi}_{S,t} = \frac{1}{\gamma}\frac{\varphi}{\sqrt{1-\rho^2}}\{W_t + h(r_t,t)\}. \tag{3.34}$$

3.2.6 Debt-repayment Term Decision

Given an initial spot rate r_0 at $t = 0$, the individual choose optimal debt repayment term τ based on (3.17). From (3.26), we get

$$J^{(\tau)}(w_0, r_0, 0) = \frac{1}{1-\gamma}\ell(r_0, 0)^{\gamma}\{w_0 + h(r_0, 0)\}. \tag{3.35}$$

Since τ affects the value function only through the function of $h(r_0, 0)$:

$$\begin{aligned} h(r_0, 0) &= \int_0^T \alpha_s B^s(r_0, 0) ds \\ &= y \int_0^T B^s(r_0, 0) ds - \int_0^\tau \psi(r_0, \tau) B^s(r_0, 0) ds, \end{aligned} \tag{3.36}$$

we only need to maximize $h(r_0, 0)$ to obtain (3.17).

In sum, the optimal debt repayment term $\hat{\tau}$ can be determined by minimizing the second term of (3.36), which by using (3.4), (3.10), (3.11), and (3.12), is expressed as

$$\begin{aligned} \hat{\tau} &= \arg\min_{\tau \geq 0} \left\{ \int_0^\tau \psi(r_0, \tau) B^s(r_0, 0) ds \right\} \\ &= \arg\min_{\tau \geq 0} \left\{ D_0 \left[\int_0^\tau e^{-\frac{1}{\tau}[a(\tau)+b(\tau)r_0+\eta_\tau]s} ds \right]^{-1} \int_0^\tau e^{-a(s)-b(s)r_0} ds \right\} \end{aligned} \tag{3.37}$$

Note that from (3.10) and (3.11), $\frac{1}{\tau}[a(\tau) + b(\tau)r_0 + \eta_\tau]$ represents the sum of the yield to maturity τ and a lending premium η_τ. For the case of increasing yield curve, the optimal debt repayment $\hat{\tau}$ is as short as possible to save interest rate payments. On the other hand, for the decreasing or hump-shaped yield curve, the optimal debt repayment term τ may be found between $t = 0$ and T, depending on the parameters of the spot rate process and η_τ.

From a practical viewpoint, the debt repayment term is determined by various factors such as income profile, tax reduction incentives, and her retirement period, even when we find the optimal debt repayment term from (3.37). For instance, the individual might have to choose her debt repayment term τ under a restricted repayment condition such as

$$\psi(r_0, \tau) \leq \beta y, \quad 0 \leq t \leq \tau$$

where $0 < \beta < 1$. This condition implies that the individual cannot repay $\psi(r_0, \tau)$ which is larger than the ratio of her regular income. In the following sections, we therefore suppose that the debt repayment term is exogenously given, by taking those practical factors into account.

3.3 Numerical Analysis

Once the debt repayment term $\hat{\tau}$ is exogenously given, we can calculate the individual's value function and her human wealth $h(r,t)$, which determines the optimal equity investment $\hat{\pi}_t$ and consumption $\hat{c}_t$. Given the same financial wealth w_t and the same future income profile y_t, committed payments reduce the individual's net human wealth $h(r,t)$, which in turn decreases her equity investment. At the same time, however, the committed debt repayment also dampens the individual's financial wealth accumulation, which is likely to increase the equity share in total financial wealth, $\hat{\pi}_{S,t}/w_t$.

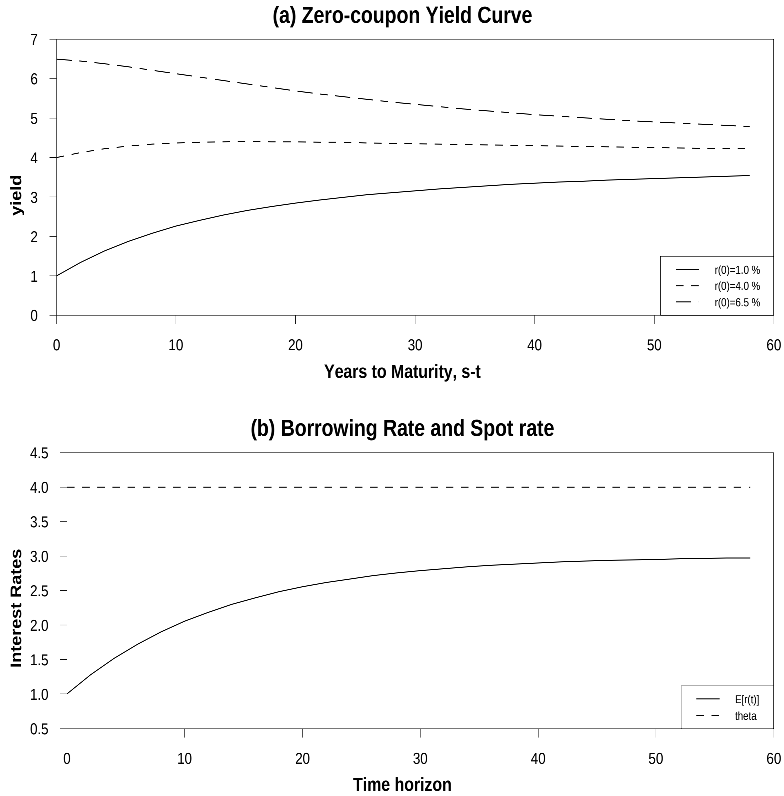

Figure 3.1: Yield Curve and Example of an Interest Rate Process

Note: Top compares the zero-coupon yield curve for different values of the spot rate at $t = 0$. The parameter values are $\kappa = 0.15$, $\bar{r} = 0.03$, $\sigma_r = 0.03$, and $\lambda_r = -0.15$. The long rate is then $\bar{R} = 0.06$. The yield curve is increasing for $r_0 < 0.03$, decreasing for $r_0 > 0.06$, and humped for intermediate values of r_0. Bottom plots an expected spot rate process $E(r_t)$ and the mortgage rate θ for $r_0 = 0.01$, $\tau = 25$, and $\eta_\tau = 0.01$. The expected spot rate is calculated by using the formula for Vasicek's model, $E_{r,0}[r_t] = \bar{r} + (r_0 - \bar{r})e^{-\kappa t}$.

Figure 3.1 represents the examples of yield curve and an example of the spot rate process we choose for numerical analysis. As Figure 3.1(a) demonstrates, the yield to the maturity will be a decreasing, humped, or increasing curve, depending on the parameters. Figure 3.1(b) plots an expected spot rate process starting at $r_0 = 0.01$ (1 percent) and the mortgage rate θ for 25 years fixed mortgage ($\tau = 20$) calculated with parameter values for $\kappa = 0.15$, $\bar{r} = 0.03$, $\sigma_r = 0.03$, and $\eta = 0.01$.

Figure 3.2 compares an individual's wealth profile without debt and with a 25 years mortgage debt. Without debt repayment, the individual's wealth increases with age and peaks in her fifties while optimal equity investment decreases with age. The equity share in total wealth also decreases with age. With debt repayment, on the other hand, the amount of wealth in each period is much smaller than that without debt. Moreover, the wealth profile over the life-cycle is linked to the debt repayment term, because the investor's committed debt repayment depresses her asset accumulation until she has paid off all her entire loan. It is noteworthy that although the amount of equity investment for the household with debt is smaller than that without debt, the financial wealth profile is much smaller than that without debt.

As a result, the equity share in total financial wealth during repayment can be bigger than that without repayment. Such counter intuitive feature comes from the structure of the optimal investment policy, which depends not only on financial wealth w_t but also on human wealth $h(r, t)$. To make this point clearer, we rewrite equation (3.34) with $\rho = 0$ as the optimal equity share in the total financial wealth

$$\frac{\hat{\pi}_{St}}{w_t} = \frac{1}{\gamma}\left(\frac{\mu_t - r_t}{\sigma_{St}^2}\right)\left(1 + \frac{h(r,t)}{w_t}\right). \tag{3.38}$$

As this equation shows, given human wealth $h(r, t)$, a smaller current financial wealth will induce a larger equity share in the total financial wealth $\hat{\pi}_{St}/w_t$. Depending on the exogenously determined debt repayment τ, an accelerated debt repayment is likely to dampen the 'current' financial wealth w_t, while the debt repayment term itself would not change much the future income wealth $h(r, t)$, once the initial debt and income profile is given. Figure 3.3 clarifies this point by comparing the equity share profile without debt, with a 25 years mortgage, and a 30 years mortgage.

The base case parameter values are set as follows: $\gamma = 0.5$, $\mu_t - r_t = 0.03$, and $\sigma_S^2 = 0.25$ with constant income profile $y = 500$. The line represents the equity share to financial wealth without debt repayment. The other two dotted lines represent the evolution of equity asset share with debt for debt repayment term $\tau = 25$ (tilted hump-shaped dashed line) and for $\tau = 30$ (double-dashed dotted line).

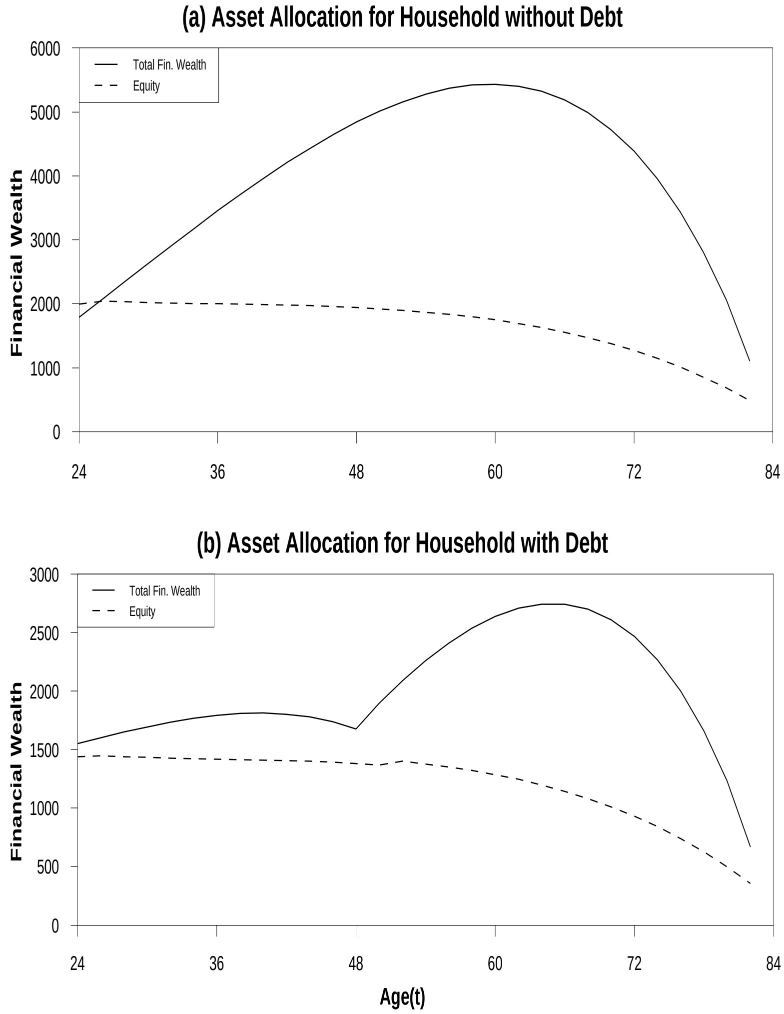

Figure 3.2: Comparison of Asset Allocation with Debt and without Debt

Note: The top graph represents optimal equity holding and total financial wealth evolution for a household without debt. The bottom one represents the same for a household with debt. The parameter values for both cases are set as follows : $\gamma = 0.5$, $\mu_t - r_t = 0.03$, $\sigma_S = 0.5$, and $\rho = 0.0$. Household's initial wealth w_0 and initial debt D_0 are 1,500 and 3000, respectively. Income y_t is assumed to be fixed to 500 for simplicity.

Figure 3.3 highlights the properties of the lifetime investment profile for the individual with and without debt. First, the equity share in total financial wealth ($\hat{\pi}_{St}/w_t$) decreases with age. Second, the equity share with debt-payment is bigger than that without debt repayment and has a hump-shaped profile. Third, the equity share with a shorter debt repayment term has a more tilted hump-shaped profile. The main driving factor that causes such differences is the evolution of the individual's financial wealth. Given an initial wealth and debt, the accelerated debt repayment reduces financial wealth accumulation, which increases the ratio of risky assets to financial wealth.

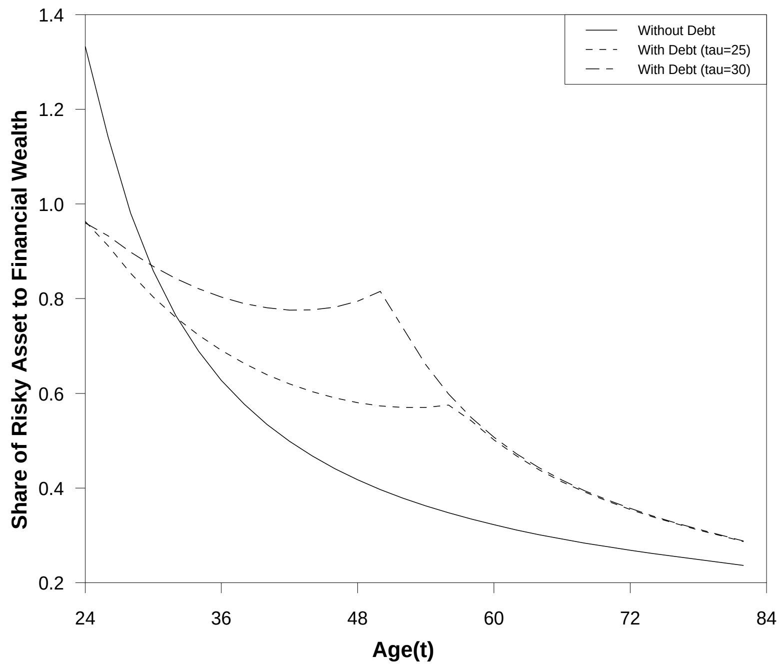

Figure 3.3: Equity Share as a Fraction of Total Wealth

Note: The figure plots the evolutions of equity share in various cases. Line plots the share of equity to financial wealth for household without debt. Dashed line plots the equity share to financial wealth for household who holds 25 years mortgage debt and double-dashed dotted line plots for 30 years mortgage, both of which are calculated for the same parameter values in Fig 3.2.

In sum, these numerical examples indicate that the observed income-wealth ratio, y/w_t characterizes the optimal equity share profile to financial wealth and debt repayment policy directly, and indirectly affects the shape of y/w_t over the life-cycle.

3.4 Empirical Analysis

To examine the effect of debt repayment on the portfolio choice over the life-cycle, we specify an equity share function. As the discussion in Section 3.3 suggests, the basic factors determining a household's equity share profile are the household's investment time t (age), financial wealth w_t, human wealth $h(r,t)$, and debt repayment term $\hat{\tau}$. An important question is whether the implication of the model actually matches the investment behavior observed in Japanese individual data.

3.4.1 Estimation Strategy

To assess how the repayment term affects portfolio choices, we estimate the following basic regression:

$$\left(\frac{\pi}{w}\right)_i = \alpha + \beta_1 age_i + \beta_2 \left(\frac{y}{w}\right)_i + \beta_3 \tau_i + \gamma X_i + \varepsilon_i \tag{3.39}$$

where $(\pi/w)_i$ is household i's equity share in total financial wealth, $(y/w)_i$ is the ratio of annual labor income to financial wealth, which is used as a proxy of human wealth to financial wealth ratio, $(h(r,t)/w)_i$ in (3.38). τ_i is a predetermined debt repayment term of an individual who holds mortgage debts. The vector X_i contains control variables such as demographic factors as well as time dummies to indicate the survey year, and ϵ_t is the error term. Time dummies are added to control time-specific effects. As discussed in Ameriks and Zeldes (2001), this approach presumes that cohort effects are equal to zero.

The implication of the model predicts that the equity share in total financial wealth is bigger on average for households with debt ($\alpha_{Debt} > \alpha_{NoDebt}$). The model also predicts that higher human wealth relative to financial wealth has a positive effect on the share ($\beta_2 > 0$), and a longer (shorter or accelerated) debt repayment term has a negative (positive) effect on the share ($\beta_3 < 0$). It is worthwhile to mention that households' equity shares in the total financial wealth, $(\pi/w)_i$, in equation (3.39) are observed only for equity holders ($\pi_i > 0$). Moreover, both the households' debt repayment term, τ_i, and income-wealth ratio $(y/w)_i$ associated with depressed financial wealth are observed only for households who hold mortgage debt.

To cope with such sample selection biases, we adopt a bivariate probit sample selection approach to estimate equity share regression, conditional on equity and debt holding. Specifically, we first estimate a bivariate probit regression to explain binary choice behavior showing whether a household holds equity and mortgage debt or not :

$$I_{i1}^* = \delta_1' Z_{i1} + u_{i1} \tag{3.40}$$
$$I_{i2}^* = \delta_2' Z_{i2} + u_{i2} \tag{3.41}$$
$$I_{ij} = 1 \text{ if } I_{ij}^* > 0 \text{ and } 0 \text{ otherwise for } j\text{=1,2}, \tag{3.42}$$

where I_{ij}^* is an unobserved decision variable to imply the difference of the household utility between holding and no-holding. I_{ij} is an observable variable which is defined by a binary dependent variable implying households i's equity and debt holding states j. The vector Z_{ij} represents the explanatory variables. If a household i holds both equity ($I_{i1}^* > 0$) and mortgage debt ($I_{i2}^* > 0$), then $I_{i1} = I_{i2} = 1$. On the other hand, if the household owns equity, but not mortgage debt, then $I_{i1} = 1$ and $I_{i2} = 0$. We next apply the bivariate probit regression results in (3.40)-(3.42) to estimate the equity share regression in (3.39), conditional on both equity and debt holding. In such a recursive structure, we assume that ϵ, u_1, and u_2 have a trivariate normal distribution with variances σ^2, 1, and 1, respectively. Finally, to explain the variation in the ratio to equity to wealth, we pay attention to two wealth variables: total financial wealth and total net worth, which is the sum of total financial wealth and home equity.

3.4.2 Data

The households' data used in this study constitute a pooled data from the annual survey from 1993 to 2000, published by Nihon Keizai Shinbun, which is known as Nikkei Radar. The annual survey consists of about 2,700 households which have been selected at random in the Tokyo metropolitan area. The survey contains information about the portfolio allocation of households and demographic factors such as the age of household head, marital status, and income. The questions were mainly answered by the heads of households aged between 22 and 85.

We aggregate each household's total financial wealth by adding up all liquid financial wealth in each category. Liquid wealth is the sum of bank account, trust funds, bonds, stocks, and mutual funds. Mutual funds consist of bonds-only mutual funds and those containing equity. In this study, we define equity as the sum of stocks and mutual funds containing any stocks. Although the simplest definition of equity is direct equity-holding, we consider that many stocks are held indirectly through mutual funds. We also define equity shares as the ratio of equity to two wealth variables: financial wealth and net worth. The net worth is the sum of total financial wealth and home equity, which is the market value of both home and other real estate minus the value of outstanding mortgage debt. We estimate the outstanding mortgage debt based on the annual debt payment and expected future repayment term. Excluded from the analysis are 1,516 households that have no financial assets, whose total financial assets or debt repayments have extremely large values, and whose head is aged under 24 and over 70.

Table 3.1: Sample Characteristics for Households with and without Debt

Variable	All	With Debt	Without Debt
Median annual income (10,000 yen)	650	850	550
Median financial assets	530	520	530
Median total assets	1460	3310	810
Median net worth	860	1100	810
Median equity share	0.0	0.0	0.0
Mean equity share	0.08	0.10	0.07
Mean equity share to net worth	0.04	0.03	0.04
Mean value of financial assets	1118.87	925.61	1194.65
Mean value of equity	140.46	143.46	138.84
Proportion of equity holders	0.27	0.32	0.25
Median age	45	47	44
Proportion married	0.80	0.95	0.74
Average number of children	0.97	1.38	0.81
Proportion with college education	0.43	0.49	0.41
Number of observations	19,981	5,638	14,343
Percent share of total observations	(100.0)	(28.2)	(71.8)

Note: Table 3.1 compares summary statistics of regular income, financial assets, and demographic factors between households with and without mortgage debt. All statistics are calculated by using Japanese pooled survey data, 'Nikkei Rader', from 1993 to 2000. The unit value of income and assets in the table is ten thousand yen. Excluded from the statistics are 1,516 households that have no financial asset, that have an outlier in the value of financial asset or debt repayment, and whose head's age is under 24 or over 70.

Table 3.1 summarizes the basic sample statistics for the total sample of 19,981 households and two sub-groups classified by debt holding status. In terms of equity share, Table 3.1 compares the median, the mean, and the fraction of households investing equity. The proportion of equity holding is 27 percent for all households, 32 percent for the households with mortgage debt, and 25 percent for the households without debt. Mean equity shares in total financial wealth are 8 percent for total sample, 10 percent for those with debt, and 7 percent for those without debt.

Table 3.1 also compares median income, financial assets, and demographics among subsamples. The median of financial assets for the households with mortgage debt is 5.2 million yen, which is smaller than 5.3 million yen of median financial assets for those without debt. Their median annual income is higher than those without debt, partly due to a higher median age in the sample. These findings suggest that we need to control age and income to estimate and compare equity share regression.

3.4.3 Empirical Results

Table 3.2 presents the coefficient estimates on explanatory variables from the bivariate probit regression for equity and debt holding. While decision about equity share is our main focal point, joint decisions about holding risky assets and liabilities are also of great interest. To investigate joint decisions, we pick up age of household's head, both annual labor income and financial wealth, taken by logarithm, as basic explanatory variables. Since debt repayment directly affects financial wealth accumulation, we drop the log of financial wealth for debt holding regression. As a control for general household characteristics, we include demographic factors such as number of children, education, and marital status. We also include a time dummy to control the time effect.

The bivariate results show the positive effects of age and income on equity and debt holding behavior. In particular, the coefficient on the log of income is significantly positive and greater for debt holding than for equity holding decision. The number of children and the dummy variable of marriage have also significantly positive effect on debt holding, which suggests that both are important driving factors for homeownership decision. On the other hand, these variables have no significant effect on equity holdings. The coefficients on the log financial wealth and on college degree dummy are significantly positive. As shown at the bottom of Table 3.2, the correlation, ρ, estimated by the error terms in both equity and mortgage debt holding bivariate probit regression, is significantly positive. This indicates that unobservable household-specific factors for explaining equity and debt holding behavior is significantly positive.

Table 3.3 reports the results of equity share in total financial wealth regressions for households with debt and those without debt. We performed three regressions by combining constant with basic explanatory variables, such as age, income-wealth ratio, debt repayment term, and net worth-financial wealth ratio. The first column (I) in Table 3.3 compares the estimation results for the specification that has only the constant term as independent variables in equation (3.38). Note that it provides the sample

Table 3.2: Bivariate Probit Regression for Equity and Mortgage Debt Holding Decision

	Equity Holding			Debt Holding		
	Coeff	t-stat		Coeff	t-stat	
Constant	-3.903	-42.658	***	-4.360	-45.109	***
Age/10	0.049	4.386	***	0.027	2.439	***
Log of income	0.217	11.059	***	0.701	42.681	***
Log of fin. wealth	0.489	44.289	***			
Number of children	0.006	0.505		0.190	18.910	***
College degree	0.313	13.966	***	-0.107	-5.008	***
Self-employed	-0.165	-5.572	***	-0.195	-7.284	***
Double-income	0.064	1.131		0.130	2.079	**
Married	-0.005	-0.120		0.553	19.978	***
Time dummy 1994	-0.069	-1.646		0.017	0.427	
1995	-0.092	-2.190	**	0.049	1.300	
1996	-0.139	-3.288	***	0.047	1.197	
1997	-0.111	-2.609	***	0.104	2.565	***
1998	0.118	2.873	***	0.157	3.919	***
1999	0.010	0.239		0.121	2.932	***
2000	0.076	1.772		0.160	3.930	***
ρ				0.112	7.550	***
Number of obs	$19,981$					
Log likelihood	$-19,540.00$					
Percent correct	35.7			57.9		

Note: ***,** represent statistical significance at 1 percent and 5 percent levels, respectively. Table 3.2 provides the determinants of the risky asset and mortgage debt holding behavior for households from the pooled survey data from 1993 to 2000 Surveys of 'Nikkei Rader'. The dependent variables are indicator variables of (1) whether households have risky financial assets, and (2) whether households hold mortgage debt. The table reports the estimated coefficients and t-statistics of the bivariate probit regression. Demographic factors and years shown in explanatory variables represent dummy variables. The bottom line shown in the explanatory variables, ρ is the correlation of the error terms in the two probit regressions. This correlation would be zero if risky asset holding and mortgage debt holding decisions were determined independently. Percent correct is calculated by the percentage of the number of matched data between actual and predicted value, based on the rule that predicted holding means its estimated probability is over 0.5.

mean of equity share, adjusted by sample selection biases by equity holding behavior. The result indicates that the equity share for households with debt is 29.0 percent, which is almost twice as much as that of those without debt. It is worthwhile to note that the difference is also larger than the difference in the simple mean reported in Table 3.1. The simple mean of equity share is 10 percent for households with debt, and 7 percent for those without debt. The second column (II) in Table 3.3 adds age of household head and financial wealth as additional explanatory variables. The coefficient on the age of household head and the coefficient of financial wealth are both positive but not significant. The third column (III) adds income-wealth ratio, (y/w) and debt repayment term, τ, both of which are expected to be important factors suggested by numerical analysis. The income-wealth ratio has a significantly positive effect on equity share for both households with debt and those without debt. In addition, the parameter estimate for debt holder is twice as large as that for those without debt. Lastly, as the upper panel of Table 3.3 shows, the coefficient on the debt repayment term is significantly negative, which is consistent with the prediction of our model. It is also noteworthy that the ratio of net worth to financial wealth is significantly positive. Accelerating debt repayment increases home equity and decreases liquid financial wealth, which results in a higher equity shares in total financial wealth.

In sum, these results support the main implications described by the model and numerical analysis in previous sections. First, the equity share in financial wealth, conditional on asset holding, is larger for households with debt than for those without debt. Moreover, this result still holds after controlling other factors. Second, the income-wealth ratio has a significantly positive effect on the household's equity share. Third, the accelerated or shorter debt repayment term would increase the equity share, which is suggested by the numerical analysis using an explicit form in the previous sections.

Finally, additional results are worth being mentioned. Table 3.4 reports the results of equity share in net worth regressions for households with debt and those without debt. Several interesting characteristics are apparent. First, the equity share in net worth, conditional on owning, are almost same among households with and without debt. Based on the first column (I) in Table 3.4, the equity share for households with debt is 10.0 percent and for households without debt is 6.3 percent. Second, as the second (II) and third columns (III) in Table 3.4 show, equity shares in net worth decrease with age, which is in contrast to the results for the equity shares in financial wealth. Third, based on the adjusted R squared, the explanatory power of the regression is higher than those of the regression for equity shares in financial wealth.

Table 3.3: Equity Share in Financial Wealth Regression Based on Bivariate Probit Sample Selection Model.

	Household with Mortgage Debt								
	I			II			III		
	Coefficient	t-stat		Coefficient	t-stat		Coefficient	t-stat	
Constant	0.290	11.406	***	0.266	5.735	***	0.293	6.279	***
age				0.004	0.602		0.003	0.373	
Annual income/Financial wealth							0.019	5.165	***
Debt repayment term/10							−0.022	−2.101	***
Net worth/Financial wealth							0.001	1.847	*
lambda of risky asset holding	0.106	6.946	***	0.109	6.608	***	0.033	1.467	
lambda of debt holding	−0.063	−3.202	***	−0.064	−3.251	***	−0.024	−1.140	
Number of observations		1,798			1,798			1,798	
Adjusted R squared		0.028			0.028			0.042	
	Household without Mortgage Debt								
	I			II			III		
	Coefficient	t-stat		Coefficient	t-stat		Coefficient	t-stat	
Constant	0.152	9.112	***	0.129	4.735	***	0.156	5.317	***
age				0.003	0.864		0.003	0.864	
Annual income/Financial wealth							0.010	2.360	**
Net worth/Financial wealth							0.001	0.049	
lambda of risky asset holding	0.125	13.265	***	0.129	12.231	***	0.108	1.193	
lambda of debt holding	−0.015	−1.042		−0.017	−1.138		0.001	0.002	
Number of observations		3,615			3,615			3,615	
Adjusted R squared		0.052			0.053			0.053	

Note: ***,**,* represent statistical significance at 1 percent, 5 percent, and 10 percent levels, respectively. The results of risky asset share regression based on bivariate probit sample selection. The dependent variable is the share of equity in total financial asset. Equity is defined by the sum of stock and mutual funds. The upper panel provides the results for households with mortgage debt, and the lower for household without debt. 'Lamda' is the product of the correlation of the error terms in 'risky asset holding probit' or 'mortgage debt holding probit' regression, and the variance of the error term in risky asset share regression. This would be zero if risky asset holding households were a random subgroup of the population. Time dummies are included in explanatory variables to control time effect.

Table 3.4: Equity Share in Net Worth Regression Based on Bivariate Probit Sample Selection Model.

	Household with Mortgage Debt								
	I			II			III		
	Coefficient	t-stat		Coefficient	t-stat		Coefficient	t-stat	
Constant	0.100	2.603	**	0.361	1.173		0.312	1.287	
age				−0.049	−2.012	**	−0.055	−2.334	**
Annual income/Net worth							0.206	33.241	***
Debt repayment term/10							−0.004	−0.078	
lambda of risky asset holding	−0.258	−2.561	**	−0.303	−2.752	***	−0.239	−2.763	***
lambda of debt holding	0.344	2.654	***	0.361	2.754	***	0.360	3.497	***
Number of observations		1,798			1,798			1,798	
Adjusted R squared		0.046			0.046			0.385	
	Household without Mortgage Debt								
	I			II			III		
	Coefficient	t-stat		Coefficient	t-stat		Coefficient	t-stat	
Constant	0.063	4.377	***	0.299	12.990	***	0.342	15.119	***
age				−0.037	−12.950	***	−0.033	−11.830	***
Annual income/Net worth							0.048	13.910	***
lambda of risky asset holding	0.143	17.652	***	0.089	9.968	***	0.020	1.957	*
lambda of debt holding	0.046	3.727	***	0.062	5.037	***	0.103	8.339	***
Number of observations		3,615			3,615			3,615	
Adjusted R squared		0.105			0.145			0.189	

Note: ***,**,* represent statistical significance at 1 percent, 5 percent, and 10 percent levels, respectively. The results of risky asset share regression based on bivariate probit sample selection. The dependent variable is the share of equity in total net worth. Equity is defined by the sum of stock and mutual funds. The upper panel provides the results for households with mortgage debt, and the lower for household without debt. 'Lamda' is the product of the correlation of the error terms in 'risky asset holding probit' or 'mortgage debt holding probit' regression, and the variance of the error term in risky asset share regression. This would be zero if risky asset holding households were a random subgroup of the population. Time dummies are included in explanatory variables to control time effect.

These empirical findings provide one explanation for the variations of the ratio of equity to financial wealth. In general, households who decide to purchase their own house usually enter a borrowing contract with debt repayment term and committed payments. As the simple consumption-portfolio choice model combined with debt repayment term decision suggests, the equity share in total financial wealth tends to be larger during the debt repayment term, due to depressed current financial wealth and hence a higher income-wealth ratio. Dampened financial wealth, on the other hand, deters households from holding equity. As they finish paying off all of their mortgage debt, their financial wealth will begin to increase at a higher pace while human wealth will peak out in the fifties and sixties age group, which induces a higher probability of equity holding but smaller equity share to financial wealth. These combined effects can induce the hump-shaped profile of the risky asset investment pattern observed in households.

On the other hand, the empirical findings in the variations of the ratio of equity to net worth are almost consistent with the implication for the classical Merton-Samuelson life-cycle portfolio theory. Based on their model with human wealth, the optimal equity share is expected to decrease with age. Such implication is supported by the empirical analysis for equity shares in net worth, which imply that any serious attempt to explain household dynamic portfolio choice should consider the variations of asset shares in net worth.

3.5 Concluding Remarks

In this chapter, we study debt-repayment and optimal consumption-investment decisions to analyze the heterogeneity in the age-related pattern of household portfolio choice. Building on Merton's (1971) consumption-portfolio problems and Munk and Sorensen's (2007) model under stochastic interest rate, we develop a framework to examine the portfolio choice problem with debt repayment schedule. We obtain an explicit link between the debt-repayment term and the consumption-investment policy for households that hold long-term mortgage debt.

The main analytical result of this chapter is that debt-repayment term decision can be determined independently of optimal consumption and investment policies. More precisely, the individual investor's value function can be expressed as the function of financial wealth, interest rate and debt-repayment term. Given individual's wealth and interest rate at an initial period, the individual can choose an optimal debt-repayment term to maximize her value function, which produces her optimal consumption and investment policy. The individual's debt repayment term affects optimal consumption-portfolio choice through both the future net human wealth and the current financial wealth accumulation. As a result, the individual's past debt repayment dampens her financial wealth accumulation, which results in a larger equity share in total financial wealth for the individual during debt repayment term. To assess the implication of the model for investment behavior, we use Japanese micro data on households to estimate the equity share regression. We find that accelerated debt-repayment has a positive effect on the equity share, conditional on asset holding. We also find that such dampened financial wealth deters ownership of risky assets. Therefore, these empirical findings support the prediction of the model and provide a qualitative explanation for the hump-shaped age-related pattern in equity investment, particularly observed among households with mortgage debt.

In the next chapter, we turn to the individual's optimal mortgage refinancing problem by introducing the regime switches in stochastic interest rate process.

A.1 Derivation of Eq. (3.20)

We rewrite (3.16) as

$$J(w,r,t) = \max_{\{c_s\},\{\Pi_s\}} E_{w,r,t}\left[\int_t^T e^{-\delta s}u(c_s)ds + e^{-\delta(T-t)}U(W_T)\right] \tag{3.43}$$

where $E_{w,r,t}[\cdot]$ denotes the conditional expectation given that $W_t = w$ and $r_t = r$. By using the principle of optimality, we can rewrite (3.43) as the discrete-time approximation

$$J(w,r,t) = \max_{c_t,\Pi_t}\{u(c_t)dt + e^{-\delta dt}E_{w,r,t}[J(W_{t+dt}, r_{t+dt}, t+dt)]\} \tag{3.44}$$

where c_t and Π_t is held fixed over the interval $[t, t+dt)$. Multiplying by $e^{\delta dt}$, subtracting $J(w,r,t)$, and dividing by dt, we obtain

$$\begin{aligned}\frac{e^{\delta dt}-1}{dt}J(w,r,t) &= \max_{c_t,\Pi_t}\Big\{e^{\delta dt}u(c_t) \\ &+ \frac{1}{dt}E_{w,r,t}[J(W_{t+dt}, r_{t+dt}, t+dt) - J(w,r,t)]\Big\}\end{aligned} \tag{3.45}$$

When we let $dt \to 0$, we have that

$$\frac{e^{\delta dt}-1}{dt} = \delta + o(dt) \to \delta$$

and that

$$\frac{1}{dt}E_{w,r,t}[J(W_{t+dt}, r_{t+dt}, t+dt) - J(w,r,t)]$$

will approach the drift of J at time t, which, from Ito's lemma, is given by

$$\begin{aligned} &J_t + J_w(rw + \alpha_t + \Pi_t^\top \Sigma\Lambda - c_t) \\ &\quad + \frac{1}{2}J_{ww}\Pi_t^\top \Sigma\Sigma^\top \Pi_t + J_r\kappa[\bar{r} - r] + \frac{1}{2}J_{rr}\sigma_r^2 + J_{wr}\Pi_t^\top \Sigma \boldsymbol{e}_1 \sigma_r. \end{aligned} \tag{3.46}$$

The limit of (3.45) is therefore

$$\begin{aligned} \delta J &= J_t + (rw + \alpha_t)J_w + \max_{c_t}\Big\{u(c_t, t) - J_w c_t\Big\} \\ &\quad + \max_{\Pi_t}\Big\{J_w\Pi_t^\top\Sigma\Lambda - J_{wr}\Pi_t^\top\Sigma\boldsymbol{e}_1\sigma_r + \frac{1}{2}J_{ww}\Pi_t^\top\Sigma\Sigma^\top\Pi_t\Big\} \\ &\quad + J_r\kappa[\bar{r} - r] + \frac{1}{2}J_{rr}\sigma_r^2 \end{aligned} \tag{3.47}$$

and we obtain (3.20).

A.2 Derivation of Value function (3.26)

The HJB equation associated with the problem can be rewritten as

$$0 = A_1(J) + A_2(J) + A_3(J) \tag{3.48}$$

where

$$\begin{aligned} A_1(J) &= \max_{c_t}\Big\{\frac{1}{1-\gamma}c_t^{1-\gamma} - J_w c_t\Big\}, \\ A_2(J) &= \max_{\Pi_t}\Big\{J_w\Pi_t^\top\Sigma\Lambda - J_{wr}\Pi_t^\top\Sigma\boldsymbol{e}_1\sigma_r + \frac{1}{2}J_{ww}\Pi_t^\top\Sigma\Sigma^\top\Pi_t\Big\}, \\ A_3(J) &= -\delta J + J_t + (rw + \alpha_t)J_w + J_r\kappa[\bar{r} - r] + \frac{1}{2}J_{rr}\sigma_r^2. \end{aligned}$$

Substituting (3.22) and (3.24) into (3.48), and using the following expressions for the derivatives of value function J in terms of J itself:

$$
\begin{aligned}
J_w &= \frac{(1-\gamma)J}{w+h}, \\
J_{ww} &= -\frac{\gamma(1-\gamma)J}{(w+h)^2}, \\
J_r &= (1-\gamma)J\Big[\frac{\gamma}{1-\gamma}\frac{\ell_r}{\ell} + \frac{h_r}{w+h}\Big], \\
J_{rr} &= (1-\gamma)J\Big[\frac{\gamma}{1-\gamma}\frac{\ell_{rr}}{\ell} - \gamma\Big(\frac{\ell_r}{\ell}\Big)^2 + 2\gamma\frac{\ell_r}{\ell}\frac{h_r}{w+h} - \gamma\Big(\frac{h_r}{w+h}\Big)^2 + \frac{h_{rr}}{w+h}\Big], \\
J_{wr} &= \gamma(1-\gamma)J\Big[\frac{\ell_r}{\ell}\frac{1}{w+h} - \frac{h_r}{(w+h)^2}\Big], \\
J_t &= (1-\gamma)J\Big[\frac{\gamma}{1-\gamma}\frac{\ell_t}{\ell} + \frac{h_t}{w+h}\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{J_w}{J_{ww}} &= -\frac{1}{\gamma}(w+h), \\
\frac{J_{wr}}{J_{ww}} &= h_r - \frac{\ell_r}{\ell}(w+h),
\end{aligned}
$$

we obtain,

$$
\begin{aligned}
A_1(J) &= \frac{\gamma}{1-\gamma}J_w^{\gamma-1/\gamma} = \frac{\gamma}{\ell}J, \\
A_2(J) &= (1-\gamma)J\left\{\frac{(\lambda_S^2+\lambda_r^2)}{2\gamma} + \sigma_r\lambda_r\left(\frac{h_r}{w+h} - \frac{\ell_r}{\ell} + \frac{\gamma}{2}\sigma_r^2\left(\frac{\ell_r}{\ell} - \frac{h_r}{w+h}\right)^2\right)\right\}, \\
A_3(J) &= J\left\{-\delta + \gamma\frac{\ell_t}{\ell} + \gamma\kappa[\bar{r}-r]\frac{\ell_r}{\ell} + \frac{\gamma}{2}\sigma_r^2\left[\frac{\ell_{rr}}{\ell} - (1-\gamma)\left(\frac{\ell_r}{\ell} - \frac{h_r}{w+h}\right)^2\right]\right\} \\
&\quad + \frac{(1-\gamma)J}{w+h}\left\{h_t + rw + \alpha_t + \kappa[\bar{r}-r]h_r + \frac{1}{2}\sigma_r^2 h_{rr}\right\} \\
&= J\left\{-\delta + \gamma\frac{\ell_t}{\ell} + \gamma\kappa[\bar{r}-r]\frac{\ell_r}{\ell} + \frac{\gamma}{2}\sigma_r^2\left[\frac{\ell_{rr}}{\ell} - (1-\gamma)\left(\frac{\ell_r}{\ell} - \frac{h_r}{w+h}\right)^2\right]\right\} \\
&\quad + \frac{(1-\gamma)J}{w+h}\left\{r(w+h) - \sigma_r\lambda_r h_r\right\},
\end{aligned}
$$

where we have used the partial differential equation satisfied by $h(r,t)$

$$h_t + \{\kappa[\bar{r} - r] + \sigma_r\lambda_r\}h_r + \frac{1}{2}\sigma_r^2 h_{rr} + \alpha_t = rh \tag{3.49}$$

which can be verified by direct substitution. Summing up, we get

$$\begin{aligned} & A_1(J) + A_2(J) + A_3(J) \\ &= \frac{\gamma}{\ell}J + (1-\gamma)J\left\{\frac{(\lambda_S^2 + \lambda_r^2)}{2\gamma} - \sigma_r\lambda_r\frac{\ell_r}{\ell} - \frac{\delta}{1-\gamma} + \frac{\gamma}{1-\gamma}\frac{\ell_t}{\ell}\right. \\ &\quad \left. + \frac{\gamma}{1-\gamma}\kappa[\bar{r} - r]\frac{\ell_r}{\ell} + \frac{\gamma}{2(1-\gamma)}\sigma_r^2\frac{\ell_{rr}}{\ell} + r\right\}, \end{aligned}$$

so that the full HJB equation (3.48) is expressed as

$$0 = \gamma J\frac{1}{\ell}\left\{1 + \frac{1}{2}\sigma_r^2\ell_{rr} + \left(\kappa[\bar{r} - r] + \frac{\gamma-1}{\gamma}\sigma_r\lambda_r\right)\ell_r + \ell_t + \frac{\gamma-1}{\gamma}\left(\frac{\delta}{1-\gamma} - r - \frac{\lambda_S^2 + \lambda_r^2}{2\gamma}\right)\ell\right\}. \tag{3.50}$$

The above PDE for $\ell(r,t)$ reduces to

$$\begin{aligned} 0 &= 1 - \left(\frac{\delta}{\gamma} - \frac{1-\gamma}{\gamma}r - \frac{1-\gamma}{2\gamma^2}(\lambda_S^2 + \lambda_r^2)\right)\ell(r,t) + \ell_t(r,t) \\ &\quad + \left(\kappa[\bar{r} - r] + \frac{\gamma-1}{\gamma}\sigma_r\lambda_r\right)\ell_r(r,t) + \frac{1}{2}\sigma_r^2\ell_{rr}(r,t) \end{aligned} \tag{3.51}$$

with terminal condition $\ell(r,T) = 1$. Conjecturing a solution of the form

$$\ell(r,t) = e^{-d_0(T-t) - d_1(T-t)r} + \int_t^T e^{-d_0(s-t) - d_1(s-t)r}ds \tag{3.52}$$

with $d_0(0) = d_1(0)$ to satisfy the terminal condition, the relevant derivatives of $\ell(r,t)$ are now

$$\ell_r(r,t) = \int_t^T -d_1(s-t)e^{g(r,s-t)}ds - d_1(T-t)e^{g(r,T-t)}, \tag{3.53}$$

$$\ell_{rr}(r,t) = \int_t^T -d_1(s-t)^2 e^{g(r,s-t)}ds - d_1(T-t)^2 e^{g(r,T-t)}, \tag{3.54}$$

$$\begin{aligned} \ell_t(r,t) &= \int_t^T \left\{d_0'(s-t) + d_1'(s-t)r\right\}e^{g(r,s-t)}ds - 1 \\ &\quad + \left\{d_0'(s-t) + d_1'(T-t)r\right\}e^{g(r,T-t)} \end{aligned} \tag{3.55}$$

where

$$g(r,u) = -d_0(u) - d_1(u)r.$$

Substituting these derivatives into (3.51), we can now obtain

$$d_1'(u) + \kappa d_1(u) = \frac{1-\gamma}{\gamma}, \tag{3.56}$$

$$d_0'(u) = -\frac{1}{2}\sigma_r^2 d_0(u)^2 + \Big(\kappa + \frac{\gamma-1}{\gamma}\sigma_r\lambda_r\Big)d_0(u) + \frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2}(\lambda_S^2+\lambda_r^2) \tag{3.57}$$

with $d_0(0) = d_1(0) = 0$. The solution for $d_0(u)$ and $d_1(u)$ are

$$d_1(u) = \frac{\gamma-1}{\gamma}\frac{1}{\kappa}(1-e^{-\kappa u}) = \frac{\gamma-1}{\gamma}b_\kappa(u) \tag{3.58}$$

and

$$\begin{aligned} d_0(u) &= \left(\frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2}(\lambda_S^2+\lambda_r^2)\right)u - \frac{1}{2}\sigma_r^2\left(\frac{\gamma-1}{\gamma}\right)^2\int_0^u b_\kappa(u)^2 du \\ &\quad + \left(\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_r\right)\frac{\gamma-1}{\gamma}\int_0^u b_\kappa(u)du \\ &= \left(\frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2}(\lambda_S^2+\lambda_r^2)\right)u + \left(\bar{r} + \frac{1-\gamma}{2\kappa^2\gamma}[\sigma_r^2 - 2\kappa\sigma_r\lambda_r]\right)\frac{\gamma-1}{\gamma}(u-b_\kappa(u)) \\ &\quad - \frac{1-\gamma}{4\kappa\gamma}\frac{\gamma-1}{\gamma}\sigma_r^2 b_\kappa(u)^2 \end{aligned} \tag{3.59}$$

where we have used that

$$\int_0^u b(s)ds = \frac{1}{\kappa}(u-b(u)), \tag{3.60}$$

$$\int_0^u b(s)^2 ds = \frac{1}{\kappa^2}(u-b(u)) - \frac{1}{2\kappa}b(u)^2. \tag{3.61}$$

Noting that, when $u = s-t$, the bond price is given by

$$\begin{aligned} B^s(r,t) &= e^{-a(s-t)-b(s-t)r} \\ &= \exp\left\{-b(u)r - \left[\Big(\bar{r} - \frac{\sigma_r}{\kappa}\lambda_r - \frac{\sigma_r^2}{2\kappa}\Big)(u-b(u)) + \frac{1}{4\kappa}\sigma_r^2 b(u)^2\right]\right\} \end{aligned}$$

and using (3.58) and (3.59), (3.52) can be written as

$$\begin{aligned} \ell(r,t) &= e^{-d_0(T-t)-d_1(T-t)r} + \int_t^T e^{-d_0(s-t)-d_1(s-t)r}ds \\ &= k(T-t)(B^T(r,t))^{\frac{\gamma-1}{\gamma}} + \int_t^T k(s-t)(B^s(r,t))^{\frac{\gamma-1}{\gamma}} \end{aligned} \tag{3.62}$$

with

$$
\begin{aligned}
k(u) \quad = \quad & \exp\Bigg\{\left(-\frac{\delta}{\gamma}+\frac{1-\gamma}{2\gamma^2}(\lambda_S^2+\lambda_r^2)\right)u \\
& + \frac{1-\gamma}{\gamma^2}\left((\bar{r}-\bar{R})(u-b(u))-\frac{\sigma_r^2}{4\kappa}b(u)^2\right)\Bigg\}. \qquad (3.63)
\end{aligned}
$$

Thus (3.48) with (3.26) satisfies the HJB equation (3.20), which verifies that (3.26) is the value function for (3.20).

Chapter 4

Optimal Mortgage Refinancing with Regime Switches

4.1 Introduction

The idea that the stochastic behavior of asset prices varies over time has attracted considerable attention among academic researchers and practitioners. After the seminal work by Hamilton (1989), the model with regime switches has been recognized as an attractive framework to analyze the asymmetric and cyclical behavior in asset returns as well as macroeconomic variables over the business cycle. Motivated by recent empirical studies on the asymmetric movements of asset prices, there have been growing literatures focusing on the role that regime switches play in the investment decision under uncertainty.

The rational models of mortgage refinancing, in contrast, generally presume that the parameters of interest rate process do not vary over time. The standard option-based models describe the behavior of refinancing as s dynamic optimal decision made by a borrower to minimize the present value of future interest payments. Analogous to the real option approach to investment under uncertainty, these models imply that the mortgage holder will delay her refinancing when the volatility of interest rates is high.

The empirical evidences on refinancing, however, have revealed that households occasionally not only delay, but also hasten their refinancing, even when it appears optimal to refinance under the standard models. For instance, Bennett, Peach, and Perostiani (1998) find that the propensity to refinance has increased significantly in the 1990s relative to the 1980s, despite the fact that the decline in mortgage rates in the 1990s was somewhat smaller than the decline that occurred in the 1980s. Their empirical finding suggests that the interest rate differential needed to induce a borrower to refinance has declined. Recently, Agarwal, Driscoll, and Laibson (2002) have noted that about one-third of the borrowers refinanced too early during the 1990s. The structural change in frequency of late and early refinancing observed in these empirical studies is a challenge to the traditional models of mortgage refinancing. To resolve these empirical puzzles, the basic option-based model has been modified to take into account the borrower-specific factors, such as heterogeneity in transaction costs, variations in housing price, consumption smoothing motive, and distracted consumer behavior.

This chapter proposes and solves a model of refinancing decision in which the drift and volatility of interest rate process shift between different regimes. Within this framework, we particularly pay attention to the effect that the regime switches have on the refinancing decision. This is interesting for several reasons. First, changes in business cycle conditions and monetary policy cause interest rates to behave quite differently in different states. Ang and Bekaert (2002) demonstrate that regime switching models forecast better out-sample movements of interest rates than single regime models. They also indicate that the regimes correspond reasonably well with the business cycles in the US. Introducing regime switches into refinancing decisions, therefore, adds an additional realistic factor to the traditional option-based refinancing models.

Second, under the regime-switching framework, the shifts in the drift and persistence parameters may have sizable effects on decisions under uncertainty. In contrast to the single-regime models focusing on the volatility of the underlying stochastic process, the model with regime switches allow us to explore the effect such parameters have on optimal policy. Consequently, combined with the effect of changes in volatility, the impact of regime switches in our model may resolve the empirical puzzles that the standard refinancing models cannot explain.

One of the puzzling features of the actual refinance activities over the past decades is the change in frequency of late and early refinancing: late refinancing was more likely observed in the 1980s, but early refinancing was relatively common in 1990s. Given the fact that the market interest rates fell rapidly in 1980s and, in contrast, fluctuated in a narrower band in 1990s, whether an option-based model with regime switches can predict both late and early refinancing is an interesting research issue.

To investigate how change in regime influences the refinancing decisions, we incorporate the regime switches in interest rates into an analytically tractable model developed by Agarwal, Driscoll, and Laibson (2002). Specifically, we assume that the market interest rates obey a Brownian motion with changes in the drift and volatility parameters. In contrast to the models focusing on the borrower-specific exogenous factors, we focus on financial factors influencing the exercising the refinancing option. From a financial viewpoint, the option is “in the money” when the borrowing rate exceeds the current market rate. Under the regime-switching framework, both future and current regimes in the interest rate process govern the refinancing decision. Changes in the interest rate differential between the borrowing rate and the current market rate motivate the borrower toward refinancing when the spread covers the loss in option value caused by refinancing, depending on the underlying regime. As a result, the optimal refinancing policy takes the form of a trigger policy that can be described by a first passage time of the interest rate differential to the different threshold for each regime.

An important question is how the regime switches in interest rates affect the optimal interest rate differential and thereby mortgage refinancing decision in each regime. To examine the impact of regime switches on optimal policy, we first numerically solve the optimal interest rate differential for different parameters such as the drift, the volatility, and the persistence in each regime. With the result based on the single-regime model as a benchmark, we next compare the threshold values of optimal interest differential derived from the two-regime model. Numerical simulations demonstrate that because of the possibility of a regime shift, the optimal refinancing threshold can be smaller or larger than the threshold under single-regime models. Finally, we calibrate the model to the optimal refinancing behavior in the US. With the estimated parameters for a simple two-regime model capturing the evolution of mortgage rates in the US, we show that the optimal refinancing thresholds are more than 200 basis points in 1980s while much less than 100 basis points in 1990s and 2000s, which result in both late and early refinancing.

This study relates to two strands in finance literature. First, it relates to the literature on the option-based approach in modeling refinancing behavior. After Dunn and McConnel (1981a, 1981b) first developed a continuous-time option-based prepayment model, there have been several approaches to attempt to explain the observed refinancing behavior. Archer and Ling (1993) and Stanton (1995) add heterogeneity in transaction costs to the standard model. Downig, Stanton and Wallace (2003) introduce the effect of variations of housing price into prepayment behavior. To explain early refinancing, Hurst and Stafford (2002) construct a model where households use their housing wealth to smooth their consumption. More recently, Agarwal, Driscoll, and Laibson (2004) emphasize the effect of a distracted consumer, who only reconsiders her refinancing decision from time to time. Introducing such infrequent behavior of households into their continuous-time analytical model, they demonstrate that the distracted refinancing decisions can induce both late and early refinancing. From an economic viewpoint, this paper extends their work in another direction, by taking into account the impact of the regime switches in interest rates on the refinancing decision. We attempt to provide a semi-analytical framework to analyze refinancing behaviors over the business cycle.

Second, this study stands on a series of recent papers on option-based models with regime shifts. Guo and Zhang (2004) study an optimal stopping time problem for pricing perpetual American put options in a regime- switching framework. Guo, Miao, and Morellec (2005) apply their closed form solution to analyze the investment decisions of the firm whose growth prospects shifts between different states. From a technical viewpoint, our approach is closely related to their work. We combine a regime-switching model with an analytical option-based refinancing model developed by Agarwal, Driscoll, and Laibson (2002). One of our contributions is the examination of an optimal refinancing decision under the regime-switching environment, providing an analytical framework to relate the cyclical movement of interest rates to the rational refinancing behavior.

The remainder of this chapter is organized as follows. Section 4.2 presents the model of rational refinancing decision with regime switches in interest rates. Section 4.3 describes the optimal refinancing policy. Section 4.4 compares the optimal policy to those under single-regime models. With an estimated two-regime model to capture the evolution of mortgage rates in the US, Section 4.5 explores the prediction of the calibrated model for the actual refinancing behavior. Section 4.6 concludes this chapter.

4.2 A Model of Mortgage Refinancing

We construct a model of mortgage refinancing that builds on the analytically tractable continuous-time model developed by Agarwal, Driscoll, and Laibson (2002), to allow for regime switches in interest rate process.

Let M denote the amount of debt to buy a mortgage which is issued at time 0. When the borrowing rate is r_t, the mortgage holder continuously pays at rate $r_t M$ per unit time until she refinances. In other words, it is an interest-only mortgage which is equivalent to an infinite-horizon mortgage. The mortgage holder can refinance at cost C at any time. When the mortgage holder refinances, the borrowing rate is changed to the market interest rate at that time. To be more specific, let $\{\mu_t\}$ denote the market interest rate process which stochastically fluctuates in time according to (4.1) below. If the mortgage holder refinances at $\tau_1, \tau_2, \ldots$, the borrowing rate $r_t = \mu_{\tau_i}$ for $t \in [\tau_i, \tau_{i+1})$. Thus, $\{r_t\}$ forms a piece-wise constant process. With a hazard rate η the mortgage holder sells her mortgage for exogenous reasons and repays the debt M. This means the payment will terminate at random time κ which follows exponential distribution with intensity η.

The market interest rate process $\{\mu_t\}$ is assumed to obey a Brownian motion with drift:

$$d\mu_t = \xi_t dt + v_t d\tilde{z}_t \tag{4.1}$$

where $\{\tilde{z}_t\}$ denotes a standard Brownian motion. To introduce regime switches into the interest rate process, $\{(\xi_t, v_t)\}$ is modeled by a continuous time Markov chain on the state space $\{(\alpha_1, \sigma_1), \ldots, (\alpha_K, \sigma_K)\}$. For this purpose, we define $\{I_t\}$ as a Markov chain on $\{1, 2, \ldots, K\}$ and let $\xi_t = \alpha_{I_t}$ and $v_t = \sigma_{I_t}$. We assume $\{\tilde{z}_t\}$ and $\{I_t\}$ (and hence $\{(\xi_t, v_t)\}$) are stochastically independent. The state of the model at t is thereby represented by a triplet (i, r, μ) if $I_t = i$, $r_t = r$ and $\mu_t = \mu$. Under the current setting, we can explicitly link optimal refinancing decisions with the changes in the state of the economy.

Throughout the paper, we assume that the mortgage holder is risk neutral. The presumptions that the mortgage is an infinite-horizon mortgage that pays only interests continuously and the borrower is risk neutral seem at first glance to be over simplistic. However, by adequately choosing η so that the expected time until future full repayment, $1/\eta$, is between twenty years ($\eta = .05$) and ten years ($\eta = .01$), the mortgage contract in our model can approximate reality. This range of values reflects the fact that most fixed-rate mortgages are commonly thirty years, and that personal exogenous reasons may cause the termination of the mortgage contract at an earlier date. The gradual repayment of principal suggests that the average duration of a 30-year fixed-rate mortgage is approximately twenty years. Agarwal, Driscoll, and Laibson (2007) extensively discuss and show that these simplifying assumptions do not make a significant difference in existing analyses and numerical results published by other researchers who do not make such simplifying assumptions.

The mortgage holder's objective is to minimize the expected net present value of future interest payments and associated refinancing costs discounted by her personal discount rate δ. If the mortgage holder does not have a refinance option, the present value of total future payments with the initial borrowing rate r is written by

$$rM \int_0^{\kappa} e^{-\delta t} dt + e^{-\delta \kappa} M. \tag{4.2}$$

Since κ is exponentially distributed with intensity η, the expectation of

(4.2) is

$$\int_0^\infty \eta e^{-\eta u}\left[rM\int_0^u e^{-\delta t}dt + e^{-\delta u}M\right]du = \frac{(r+\eta)M}{\delta+\eta}. \tag{4.3}$$

On the other hand, when the mortgage holder refinances at $\tau_1, \tau_2, \ldots$ according to certain refinancing policy, the present value of total payments starting with initial state (i, r, μ) becomes

$$\begin{aligned} U_i(r,\mu) &= rM\int_0^{\tau_1} e^{-\delta t}dt + \sum_{k=1}^{N-1}\mu_{\tau_k}M\int_{\tau_k}^{\tau_{k+1}} e^{-\delta t}dt \\ &\quad + \mu_{\tau_N}M\int_{\tau_N}^{\kappa} e^{-\delta t}dt + \sum_{k=1}^{N} e^{-\delta\tau_k}C + e^{-\delta\kappa}M \end{aligned} \tag{4.4}$$

where N is the number of refinances before κ. The first three terms in (4.4) represent instantaneous payments while the fourth and fifth terms respectively indicate refinancing costs and debt payment.

To minimize $\mathrm{E}\left(U_i(r,\mu)\right)$, the mortgage holder attempts to find a sequence $\boldsymbol{\tau} = \{\tau_1, \tau_2, \ldots\}$ of refinancing epochs where τ_i's are stopping times with respect to the filtration generated by $\{\mu_t\}$ and $\{I_t\}$. In the subsequent sections, we denote by $V_i(r,\mu) = \min_{\boldsymbol{\tau}} \mathrm{E}\left(U_i(r,\mu)\right)$ the expected present value of future interest payments under optimal refinancing policy. In the context of optimization, $V_i(r,\mu)$ is referred to as a value function of the problem.

4.3 Optimal Refinancing Policy

In this section, we state a couple of propositions that characterize the optimal refinancing policy and the resulting value functions. Agarwal, Driscoll, and Laibson (2002) have solved a special case of the borrower's problem without regime switch, i.e., $\alpha_i = 0$ and $\sigma_i = \sigma$ for $i = 1, \ldots, K$. They discovered two important findings that characterize optimal policy. These results can be extended even when we introduce regime switches into the drift and volatility of the market rate process as in Section 4.2. We state the main results in the form of the following propositions, proofs to which are given in Appendix B.1.

Proposition 1 For each regime i there exists a threshold value θ_i such that it is optimal to refinance (not to refinance, respectively) when $I_t = i$ and $x_t \le \theta_i$ ($x_t > \theta_i$) where $x_t = \mu_t - r_t$ denotes interest rate differential between market rate and borrowing rate.

Proposition 1 states that the optimal refinancing policy takes the form of a trigger policy for each regime. The trigger is of a threshold type in the sense that under the optimal policy the mortgage holder should start refinancing for the first time when x_t reaches to θ_i. An intuition behind this is that the amount of future payments reduced by refinance depends on r_t and μ_t only through differential x_t. It is worthwhile to note that from Proposition 1 the value function satisfies

$$V_i(r, r + x) = V_i(r + x, r + x) + C, \quad \forall x \leq \theta_i \tag{4.5}$$

for arbitrary r. To state the next result, we define

$$W_i(x) = V_i(0, x) - \frac{\eta M}{\delta + \eta}. \tag{4.6}$$

Proposition 2 V_i can be expressed as

$$V_i(r, r + x) = W_i(x) + \frac{(r + \eta)M}{\delta + \eta}, \quad \forall x. \tag{4.7}$$

Recall from (4.3) that $(r + \eta)M/(\delta + \eta)$ is the expected discounted total payments when the mortgage holder does not have a refinance option. Thus, $W_i(x)$ represents the value of option to refinance when the interest rate differential is x and the regime is i. It should be noted that, from (4.5) and Proposition 2, $W_i(x)$ satisfies

$$W_i(x) = W_i(0) + C + \frac{M}{\delta + \eta}x, \qquad \forall x \leq \theta_i. \tag{4.8}$$

From these theoretical results, the problem to determine the optimal policy and value functions is reduced to obtaining θ_i and $W_i(x)$. This can be achieved by solving simultaneous Bellman equations for $W_i(x)$ with appropriate boundary conditions. For simplicity of exposition, we consider the case when $K = 2$ and denote the transition rate matrix of $\{I_t\}$ by

$$\boldsymbol{Q} = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}.$$

See Section 2.3.1 for the details of the transition rate matrix $\boldsymbol{Q}$. A similar approach can be applied in principle to the case with $K \geq 3$, though the number of permutations of threshold values increases as K increases, which may cause difficulties in numerical computation.

In what follows, we assume without loss of generality that $\theta_a < \theta_b$ (either $a = 1, b = 2$ or $a = 2, b = 1$). It should be remarked that $\theta_a, \theta_b < 0$, otherwise it is worthless to refinance. Since it is not optimal to refinance in both regimes for $x > \theta_b$, $W_a(x)$ and $W_b(x)$ satisfy the following simultaneous HJB equations:

$$\begin{cases} (\delta+\eta+q_a)W_a(x) = \alpha_a W_a'(x) + \dfrac{\sigma_a^2}{2}W_a''(x) + q_a W_b(x), \\ (\delta+\eta+q_b)W_b(x) = \alpha_b W_b'(x) + \dfrac{\sigma_b^2}{2}W_b''(x) + q_b W_a(x), \end{cases} \quad x > \theta_b, \tag{4.9}$$

(See Appendix B.2). The characteristic equation associated with (4.9) is

$$g_a(x)g_b(x) = q_a q_b \tag{4.10}$$

where

$$g_i(x) = \frac{\sigma_i^2}{2}x^2 + \alpha_i x - (\delta+\eta+q_i), \quad i = a, b.$$

It is readily seen that (4.10) has four real roots two of which are negative, cf., Guo and Zhang (2004). If we denote the roots of (4.10) by $\beta_1, \beta_2 < 0$ and $\beta_3, \beta_4 > 0$, the general solution of (4.9) is given by

$$W_a(x) = \sum_{i=1}^{4} A_i e^{\beta_i x}, \quad x > \theta_b. \tag{4.11}$$

However, to satisfy the boundary conditions $\lim_{x\to\infty} W_a(x) = 0$, the positive exponents in (4.11) must vanish, i.e., $A_3 = A_4 = 0$. Substituting the resultant form of $W_a(x)$ into (4.9), we obtain

$$W_b(x) = \ell_{a,1}A_1 e^{\beta_1 x} + \ell_{a,2}A_2 e^{\beta_2 x}, \quad x > \theta_b \tag{4.12}$$

with $\ell_{i,j} = -q_i/g_i(\beta_j)$.

Next we consider the range $x < \theta_a$. In this range, both $W_a(x)$ and $W_b(x)$ satisfy (4.8). This together with (4.11) and (4.12) imply

$$\begin{aligned} W_a(x) &= A_1 + A_2 + \psi(x), \\ W_b(x) &= \ell_{a,1}A_1 + \ell_{a,2}A_2 + \psi(x), \end{aligned} \quad x < \theta_a \tag{4.13}$$

where

$$\psi(x) = \frac{M}{\delta+\eta}x + C.$$

Finally, for $\theta_a < x < \theta_b$, $W_b(x)$ is given by (4.13) and $W_a(x)$ satisfies the HJB equation

$$\begin{aligned} (\delta+\eta+q_a)W_a(x) &= \alpha_a W_a'(x) + \frac{\sigma_a^2}{2}W_a''(x) \\ &\quad + q_a\left[W_b(0) + C + \frac{M}{\delta+\eta}x\right], \quad \theta_a < x < \theta_b. \end{aligned} \tag{4.14}$$

Let γ_1 and γ_2 be the two roots of $g_a(x) = 0$. Then, the solution of (4.14) is given as

$$W_a(x) = B_1 e^{\gamma_1 x} + B_2 e^{\gamma_2 x} + \phi_b(x), \quad \theta_a < x < \theta_b$$

where

$$\eta_i(x) = \frac{q_i}{\delta + \eta + q_i}\left\{\psi(x) + \ell_{i,1}A_1 + \ell_{i,2}A_2 + \frac{\alpha_i M}{(\delta + \eta + q_i)(\delta + \eta)}\right\}, \quad i = a, b.$$

In summary, the forms of $W_a(x)$ and $W_b(x)$ are identified as follows.

$$W_a(x) = \begin{cases} A_1 e^{\beta_1 x} + A_2 e^{\beta_2 x}, & x > \theta_b \\ B_1 e^{\gamma_1 x} + B_2 e^{\gamma_2 x} + \phi_a(x), & \theta_a < x < \theta_b \\ \psi(x) + A_1 + A_2, & x < \theta_a \end{cases} \tag{4.15}$$

$$W_b(x) = \begin{cases} \ell_{a,1} A_1 e^{\beta_1 x} + \ell_{a,2} A_2 e^{\beta_2 x}, & x > \theta_b \\ \psi(x) + \ell_{a,1}A_1 + \ell_{a,2}A_2, & x < \theta_b. \end{cases} \tag{4.16}$$

The set of the value of options (4.15) and (4.16) contain four unknown coefficients and two unknown threshold values. As the boundary conditions to determine these parameters, we invoke the value matching and the smooth pasting conditions which are widely known as optimality condition, cf., Chang (2004). Specifically, $W_a(x)$ and $W_b(x)$ must satisfy the following six equations:

$$\lim_{x\uparrow\theta_b} W_a(x) = \lim_{x\downarrow\theta_b} W_a(x), \qquad \lim_{x\uparrow\theta_b} W_a'(x) = \lim_{x\downarrow\theta_b} W_a'(x), \tag{4.17}$$

$$\lim_{x\uparrow\theta_a} W_a(x) = \lim_{x\downarrow\theta_a} W_a(x), \qquad \lim_{x\uparrow\theta_a} W_a'(x) = \lim_{x\downarrow\theta_a} W_a'(x), \tag{4.18}$$

$$\lim_{x\uparrow\theta_b} W_b(x) = \lim_{x\downarrow\theta_b} W_b(x), \qquad \lim_{x\uparrow\theta_b} W_b'(x) = \lim_{x\downarrow\theta_b} W_b'(x). \tag{4.19}$$

Unfortunately, (4.17)–(4.19) are nonlinear simultaneous equations which cannot be solved explicitly. Instead, we develop an efficient numerical algorithm to solve them, the details of which are described in Appendix B.3.

4.4 Numerical Results and Discussion

In this section, we investigate the effect that the regime switches have on the optimal refinancing policy. An important question is how regime switches in interest rates actually affect the optimal interest rate differential and thereby mortgage refinancing decision in each regime. For this purpose, we numerically solve the optimal threshold of interest rate differential, using the value functions with associated boundary conditions described in the previous section.

Table 4.1: Optimal threshold ($-\theta$) in basis points under single-regime model

	σ		
	0.0060	0.0120	0.0240
$\alpha = -0.0100$	209	214	229
$\alpha = 0.0000$	81	111	153
$\alpha = 0.0100$	35	60	104

Note: Thresholds calculated for $\eta = 0.05$ and $\delta = 0.05$. α denotes the drift and σ denotes the volatility parameter.

4.4.1 Results for the Benchmark Model

Following Agarwal, Driscoll, and Laibson (2002), we first replicate the optimal refinancing threshold based on their single-regime model as a benchmark. On the choices of the parameter values, we follow the work in Agarwal, Driscoll, and Laibson (2002). Specifically, we assume the refinancing cost of mortgage size $M = 200,000$ is fixed as $C = 0.01M + 2,000 = 4,000$. We adopt a standard exponential discount rate with a discount rate of $\delta = 0.05$. We set the hazard rate to $\eta = 0.05$. The hazard rate $\eta = 0.05$ implies that the expected time until exogenous future full repayment of mortgage is twenty years. As the base case volatility parameter, we pick the same volatility parameter as they use, $\sigma = 0.012$, corresponding to the observed standard deviation of the first difference of the 30-year mortgage rate.

Table 4.1 represents the optimal refinancing thresholds for single-regime models with different combinations of the drift and volatility parameters. For comparative analysis, we use three different values for both volatility and drift parameters. Consistent with most option-based models, a higher volatility increases the optimal refinancing threshold. As each column in the middle row of Table 4.1 shows, the threshold under no drift case ($\alpha = 0.0000$) increases from 81 basis points to 153 basis points as the volatility increases. The threshold also increases as the drift shifts from positive to negative when we compare the results for $\alpha = -0.0100$ and those for $\alpha = 0.0100$. The drift parameter of $\alpha = 0.0100$ implies the annualized change of market interest rates is 100 basis points on average. Because the sign of the drift parameter affects the option value for borrowers to wait to refinance, the upper and lower rows and columns reveal that there is a much wider range across the refinancing thresholds. These results suggest that the future changes in the combination of the drift and volatility parameters in interest rate process may produce both late and early refinancing.

4.4.2 The Effect of Regime Switches

We next solve the optimal threshold of interest rate differential under the assumption that interest rate process switches between two regimes. To examine how regime switches in interest rate affect the optimal threshold, we use the same combinations of the volatility and drift parameters shown in Table 4.1. For simplicity, we fix the volatility in regime 2 as the base case parameter, $\sigma_2 = 0.012$. As the parameter of persistence in each regime, we use $q_1 = q_2 = 0.5$, which implies that each regime switches to the other in two years on average.

The upper rows and columns in Table 4.2 represent the optimal thresholds with volatility switches and no drift. Several interesting patterns are apparent. First, the optimal thresholds are characterized by the different interest differentials in each regime, depending on parameter values of volatility in each regime. For instance, the trigger threshold for the fixed volatility in regime 2 ($-\theta_2$ with $\sigma_2 = 0.012$) increases from 96 to 176 basis points, as the volatility in the other regime (σ_1) increases. Second, each row and column in the upper part of Table 4.2 also reveals that depending on parameter values, the two trigger thresholds even switch orders. More precisely, the threshold for a higher volatility in regime 1 ($-\theta_1$ with $\sigma_1 = 0.024$) is 128 basis points while the threshold for a lower volatility in regime 2 ($-\theta_2$ with $\sigma_2 = 0.012$) is 176 basis points. It is worthwhile to note that the threshold in regime 1 is much smaller than that in regime 2. This counterintuitive result arises because of the possibility of a regime shift.

The lower rows and columns in Table 4.2 report the results for both drift and volatility parameters switching between the two regimes. The gaps of the two thresholds between the positive and negative drift are smaller than those based on the single-regime model. For example, the two thresholds for a pair of same volatility ($-\theta_2$ and $-\theta_1$with $\sigma_1 = \sigma_2 = 0.012$) are 79 and 199 basis points while the two independent thresholds under the single-regime model with the same volatility ($\sigma = 0.012$), positive and negative drift parameters ($\alpha = 0.0100$, $\alpha = -0.0100$) are 60 and 214 basis points, respectively. This pattern also demonstrates that the future drift in the other regime affects the optimal trigger threshold in the current regime. Reflecting the possibility of a regime shift, the impact of the drift and volatility parameters on the value-maximizing refinancing thresholds in two-regime model is not as important as in traditional option-based models.

Table 4.2: Optimal threshold ($-\theta_i$) in basis points with regime switches

			σ_1		
			0.0060	0.0120	0.0240
$\alpha_1 =$	0.0000	$-\theta_1$	102	111	128
$\alpha_2 =$	0.0000	$-\theta_2$	96	111	176
$\alpha_1 =$	-0.0100	$-\theta_1$	153	199	217
$\alpha_2 =$	0.0100	$-\theta_2$	72	79	135

Note: Thresholds calculated for $\eta = 0.05$, $\delta = 0.05$. α_i denotes the drift parameter and σ_i denotes the volatility parameter in each regime i. The volatility in regime 2 is fixed as $\sigma_2 = 0.0120$. The parameter of persistence q_i is set as $q_1 = q_2 = 0.5$.

4.4.3 Implication for Late and Early Refinancing

To conclude the simulation section, we consider the implication for the empirical puzzles of late and early refinancing observed in the actual refinancing behavior. Table 4.3 provides the differentials between the thresholds with regime switch in Table 4.1 and those under the single-regime with the same drift and volatility parameters in Table 4.2. Negative differential of the thresholds in Table 4.3 means that the trigger thresholds with regime switches are smaller than those under the single-regime models. In other words, the negative differential implies the gain of the possibility of early refinancing. Similarly, positive differential suggests the tendency of late refinancing.

Table 4.3: Differentials of optimal thresholds between two-regime and single-regime model

			σ_1		
			0.0060	0.0120	0.0240
$\alpha =$	0.0000	$\Delta\theta_1$	21	0	-25
		$\Delta\theta_2$	-15	0	65
$\alpha_1 =$	-0.0100	$\Delta\theta_1$	-56	-15	-12
$\alpha_2 =$	0.0100	$\Delta\theta_2$	12	19	31

Note: The differentials calculated by subtracting the thresholds in the single regime model in Table 4.1 from the optimal thresholds based on the two-regime model in Table 4.2, for the same drift, volatility, and persistence parameter. The volatility in regime 2 is fixed as $\sigma_2 = 0.0120$.

As the upper rows and columns in Table 4.3 show, the early refinancing arises when the volatilities in the other regime are lower while the late refinancing arises when the volatilities in the other regime are higher. Each row and column in the lower of Table 4.3 reports the results for both the drift and volatility switches. As the negative differentials in Table 4.3 show, the early refinancing tends to arise when the drift will switch from negative in that regime to positive in the other regime. The late refinancing, on the other hand, tends to arise when the drift will switch from positive to negative.

Lastly, an additional result is worth being mentioned. We consider the effects of the parameter of persistence, q_i in each regime on the difference of optimal thresholds between a two-regime model and a single-regime model. Table 4.4 reports the same comparison with the same drift and volatility parameters in Table 4.3 but the persistence parameter, $q_i = 0.5$ changed to alternatives. We use three sets of different parameter as the alternatives, $q_i = 2.0, 1.0$, and 0.25, each of which implies that the current regime persists in 0.5 years, 1 year and 4 years, respectively. As expected, the tendency toward both early and late refinancing is enhanced as the persistence in that regime decreases. The size of negative or positive differentials of the optimal thresholds between the two-regime and the single-regime increases with the gain in the frequency of regime switch from $q_i = 0.25$ to $q_i = 2.0$.

In sum, the regime switches in interest rates play an indisputable role in the optimal refinancing policy. Because of the possibility of a regime shift, the optimal refinancing threshold can be larger or smaller than the threshold based on the standard option-based model, depending on the drift, volatility and persistence parameters. These numerical simulations lead us to reconsider both late and early refinancing behavior within the basic rational refinancing framework.

Table 4.4: Differentials of optimal thresholds between two-regime and single regime model with changes in the persistence parameters (q_i)

			σ_1		
			0.0060	0.0120	0.0240
$q_1 = q_2 = 2.0$	$\alpha_1 = -0.0100$	$\Delta\theta_1$	−72	−22	−17
	$\alpha_2 = 0.0100$	$\Delta\theta_2$	21	31	56
$q_1 = q_2 = 1.0$	$\alpha_1 = -0.0100$	$\Delta\theta_1$	−68	−19	−15
	$\alpha_2 = 0.0100$	$\Delta\theta_2$	16	25	49
$q_1 = q_2 = 0.25$	$\alpha_1 = -0.0100$	$\Delta\theta_1$	−42	−7	−4
	$\alpha_2 = 0.0100$	$\Delta\theta_2$	8	14	25

Note: Differentials are calculated by subtracting the thresholds for the single regime-model in Table 4.1 from the optimal thresholds based on the two-regime model with different persistence parameters. The volatility in regime 2 is fixed as $\sigma_2 = 0.0120$.

4.5 Calibration

Our main interest is whether the option based refinancing model with regime switches can resolve both late and early refinancing puzzles. To quantitatively evaluate the predictions of our model, we proceed in two steps. First, we estimate a two-regime model for the actual evolution of the mortgage rate in the US. Second, we calibrate the model with the estimated drift, volatility, and persistence parameters in each regime, in order to compare the optimal refinancing thresholds across the sub samples from January 1984 to December 2006. This numerical analysis allows us to assess the predictions for late and early refinancing, both of which have been observed across the decades.

The data set for estimation is taken from Freddie Mac's primary mortgage market survey (PMMS). The weekly mortgage rates in that survey are the average of 125 lender's rates, who contribute rates to Freddie Mac. These rates are based on 30-year fixed mortgage rates with 20 percent down payment and 80 percent financed over the life of the loan. We divide the whole observations into four sub samples, two of which correspond to the early 1980s and 1990s, and exactly the same sample period during which Bennett, Peach, and Perostiani (1998) compare the propensity to refinance in their empirical study.

Table 4.5 represents the estimated results of the mortgage rates process, based on a simplifying assumption that changes in weekly mortgage rates are driven by a Brownian motion with regime switches in the drift and volatility parameters. All parameters are estimated by likelihood maximization, and standard errors are retrieved by inverting the Hessian matrix. The result of the simple two-regime models reveals several interesting characteristics in the evolution of weekly mortgage rates in the US. First, the size of both positive and negative drifts declines during the sample periods. Annualized weekly drifts in the 1980's are greater than 250 basis points in absolute value while they stand at almost 100 basis points in the late 1990s and less than 100 basis points in the 2000s. Second, the volatility parameters in the negative drift regime (σ_1) are commonly smaller than those in the positive drift regime (σ_2). Third, the persistence parameter varies across the sample. Most of the estimated values q_i are greater than the value of 2.0, implying that the regimes tend to switch within half a year. The lower rows and columns in Table 4.5 also report the sample average of the drift and volatility. In contrast to the results based on the two-regime model, sample volatilities are almost stable at around 70 basis points. Throughout the whole samples, the sample averages of the drift appear to be negative.

Next we evaluate the predictions of the refinancing model with regime switches. The upper lows and columns in Table 4.6 compare the optimal

Table 4.5: Maximum likelihood estimation results of mortgage rates process with two regimes

Variable	1984–2006	1984–1990	1991–1994	1995–1999	2000–2006
α_1	−0.0121	−0.0287	−0.0188	−0.0101	−0.0036
	(−0.0025)	(−0.0046)	(−0.0045)	(−0.0067)	(−0.0030)
α_2	0.0092	0.0375	0.0147	0.0100	0.0024
	(0.0034)	(0.0077)	(0.0060)	(0.0081)	(0.0040)
σ_1	0.0046	0.0063	0.0045	0.0048	0.0053
	(0.0003)	(0.0003)	(0.0004)	(0.0006)	(0.0002)
σ_2	0.0089	0.0067	0.0079	0.0095	0.0072
	(0.0004)	(0.0005)	(0.0006)	(0.0008)	(0.0005)
q_1	4.51	4.53	2.91	3.62	0.98
	(1.34)	(1.53)	(1.67)	(2.00)	(0.70)
q_2	4.99	8.26	2.18	3.95	0.88
	(1.39)	(2.56)	(1.42)	(2.21)	(0.79)
Log likelihood	1131.54	312.79	205.89	232.10	400.21
sample					
mean α	−0.0044	−0.0089	−0.0067	−0.0032	−0.0035
volatility σ	0.0071	0.0071	0.0072	0.0075	0.0061
Observation	1200	365	209	261	365

Note: Maximum likelihood estimates with standard errors in parentheses for the two-regime model with the standard diffusion specification are presented. α denotes the sample mean of drift and σ denotes the standard deviation of mortgage rate movements.

Table 4.6: Optimal threshold ($-\theta_i$) for the estimated parameters

Threshold		1984–2006	1984–90	91–94	95–99	2000–06
Two-regime	$-\theta_1$	109	376	140	76	117
	$-\theta_2$	107	43	139	67	78
One-regime ($\alpha = 0$)	$-\theta$	98	98	99	101	92
One-regime ($\alpha \neq 0$)	$-\theta$	165	223	197	149	150

Note: Thresholds are calculated using the estimated parameters in Table 4.5. The results in lower row and column are based on the single regime model without drift ($\alpha = 0$) and with the sample mean of drift α in the lower row and column in Table 4.5.

refinancing thresholds under the two-regime model with the calibrated parameters capturing regime switches. The lower rows and columns in Table 4.6 report the threshold under the single-regime model with the sample average of drift and volatility. Several interesting predictions are apparent. First, the optimal thresholds vary and decline over the sub sample periods. The threshold in the negative drift regime in the 1980s is over 300 basis points, which is two times greater than that in the early 1990s and three times greater than those in both the late 1990s and 2000s. This result predicts that late refinancing was common in the 1980s. Second, in contrast, the thresholds under the single-regime without drift are about 100 basis points and stable across the decades. Third, as a result, when comparing the optimal thresholds to those under the single-regime model, the differences suggest early refinancing tends to arise in the late 1990s. The optimal thresholds under the two-regime model range from 67 to 76 basis points, both of which are much smaller than the threshold of 101 basis points under the single-regime without drift. It is worthwhile to note that the optimal refinancing thresholds in the 1990s and 2000s are also smaller than those under the single-regime with the negative sample average as the drift parameters. These predictions are almost consistent with the empirical findings: late refinancing is common in the 1980s, early refinancing arises in the 1990s and 2000s, and the changes in frequency of late and early refinancing are observed.

4.6 Concluding Remarks

In this chapter, we investigate the effect that the regime switches in interest rate process have on refinancing decision under uncertainty. We extend a model of mortgage refinancing developed by Agarwal, Driscoll, and Laibson (2002) by allowing the drift and volatility of interest rate process to switch between regimes. The main analytical result is that because of the possibility of a regime shift, the optimal refinancing policy takes the form of the different trigger threshold of interest differential for each regime. Numerical simulations demonstrate that the optimal refinancing thresholds can be smaller or larger than the threshold under the single-regime model, depending on parameter values. Finally, we evaluate the predictions of the model, based on the estimated parameters for a two-regime model to capture the evolution of the mortgage rates in the US. Our model explain the late refinancing in the 1980s as well as the tendency toward early refinancing in recent periods, both of which have been documented empirically. Therefore, the regime switch in interest rates is one of the likely contributors to both late and early mortgage refinancing observed in the actual behavior.

B.1 Proofs of Propositions in Chapter 4

We will first prove Proposition 2 and then prove Proposition 1. In what follows, we may represent the market rate process as

$$\mu_t^{(a)} = a + \int_0^t \alpha_{I_s} ds + \int_0^t \sigma_{I_s} dz_s$$

to explicitly indicate initial rate $\mu_0 = a$. It is noted that, if $\{\mu_t^{(a)}\}$ and $\{\mu_t^{(b)}\}$ are constructed by using the same sample paths of $\{z_t\}$ and $\{I_t\}$,

$$\mu_t^{(a)} - \mu_t^{(b)} = a - b, \qquad \forall t \geq 0. \tag{4.20}$$

Proof of Proposition 2. We will prove

$$V_i(r + x, \mu + x) = V_i(r, \mu) + \frac{xM}{\delta + \eta}, \quad \forall x. \tag{4.21}$$

The desired result is readily obtained by substituting $r = 0, x = r$ and $\mu = x$ into (4.21).

Let $U_i(r, \mu)$ be given by (4.4) and let $U_i(r+x, \mu+x)$ denote the discounted total payments for the initial state $(i, r+x, \mu+x)$. The same τ_i, κ, $\{\mu_t\}$ and $\{I_t\}$ are used to define both $U_i(r, \mu)$ and $U_i(r + x, \mu + x)$. From (4.20), the borrowing rate of $U_i(r + x, \mu + x)$ is larger than $U_i(r, \mu)$ by x at any time. Thus, we obtain

$$U_i(r + x, \mu + x) = U_i(r, \mu) + xM \int_0^\kappa e^{-\delta t} dt. \tag{4.22}$$

Taking expectation of both sides of (4.22) yields

$$\mathrm{E}\left(U_i(r+x,\mu+x)\right) = \mathrm{E}\left(U_i(r,\mu)\right) + \mathrm{E}\left(xM\int_0^{\kappa} e^{-\delta t}dt\right) = \mathrm{E}\left(U_i(r,\mu)\right) + \frac{xM}{\delta+\eta}. \tag{4.23}$$

If τ_i's are optimal refinancing epochs of $U_i(r,\mu)$, we obtain from (4.23) that

$$\mathrm{E}\left(U_i(r+x,\mu+x)\right) = V_i(r,\mu) + \frac{xM}{\delta+\eta}$$

which in turn implies

$$V_i(r+x,\mu+x) \leq V_i(r,\mu) + \frac{xM}{\delta+\eta}.$$

Since reversed inequality $V_i(r+x,\mu+x) \geq V_i(r,\mu) + \frac{xM}{\delta+\eta}$ holds if τ_i's are optimal for $U_i(r+x,\mu+x)$, (4.21) has been proved. □

Proof of Proposition 1. For $x > y$, suppose to the contrary that an immediate refinance is optimal for the initial state $(i,r,r+x)$ while it is not optimal for $(i,r,r+y)$. To be more specific,

$$V_i(r,r+x) = V_i(r+x,r+x) + C, \tag{4.24}$$
$$V_i(r,r+y) < V_i(r+y,r+y) + C. \tag{4.25}$$

Let τ be the first refinancing epoch under optimal policy for the initial state $(i,r,r+y)$. From (4.7) and (4.25), optimality of τ implies

$$V_i(r,r+y) = \mathrm{E}_{(i,r,r+y)}\left(A + 1_{\{\tau<\kappa\}}e^{-\delta\tau}\{V_{I_\tau}(\mu_\tau^{(r+y)},\mu_\tau^{(r+y)}) + C\}\right) \tag{4.26}$$
$$< W_i(0) + \frac{(r+y+\eta)M}{\delta+\eta} + C \tag{4.27}$$

where $\mathrm{E}_{(i,r,r+y)}(\cdot)$ denotes expectation conditional on the initial state $(i,r,r+y)$,

$$A = rM\int_0^{\min(\tau,\kappa)} e^{-\delta t}dt + 1_{\{\tau\geq\kappa\}}e^{-\delta\kappa}M$$

and $1_B = 1$ if B is true and 0 otherwise. For the same τ, (4.24) implies that

$$V_i(r,r+x) = W_i(0) + \frac{(r+x+\eta)M}{\delta+\eta} + C \tag{4.28}$$
$$< \mathrm{E}_{(i,r,r+x)}\left(A + 1_{\{\tau<\kappa\}}e^{-\delta\tau}\{V_{I_\tau}(\mu_\tau^{(r+x)},\mu_\tau^{(r+x)}) + C\}\right) \tag{4.29}$$

holds for the initial state $(i, r, r + x)$. From (4.26)–(4.29), we obtain

$$(4.29) - (4.26) > (4.28) - (4.27) = \frac{(x-y)M}{\delta + \eta}. \tag{4.30}$$

However, since (4.26) and (4.29) share the same τ, κ and A which are independent of x, and $V_i(r, r) = W_i(0) + \frac{(r+\eta)M}{\delta+\eta}$ from (4.7), (4.20) implies

$$\begin{aligned} (4.29) - (4.26) &= \mathrm{E}_{(i,r,r+y)}\left(1_{\{\tau<\kappa\}} e^{-\delta\tau} \frac{(\mu_\tau^{(r+x)} - \mu_\tau^{(r+y)})M}{\delta + \eta}\right) \\ &= \mathrm{E}_{(i,r,r+y)}\left(1_{\{\tau<\kappa\}} e^{-\delta\tau} \frac{(x-y)M}{\delta+\eta}\right) < \frac{(x-y)M}{\delta+\eta}. \end{aligned} \tag{4.31}$$

Since (4.31) contradicts (4.30), the proof has been completed. □

B.2 Derivation of Eq. (4.9)

We will derive the simultaneous Bellman equations (4.9). Suppose that $I_t = a$ and $x > \theta_b$ at time t. Since it is not optimal to refinance in both regions for $x > \theta_b$, the state is in the continuation region where the borrower will wait to refinance her mortgage. Partitioning by the events in $(t, t+dt)$, we obtain

$$W_a(x) = e^{-\delta dt}\Big\{(1-(\eta+q_a)dt)E[W_a(x+d\mu_t)|\mu_t = r_t+x, I_t = a]+q_b dt W_b(x)\Big\}. \tag{4.32}$$

Note that the probability of concurrent transition of μ_t and I_t is $o(dt)$. From Ito's lemma and (4.1), we have

$$W_a(x + d\mu_t) = W_a(x) + \alpha_a W_a^{'}(x)dt + \sigma_a W_a^{'}(x)d\tilde{z}_t + \frac{\sigma_a^2}{2} W_a^{''}(x)dt \tag{4.33}$$

Taking expectation of (4.33) and substituting $e^{-\delta dt} = 1 - \delta dt + o(dt)$ into (4.33), we obtain the following Bellman equation of $W_a(x)$ after dividing by dt,

$$(\delta + \eta + q_a)W_a(x) = \alpha_a W_a^{'}(x) + \frac{\sigma_a^2}{2} W_a^{''}(x) + q_b W_b(x), \quad x > \theta_b. \tag{4.34}$$

In the same way, $W_b(x)$ satisfies

$$(\delta + \eta + q_b)W_b(x) = \alpha_b W_b^{'}(x) + \frac{\sigma_b^2}{2} W_b^{''}(x) + q_a W_a(x), \quad x > \theta_b. \tag{4.35}$$

□

B.3 Numerical Algorithm for Computing Value Functions

We describe a numerical algorithm for computing unknown coefficients of $W_i(x)$ and the threshold θ_i. We will use the same notations as in Section 4.3.

By plugging (4.15) and (4.16) into (4.17)–(4.19), these boundary conditions can be rewritten in a vector form as

$$\boldsymbol{A}\boldsymbol{E}_{\beta_1,\beta_2}(\theta_b) = \boldsymbol{B}\boldsymbol{E}_{\gamma_1,\gamma_2}(\theta_b) + \boldsymbol{\phi}_a(\theta_b), \tag{4.36}$$

$$\boldsymbol{B}\boldsymbol{E}_{\gamma_1,\gamma_2}(\theta_a) + \boldsymbol{\phi}_a(\theta_a) = \boldsymbol{\psi}(\theta_a) + \boldsymbol{A}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.37}$$

$$\boldsymbol{A}\boldsymbol{L}_a\boldsymbol{E}_{\beta_1,\beta_2}(\theta_b) = \boldsymbol{\psi}(\theta_b) + \boldsymbol{A}\widetilde{\boldsymbol{L}}_a \tag{4.38}$$

where

$$\boldsymbol{A} = (A_1, A_2), \qquad \boldsymbol{B} = (B_1, B_2), \qquad \boldsymbol{L}_i = \begin{pmatrix} \ell_{i,1} & 0 \\ 0 & \ell_{i,2} \end{pmatrix}$$

$$\boldsymbol{E}_{c_1,c_2}(x) = \begin{pmatrix} e^{c_1 x} & c_1 e^{c_1 x} \\ e^{c_2 x} & c_2 e^{c_2 x} \end{pmatrix}, \qquad \widetilde{\boldsymbol{L}}_i = \begin{pmatrix} \ell_{i,1} & 0 \\ \ell_{i,2} & 0 \end{pmatrix}$$

$$\boldsymbol{\phi}_i(x) = \left(\phi_i(x), \frac{q_i M}{(\delta+\eta+q_i)(\delta+\eta)}\right), \qquad \boldsymbol{\psi}(x) = \left(\psi(x), \frac{M}{\delta+\eta}\right).$$

To solve nonlinear equations (4.36)–(4.38), we propose the following numerical procedure which reduces the original problem to find six unknowns to a simpler problem with single variable.

1. Set a trial value of θ_b.
2. Compute $\boldsymbol{A}$ from (4.38) by $\boldsymbol{A} = \boldsymbol{\psi}(\theta_b)\left[\boldsymbol{L}_a\boldsymbol{E}_{\beta_1,\beta_2}(\theta_b) - \widetilde{\boldsymbol{L}}_a\right]^{-1}$.
3. Compute $\boldsymbol{B}$ from (4.36) by $\boldsymbol{B} = [\boldsymbol{A}\boldsymbol{E}_{\beta_1,\beta_2}(\theta_b) - \boldsymbol{\phi}_a(\theta_b)]\boldsymbol{E}^{-1}_{\gamma_1,\gamma_2}(\theta_b)$.
4. Compute θ_a from one of the equations in (4.37).
5. Check if θ_a obtained in Step 4 satisfies another equation in (4.37). If the equation is not satisfied within a prescribed error, goto Step 1 and revise the trial value of θ_b.

It should be noted that the above procedure converges only when we choose the correct order of thresholds, i.e., $\theta_1 < \theta_2$ or $\theta_1 > \theta_2$. If the procedure does not converge, we will again implement it for different order of threshold values. Judging from the numerical results we exhibit in Sections 4.4 and 4.5, the procedure performs quite well. It is numerically stable and converges quickly.

Chapter 5

Optimal Investment with Regime Switches and Higher Borrowing Rate

5.1 Introduction

Conventional wisdom suggests that people care about not only consumption but also their wealth status. Individuals then accumulate wealth both for future consumption (the consumption motive) and for their social status (the wealth accumulation motive). The most straightforward way to include this notion into the standard consumption and portfolio choice problem is to modify the utility function to have direct preference for wealth. Obviously, the wealth accumulation motive changes the individual's saving behavior, which has important implications both for asset demand, asset pricing, and economic growth. After Bakshi and Chen (1996) apply this notion to explain the equity premium puzzle, there has been growing attention to researches in economics and finance, such as growth theory [e.g. Futagami and Shibata (1998), Clemens (2003)], equity premium puzzle [e.g. Bakshi and Chen (1996), Smith (2001)], and household saving behavior [e.g. Zhou (1995)]. Natural implications studied by these researches are that the model with direct preference can induce excess accumulation on capital, a higher equity premium, and a higher saving rate, all of which have advantages to explain empirical observations in the economy and financial markets.

In this chapter, we extend the consumption and investment choice problem for an individual investor with preference for wealth among two dimensions. First, the borrowing rate is allowed to be higher than the risk-free rate. Second, rather than being constant, the parameters of return processes are allowed to vary over time. By introducing regime switches into the return processes, the dynamic property of the decision variables is subject to discrete regime shifts at random times. Building on the work of Xu and Chen (1998a,1998b) and of Bakshi and Chen (1996), we develop an analytically tractable model which encompasses the model with the standard CRRA preference, the model with a borrowing rate equal to the risk-free rate, and the model with a constant regime. We obtain a semi-analytical form and characterize the optimal policy for the consumption and investment decision with regime switches. The semi-analytical form is simple and easily solved for optimal policies numerically and yields the closed form solution for a constant regime as the special case of our general model.

The closed form solutions allow us to do explicit comparative statics, and clarify precise properties for the optimal consumption and investment policies. In particular, we pay attention to the effect that the regime switches has on the optimal investment policy under a preference for wealth and a higher borrowing rate. To investigate these effects, we numerically solve the optimal policies, by changing the associated parameters. The main analytical result shows that the optimal investment policy depends only on the current regime, while optimal consumption depends on both the future and the current regime. The combination of these effect results in a time-varying risky investment share and a relatively stable consumption-wealth ratio, both of which have been observed in aggregate time series data over the business cycle.

Introducing a higher borrowing rate than the deposit rate into the standard model is a natural extension for an investor's problem. As Davis, Kubler, and Willen (2005) pointed out, household borrowing costs on unsecured loans exceed the risk-free return by about six to nine percentage points on an annual basis. This motivates our study on the implication of higher borrowing rate for individual's investment behavior under an analytically tractable regime switching framework. Allowing the states to vary across regimes has the advantage of capturing stochastic investment opportunities observed in the asset returns. Building on the model with external social wealth proposed by Bakshi and Chen (1996), we also incorporate regime switches into an external social wealth process. External social wealth can be modeled to reflect the average wealth level of a specific socioeconomic group to which the investors belong. Regime switches in the external social wealth can be interpreted to represent changes in the state of the macroeconomy or the changes in the standards of living of "the Joneses ".

This chapter stands on a substantial number of literatures on consumption and portfolio choice problems first developed by Merton (1969). The analytical studies focusing on the wedge between the borrowing rate and risk-free rate are conducted by Fleming and Zariphopoulou (1991) and Xu and Chen (1998). They derived a closed form solution for the consumption and portfolio choice problem with a higher borrowing rate under the standard CRRA utility. The studies focusing on modifying the investor's utility are growing. Sundaresan (1989) and Constantinides (1990) introduce utility with habit formation where an individual's utility depends on the level of excess consumption to the habit or past standard of living level. Abel (1990) proposes the model with external habit where he defines the period utility as a function of the ratio of one's own consumption to aggregate consumption. Bakshi and Chen (1996) modify the preference for wealth to contain not only one's wealth but also the external average wealth level.

The motivation of these studies is mainly to resolve the equity premium puzzle. From a technical view point, the advantages of the model with direct preference for wealth, more specifically, with preference for the "ratio" of investor's wealth is that it admits an analytical solution with a higher borrowing rate. From an economic view point, the other advantage is that the wealth accumulation motive helps to understand the saving behavior. Zou(1995) argues that the accumulation motive explains "the saving puzzle", namely, why wealth increases with age and why individuals do not reduce their wealth after retirement. Our work is most closely related to the above work of Xu and Chen (1998) and of Bakshi and Chen (1996). We extend these studies to allow the investment opportunities to vary across regimes. As a result, each model can be independently nested in our more general model as a special case with changing model parameters.

Finally, this work is related to a number of studies on investment under uncertainty with regime switches. The most attractive characteristic in the regime switching framework is that the model with regime shifts can capture a number of stylized features of asset returns documented by empirical finance literature: stock and bond returns are time-varying and partly predictable [Campbell (1987); Fama and French (1988,1989)], their volatility change over time [Bollerslev, Chou, and Kroner (1992)], and correlations behave quite differently during a bear market [Ang and Chen (2002)]. Our model setup where drift and volatility are governed by the Markov regime switching is applied to several works. Notable analytical studies include option pricing [Guo and Zhang (2004)], firm's investment decision [Guo, Miao, and Morellec (2005); Makimoto (2008)], and mortgage refinancing [Kimura and Makimoto(2008)].

The structure of this chapter is as follows. Section 5.2 contains the model setup. In Section 5.3 we solve the model and describe the analytical results with and without regime switches. Section 5.4 discusses the numerical simulation of the model. Section 5.5 concludes.

5.2 The Model

We consider an investor who maximizes the expected value of her utility, where in each period the utility function is of the constant relative risk aversion (CRRA) form with the preference for her wealth ratio to an external social wealth. The investor faces stochastic investment opportunities with a higher borrowing rate than the deposit rate, where the drift and the volatility of asset returns shift between different states at random times.

5.2.1 Preference

The investor maximizes

$$E\left[\int_t^\infty e^{-\delta s} u(c_s, w_s, v_s) ds\right] \tag{5.1}$$

where $u(c_s, w_s, v_s)$ is a utility function over the investor's consumption c_s, her wealth w_s, and an external reference wealth v_s. We assume the utility function has a general form,

$$u(c, w, v) = \frac{c^{1-\gamma}}{1-\gamma}\left(\frac{w}{v}\right)^{-\epsilon} \tag{5.2}$$

where $\gamma > 0$, and $\epsilon \geq 0$ when $\gamma \geq 1$ and $\epsilon < 0$ otherwise. It encompasses a popular CRRA utility function when $\epsilon = 0$, and that with the individual investor's utility to absolute wealth when v is a constant.

The external wealth v can be interpreted to represent the wealth of the economy, which contains both tangible and intangible wealth such as real estate, or a social wealth standard for the investor's group. Then, the ratio of the investor's wealth to the social wealth w_t/v_t determines her status in the specific group. Specifically, an investor is said to be in the middle class if $w_t/v_t = 1$, in the lower class if $w_t/v_t < 1$ and the upper class otherwise. The utility structure of the model thereby shares the same spirit with Abel's (1990) "catching up with the Jones" model, where the investor's preference for consumption is not absolute consumption but the ratio of her consumption to the external consumption standards (i.e. the consumption of "the Jones").

5.2.2 Return Processes

The investor's wealth is held in two assets. One asset is a money market account whose price satisfies

$$\frac{dX_t}{X_t} = \begin{cases} r(t)dt, & X_t \geq 0 \\ R(t)dt, & X_t < 0 \end{cases} \tag{5.3}$$

where $r(t)$ is the risk-free interest rate (i.e. the deposit rate of a money market account) and $R(t)$ is the borrowing rate. The other asset is a nondividend-paying stock with price S_t that follows a stochastic differential equation

$$\frac{dS_t}{S_t} = \mu_S(t)dt + \sigma_S(t)dz_t \tag{5.4}$$

where we assume $\mu_S(t) > R(t) > r(t)$ and $\sigma_S(t) > 0$. The parameter $\sigma_S(t)$ represents the volatility of the stock. The external wealth index V_t follows

$$\frac{dV_t}{V_t} = \mu_V(t)dt + \rho_{SV}\sigma_V(t)dz_t + \sqrt{1-\rho_{SV}^2}\sigma_V(t)\hat{z}_t \tag{5.5}$$

where $\{\hat{z}_t\}$ is a standard Brownian motion independent of $\{z_t\}$. The correlation between stock market returns and social wealth returns is $\rho_{SV}dt$, where $-1 \leq \rho_{SV} \leq 1$, and the parameter $\sigma_V(t)$ is the volatility of the social wealth index.

5.2.3 Regime Switches

We assume that the dynamics of the drift and volatility parameters follow a two-state, continuous time Markov chain. The two states might capture different states of the economy where the low (high, respectively) mean returns with high (low) volatility in contraction (expansion) over the business cycles.

Let $\{I_t\}$ be a continuous time Markov chain on $\{1, 2\}$, which represents "the state of the economy". Let $(r_i, R_i, \mu_{S,i}, \sigma_{S,i}, \mu_{V,i}, \sigma_{V,i})$ denote the set of six parameters of the model when the state of the economy $I_t = i$ $(i = 1, 2)$. More specifically, $(r_1, R_1, \mu_{S,1}, \sigma_{S,1}, \mu_{V,1}, \sigma_{V,1})$ might represent the parameters of return processes in contraction state and $(r_2, R_2, \mu_{S,2}, \sigma_{S,2}, \mu_{V,2}, \sigma_{V,2})$ might represent those in expansion state. With the notation in Section 5.2.2, we can write $\Theta(t) = (r(t), R(t), \mu_S(t), \sigma_S(t), \mu_V(t), \sigma_V(t)) = (r_{I_t}, R_{I_t}, \mu_{S,I_t}, \sigma_{S,I_t}, \mu_{V,I_t}, \sigma_{V,I_t})$. The time varying parameter set therefore forms piece-wise constant processes. In what follows, we assume $\mu_{S,i} > R_i > r_i$ $(i = 1, 2)$ so as to hold $\mu_S(t) > R(t) > r(t)$ for all t with probability 1.

We assume $\{I_t\}$ is stochastically independent both of $\{z_t\}$ and $\{\hat{z}_t\}$. Hence $\{\Theta(t)\}$ is also stochastically independent of $\{z_t\}$ and $\{\hat{z}_t\}$. In other words, the parameters of return processes, such as drift and volatility, independently shift between different states at random times. The state of the model at time t is therefore represented by a triplet (w, v, i) if $W_t = w$, $V_t = v$, and $I_t = i$. As in Section 2.3.1 and Chapter 4, the regime process $\{I_t\}$ is governed by the transition rate matrix

$$\boldsymbol{Q} = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}.$$

5.2.4 Investor's Optimization Problem

Given the investor's wealth W_t at time t, she is assumed to consume the rate of c_t, invest π_t into equity and save the rest $W_t - \pi_t$ into the money market account, which allows the investor to borrow money at a higher borrowing rate. These assumptions on investment opportunities imply that the wealth dynamics for the investor are given by

$$dW_t = [(\mu_{S,I_t} - r_{I_t})\pi_t + r_{I_t}W_t - c_t - (R_{I_t} - r_{I_t})(W_t - \pi_t)^-]dt + \sigma_{S,I_t}\pi_t dz_t \quad (5.6)$$

where a function $(a)^- = -a$ if $a < 0$ and 0 otherwise.

The investor's problem is to determine optimal instantaneous consumption c_s and amount of stock to be invested π_s to maximize her expected utility,

$$E\left[\int_t^\infty e^{-\delta s}\frac{c_s^{1-\gamma}}{1-\gamma}\left(\frac{W_s}{V_s}\right)^{-\epsilon} ds \middle| W_t = w, V_t = v, I_t = i\right], \quad (5.7)$$

subject to the wealth dynamics in (5.6). For the above problem, we define $J^{(i)}(w, v)$ as the investor's value function in regime i,

$$J^{(i)}(w, v) \equiv \max_{\{c_s\},\{\pi_s\}} E\left[\int_t^\infty e^{-\delta s}\frac{c_s^{1-\gamma}}{1-\gamma}\left(\frac{W_s}{V_s}\right)^{-\epsilon} ds \middle| W_t = w, V_t = v, I_t = i\right]. \quad (5.8)$$

5.3 Solution to the Investor's Optimal Policy

Since (w, v, i) are jointly Markov, the value function $J^{(i)}(w, v)$ satisfies the following nonlinear HJB equation (See Appendix C.1 for the derivation)

$$\delta J^{(i)} = \sup_{c>0}\{u(c, w, v) - cJ_w^{(i)}\} + A_i(w, v) - q_i J^{(i)} + q_i J^{(3-i)} \quad i = 1, 2 \quad (5.9)$$

where

$$A_i(w,v) = \max\Big[\sup_{\pi \le w}\{f_i(\pi)\},\ \sup_{\pi > w}\{g_i(\pi)\}\Big], \tag{5.10}$$

$$\begin{aligned} f_i(\pi) &= r_i w J_w^{(i)} + \mu_{V,i} v J_v^{(i)} + \frac{1}{2}\sigma_{V,i}^2 v^2 J_{vv}^{(i)} \\ &\quad + (\mu_{S,i} - r_i)\pi J_w^{(i)} + \frac{1}{2}\pi^2 \sigma_{S,i}^2 J_{ww}^{(i)} + \pi\sigma_i J_{wv}^{(i)}, \qquad (5.11) \\ g_i(\pi) &= R_i w J_w^{(i)} + \mu_{V,i} v J_v^{(i)} + \frac{1}{2}\sigma_{V,i}^2 v^2 J_{vv}^{(i)} \\ &\quad + (\mu_{S,i} - R_i)\pi J_w^{(i)} + \frac{1}{2}\pi^2 \sigma_{S,i}^2 J_{ww}^{(i)} + \pi\sigma_i J_{wv}^{(i)}. \qquad (5.12) \end{aligned}$$

Here, $J_w^{(i)}$, $J_v^{(i)}$, $I_{ww}^{(i)}$, $I_{vv}^{(i)}$, and $I_{wv}^{(i)}$ are defined by $J_w^{(i)} \equiv \partial J^{(i)}/\partial w$, $J_v^{(i)} \equiv \partial J^{(i)}/\partial v$, $J_{ww}^{(i)} \equiv \partial^2 J^{(i)}/\partial w^2$, $J_{vv}^{(i)} \equiv \partial^2 J^{(i)}/\partial v^2$, and $J_{wv}^{(i)} \equiv \partial^2 J^{(i)}/\partial w \partial v$. Note that the superscript $(3-i)$ in (5.9) represents the other state of i, implying $(3-i) = 2$ for $i = 1$ and $(3-i) = 1$ for $i = 2$. The parameter σ_i in (5.11) and (5.12) is the instantaneous covariance of stock return with return on social-wealth index defined by

$$\sigma_i = \rho_{SV}\sigma_{S,i}\sigma_{V,i}. \tag{5.13}$$

Note that the HJB equation (5.9) depends on whether the investor borrows money to invest ($\pi > w$) or not, and also on the regime i. Differentiating (5.9) with respect to c gives the optimal consumption $\hat{c}$ as

$$\hat{c} \equiv \arg\sup_{c>0}\{u(c,w,v) - cJ_w^{(i)}\} = (J_w^{(i)})^{-1/\gamma}\left(\frac{w}{v}\right)^{-\epsilon/\gamma}. \tag{5.14}$$

Substituting (5.14) into the first term of the right hand side of (5.9) yields

$$\sup_{c>0}\{u(c,w,v) - cJ_w^{(i)}\} = \frac{\gamma}{1-\gamma}(J_w^{(i)})^{1-1/\gamma}\left(\frac{w}{v}\right)^{-\epsilon/\gamma}. \tag{5.15}$$

Similarly, differentiating (5.11) and (5.12) with respect to π gives

$$\begin{aligned} \pi_i^f(w) &\equiv \arg\sup_{\pi\in R} f_i(\pi) = -\frac{(\mu_{S,i} - r_i)}{\sigma_{S,i}^2}\frac{J_w^{(i)}}{J_{ww}^{(i)}} - \frac{\sigma_i}{\sigma_{S,i}^2}\frac{J_{wv}^{(i)}}{J_{ww}^{(i)}}v, \qquad (5.16) \\ \pi_i^g(w) &\equiv \arg\sup_{\pi\in R} g_i(\pi) = -\frac{(\mu_{S,i} - R_i)}{\sigma_{S,i}^2}\frac{J_w^{(i)}}{J_{ww}^{(i)}} - \frac{\sigma_i}{\sigma_{S,i}^2}\frac{J_{wv}^{(i)}}{J_{ww}^{(i)}}v. \qquad (5.17) \end{aligned}$$

In the same way, substituting (5.16) and (5.17) back into (5.11) and (5.12) yields

$$\begin{aligned}\sup_{\pi\in R} f_i(\pi) &= r_i w J_w^{(i)} + \mu_{V,i} v J_v^{(i)} + \frac{1}{2}\sigma_{V,i}^2 v^2 J_{vv}^{(i)} - \frac{1}{2}\frac{(\mu_{S,i}-r_i)^2}{\sigma_{S,i}^2}\frac{J_w^{(i)2}}{J_{ww}^{(i)}} \\ &\quad - \frac{1}{2}\frac{\sigma_i^2}{\sigma_{S,i}^2}\frac{J_{wv}^{(i)2}}{J_{ww}^{(i)}} v^2 - \frac{(\mu_{S,i}-r_i)\sigma_{S,i}}{\sigma_{S,i}^2}\frac{J_w^{(i)} J_{wv}^{(i)}}{J_{ww}^{(i)}} v, \qquad (5.18)\\ \sup_{\pi\in R} g_i(\pi) &= R_i w J_w^{(i)} + \mu_{V,i} v J_v^{(i)} + \frac{1}{2}\sigma_{V,i}^2 v^2 J_{vv}^{(i)} - \frac{1}{2}\frac{(\mu_{S,i}-R_i)^2}{\sigma_{S,i}^2}\frac{J_w^{(i)2}}{J_{ww}^{(i)}} \\ &\quad - \frac{1}{2}\frac{\sigma_i^2}{\sigma_{S,i}^2}\frac{J_{wv}^{(i)2}}{J_{ww}^{(i)}} v^2 - \frac{(\mu_{S,i}-R_i)\sigma_i}{\sigma_{S,i}^2}\frac{J_w^{(i)} J_{wv}^{(i)}}{J_{ww}^{(i)}} v. \qquad (5.19)\end{aligned}$$

It should be also noted that

$$\begin{aligned}f_i(w) = g_i(w) &= \mu_{V,i} v J_v^{(i)} + \frac{1}{2}\sigma_{V,i}^2 v^2 J_{vv}^{(i)} \\ &\quad + \mu_{S,i} w J_w^{(i)} + \frac{1}{2}\sigma_{S,i}^2 w^2 J_{ww}^{(i)} + \sigma_i w v J_{wv}^{(i)}. \qquad (5.20)\end{aligned}$$

From the assumption that $r_i < R_i < \mu_{S,i}$, (5.16) and (5.17) imply

$$\pi_i^f(w) > \pi_i^g(w). \qquad (5.21)$$

Thus, the second term of the right hand side in (5.10) is expressed as

$$A_i(w,v) = \begin{cases} g_i(\pi_i^g(w)), & w < \pi_i^g(w) \\ f_i(w) = g_i(w), & \pi_i^g(w) \le w < \pi_i^f(w) \\ f_i(\pi_i^f(w)), & \pi_i^f(w) \le w \end{cases} \qquad (5.22)$$

The resulting HJB equation (5.9) is, therefore,

$$\delta J^{(i)} = \frac{\gamma}{1-\gamma}(J_w^{(i)})^{1-1/\gamma}\left(\frac{w}{v}\right)^{-\epsilon/\gamma} + A_i(w,v) - q_i J^{(i)} + q_i J^{(3-i)}. \qquad (5.23)$$

To solve the above HJB equation, we need the following assumption on the parameters of the model. We define

$$D_i = \begin{cases} (1-\gamma-\epsilon)\Big\{R_i + \frac{(\mu_{S,i}-R_i+\epsilon\sigma_i)^2}{2(\gamma+\epsilon)\sigma_{S,i}^2} + \frac{\epsilon\mu_{V,i}+\frac{1}{2}\epsilon(\epsilon-1)\sigma_{V,i}^2}{1-\gamma-\epsilon}\Big\}, \\ \qquad \text{for} \quad (\gamma+\epsilon)\sigma_{S,i}^2 - \epsilon\sigma_i < \mu_{S,i} - R_i \\ (1-\gamma-\epsilon)\Big\{\mu_{S,i} - \frac{(\gamma+\epsilon)}{2}\sigma_{S,i}^2 + \epsilon\sigma_i + \frac{\epsilon\mu_{V,i}+\frac{1}{2}\epsilon(\epsilon-1)\sigma_{V,i}^2}{1-\gamma-\epsilon}\Big\}, \\ \qquad \text{for} \quad \mu_{S,i} - R_i \le (\gamma+\epsilon)\sigma_{S,i}^2 - \epsilon\sigma_i < \mu_{S,i} - r_i \\ (1-\gamma-\epsilon)\Big\{r_i + \frac{(\mu_{S,i}-r_i+\epsilon\sigma_i)^2}{2(\gamma+\epsilon)\sigma_{S,i}^2} + \frac{\epsilon\mu_{V,i}+\frac{1}{2}\epsilon(\epsilon-1)\sigma_{V,i}^2}{1-\gamma-\epsilon}\Big\}, \\ \qquad \text{for} \quad \mu_{S,i} - r_i \le (\gamma+\epsilon)\sigma_{S,i}^2 - \epsilon\sigma_i \end{cases} \tag{5.24}$$

and

$$\delta^* = \frac{D_1 - q_1 + D_2 - q_2 + \sqrt{(D_1 - q_1 - D_2 + q_2)^2 + 4q_1q_2}}{2}. \tag{5.25}$$

Assumption 1 The discount rate satisfies $\delta > \delta^*$.

Under the Assumption 1, we can solve the HJB equation and obtain the optimal policy as in Propositions 3 and 4. Their proofs are presented in Appendix C.2.

Proposition 3 (Value function) Let the utility be given in (5.2). Suppose the parameters satisfy Assumption 1. Then, the value function for each regime i has the form

$$J^{(i)}(w,v) = \frac{1}{1-\gamma-\epsilon} K_i w^{1-\gamma-\epsilon} v^{\epsilon} \tag{5.26}$$

and K_i is the solution for the following simultaneous non-linear equations

$$\delta K_i = \theta K_i^{\zeta} + D_i K_i - q_i K_i + q_i K_{3-i}, \quad i = 1, 2 \tag{5.27}$$

where $\zeta = 1 - 1/\gamma$ and $\theta = \frac{\gamma(1-\gamma-\epsilon)}{1-\gamma} > 0$.

Proposition 3 states that, given w and v, the investor's value function with regime switches depends on K_i, which is determined once D_i and q_i are given. In other word, to characterize optimal policy, we only need to focus on the values of D_i for $i = 1, 2$ in (5.24). Regime dependent parameters affect the value function only through D_i.

The next proposition provides the optimal consumption and investment policy, both of which are obtained straightforwardly by substituting the value function (5.26) into (5.14), (5.16) and (5.17).

Proposition 4 (Optimal Policy) Let the utility be as given in (5.2). Suppose the parameters satisfy Assumption 1. Then, the optimal consumption $\hat{c}_i$ and investment policy $\hat{\pi}_i$ for regime i to the consumption-portfolio problem in (5.8) are

$$\hat{c}_i = K_i^{-1/\gamma} w, \tag{5.28}$$

$$\hat{\pi}_i = \begin{cases} \frac{1}{\gamma+\epsilon}\left(\frac{\mu_{S,i}-R_i+\epsilon\sigma_i}{\sigma_{S,i}^2}\right)w, & (\gamma+\epsilon)\sigma_{S,i}^2 - \epsilon\sigma_i < \mu_{S,i} - R_i \\ w, & \mu_{S,i} - R_i \leq (\gamma+\epsilon)\sigma_{S,i}^2 - \epsilon\sigma_i < \mu_{S,i} - r_i \\ \frac{1}{\gamma+\epsilon}\left(\frac{\mu_{S,i}-r_i+\epsilon\sigma_i}{\sigma_{S,i}^2}\right)w, & \mu_{S,i} - r_i \leq (\gamma+\epsilon)\sigma_{S,i}^2 - \epsilon\sigma_i. \end{cases} \tag{5.29}$$

Form Proposition 4, the optimal consumption and investment policies are determined by (5.28) and (5.29). Optimal consumption-wealth ratio is equal to $K_i^{-1/\gamma}$. Optimal investment policy, on the other hand, does not directly depend on $K_i^{-1/\gamma}$ but depends on the associated parameters in the current regime: the equity premium ($\mu_{S,i} - r_i$, $\mu_{S,i} - R_i$), the volatility ($\sigma_{S,i}$, $\sigma_{V,i}$, σ_i), and the investor's preference parameter (γ, ϵ). Moreover, the investment policy takes three strategies: "Borrow money to invest into stock", "No borrowing but all in stock", and "No borrowing and partly invest into stock", depending on the investment opportunities in the current regime i.

For a single regime case, we can obtain a closed form solution to the consumption-portfolio problem in (5.8). Applying Propositions 3 and 4 to the model without regime switches straightforwardly gives the following result.

Proposition 5 (Optimal Policy without Regime Switches) Let the utility be given in (5.2). Suppose the parameters satisfy Assumption 1. Then, the optimal solution to the consumption-portfolio problem in (5.8) without regime switches is

$$J(w,v) = \frac{1}{1-\gamma-\epsilon} K w^{1-\gamma-\epsilon} v^{\epsilon}, \tag{5.30}$$

$$\hat{c} = K^{-1/\gamma} w, \tag{5.31}$$

$$\hat{\pi} = \begin{cases} \frac{1}{\gamma+\epsilon}\left(\frac{\mu_S-R+\epsilon\sigma}{\sigma_S^2}\right)w, & (\gamma+\epsilon)\sigma_S^2 - \epsilon\sigma < \mu_S - R \\ w, & \mu_S - R \leq (\gamma+\epsilon)\sigma_S^2 - \epsilon\sigma < \mu_S - r \\ \frac{1}{\gamma+\epsilon}\left(\frac{\mu_S-r+\epsilon\sigma}{\sigma_S^2}\right)w, & \mu_S - r \leq (\gamma+\epsilon)\sigma_S^2 - \epsilon\sigma \end{cases} \tag{5.32}$$

where

$$K = \left[\frac{\gamma - 1}{\gamma(\gamma + \epsilon - 1)} \Big(\delta - D \Big) \right]^{-\gamma} \tag{5.33}$$

and D is defined by (5.24).

As a final note, we compare the optimal policies for regime switches, with those for a single regime. The next corollary gives the result. Let $K^{(i)}$ $(i = 1, 2)$ be given by (5.33) for $D = D_i$. Thus, $K^{(i)}$ denotes the coefficient of the value function (5.30) for a single regime case.

Corollary 1 (Property of K_i) Without loss of generality, we suppose $D_1 < D_2$. Then, $K^{(1)} < K_1, K_2 < K^{(2)}$ where K_i $(i = 1, 2)$ is the coefficient of the value function for two regime case.

Corollary 1 implies that K_i lies between $K^{(1)}$ and $K^{(2)}$, both of which are determined by myopic value function under a single regime. A value maximizing policy with regime switches is therefore derived so that the optimal consumption policy in each regime recognizes the possibility of a regime shift.

These propositions and corollary state important properties of consumption and investment policies. The amount invested into stock is determined by the combination of the current regime-specific optimal investment strategy and both current and future regime dependent optimal consumption (saving) strategy, through the coefficient of the value function K_i.

5.4 Implication for Household Investment

The different time variations in an aggregated household's investment and consumption are a salient feature of the data. Figure 5.1 plots the aggregate risky asset shares of financial wealth invested in equity and the consumption-wealth ratio for the U.S. households. To highlight the business cycle, the contraction periods are shaded. As Figure 5.1 shows, equity share of financial wealth exhibits strong cyclical patterns, whereas the consumption-wealth ratio has a comparatively negligible cyclical component.

In this section, we explore how optimal investment and consumption policies are affected by three aspects of generalization: (1) a higher borrowing rate, (2) a direct preference for wealth, and (3) regime switches in return processes. To distinguish each effect on optimal behavior, we numerically solve optimal polices, using the semi-analytical form or the closed form expression described in the previous section. Then we compare the results to those for the standard CRRA model developed by Merton (1969). Finally, we consider the implication for household investment behavior, such as the time-variation observed in Figure 5.1.

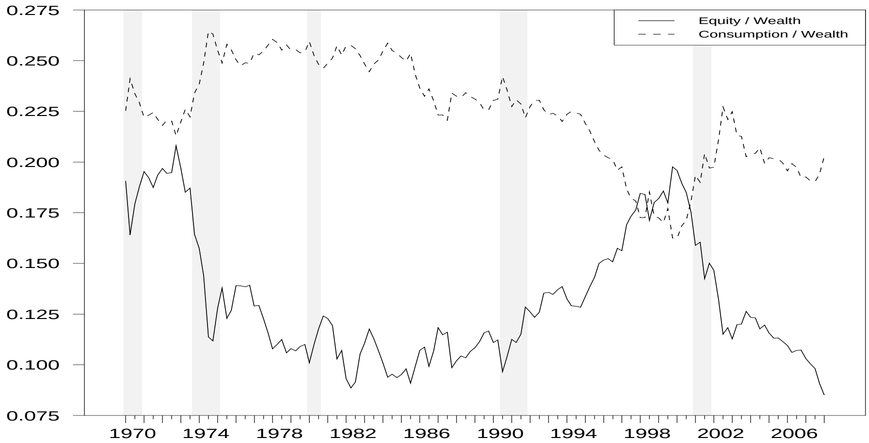

Figure 5.1: Equity Share and Consumption-Wealth Ratio in the U.S.

Note: Author's calculation by using data from Flow of Funds Account in the U.S.

5.4.1 Parameter Settings

Table 5.1 summarizes our parameter settings. We set the coefficient of relative risk aversion γ to 4 in our baseline specification. Following Campbell (1999), we set the annual expected return on equity to 8 percent, the standard deviation of equity return to 15 percent, and the risk-free rate to 2 percent. As the alternatives, we set the preference for wealth to the values ranging from 0.0 to 3.0.

Following Davis, Kubler, and Willen (2005), we adopt an average borrowing cost to be 6 percent and consider other values ranging from 5 to 7 percent, depending on the regime. Most empirical studies applying the regime switching framework have documented that the expected return and the volatility of equity are different between contraction and expansion. It has been widely recognized that the volatility of equity returns are higher during contraction (bear market) than during expansion (bull market). We first fix the volatility in regime 2 at 15 percent ($\sigma_{S,2} = .15$), and change the volatility in regime 1 ($\sigma_{S,1}$) to the values ranging from 5 percent to 25 percent. We next adopt different values as the return and volatility of equity among both regimes. We pick a lower equity return and risk-free rate in regime 1 (Bear market) than in regime 2 (Bull market) and consider the volatility values in regime 1 ranging 0.05 to 0.25.

For the parameters of social-wealth index, we set both drift and volatility in the baseline at 5 percent. We assume the drift values reflect the nominal growth rate of the economy. The drift value of 5 percent can be interpreted as the sum of 3 percent real growth rate and 2 percent inflation rate. We change the volatility values depending on regimes. We consider volatility parameter $\sigma_{V,1}$ in regime 1 ranging from 0.05 to 0.25 and set that in regime 2 to 0.05. We set the correlation of equity return and social-wealth index to 0.75. The positive correlation is the natural assumption about the relation between the asset market and aggregate economy.

Table 5.1: Parameter Settings

Parameter		Baseline	Higher Borrowing Rate		
			Alternatives	Regime 1	Regime 2
Discount factor	δ	0.25			
Risk Aversion	γ	4.00			
Preference for wealth	ϵ	0.00	1.0 to 3.0		
Risky asset return	μ_S	0.08		0.02	0.10
Volatility of dS_t/S_t	σ_S	0.15		.05 to .25	0.15
Borrowing rate	R	0.02	0.06	.05	0.07
Risk-free rate	r	0.02		0.01	0.03
Social-wealth return	μ_V	0.05	0.01 to .10		
Volatility of dV_t/V_t	σ_V	0.05		.05 to .25	0.05
Correlation	ρ_{SV}	0.75			
Transition Parameter	q			0.5	0.5

5.4.2 Effect of Volatility Switches with a Higher Borrowing Rate

We first focus on the effect of switches in volatility parameter on optimal policy. Table 5.2 compares the optimal risky investment policies for the baseline parameters to those for a higher borrowing rate, switches in volatility, and preference for wealth. Several interesting patterns are evident from this table. First, the optimal risky asset share is regime specific: the optimal risky asset share to wealth ($\hat{\pi}/w$) is different across regime, conditional on the current regime $I_t = i$ in the economy. More precisely, the risky asset share in regime 1 has a different value from that in regime 2, depending on the volatility in regime 1. Note that the volatility in regime 1 ranges from 0.05 to 0.25 while volatility in regime 2 is fixed at 0.15. Second, a higher borrow-

Table 5.2: Optimal Risky Asset Share under Changes in Volatility.

$\sigma_{S,1}$	Baseline	Higher Borrowing Rate: $R = .04 > r = .02$					
		$\epsilon = 0$		$\epsilon = 1$		$\epsilon = 2$	
		$\sigma_{S,2} = .15$		$\sigma_{S,2} = .15$		$\sigma_{S,2} = .15$	
		Regime 1	2	Regime 1	2	Regime 1	2
0.05	6.00	2.00	0.67	1.75	0.58	1.58	0.53
0.10	1.50	1.00	0.67	1.00	0.58	1.00	0.53
0.15	0.67	0.67	0.67	0.58	0.58	0.53	0.53
0.20	0.38	0.38	0.67	0.34	0.58	0.31	0.53
0.25	0.24	0.24	0.67	0.22	0.58	0.21	0.53

Note: Optimal risky asset share ($\hat{\pi}/w$) calculated for the baseline parameters and the alternative values with changes in volatility $\sigma_{S,1}$ and ϵ. $\sigma_{S,2}$ is fixed at .15.

ing rate dampens risky investment in the leverage region where the investor borrows money to invest a larger amount than her wealth w into risky assets ($\hat{\pi}/w > 1$). Compared to the baseline case in the second column in Table 5.2, the risky assets share in the leverage region is smaller than that in the baseline result where the borrowing rate is equal to the deposit rate. This result is obvious because the wedge between equity return and borrowing rate ($\mu_{S,i} - R_i$) is smaller than the equity premium ($\mu_{S,i} - r_i$). However, the share does not increase continuously. Instead the optimal risky asset share with a higher borrowing rate stays equal to 1 in specific combinations of volatility $\sigma_{S,1}$ and preference coefficient ϵ. It reflects the analytical result that the optimal investment policy has the different formulas in equation (5.29), depending on the associated parameters. Third, the risky asset share ($\hat{\pi}/w$) decreases with the preference parameter for the investor's wealth ϵ. In other words, when the investor cares about her wealth status, the wealth accumulation motive makes the investor more risk averse.

Table 5.3 highlights the effect that regime switches have on optimal consumption policy. In contrast to the optimal investment policy, the optimal consumption wealth ratio ($\hat{c}/w$) in regime 2 ($\sigma_{S,2}$ fixed to 0.15) varies, depending on the volatility in the other regime ($\sigma_{S,1}$). It should be noted that the optimal consumption wealth ratio in regime 1 has much smaller ranges than that under the single-regime baseline model. This reflects the fact that the volatility in the other regime (regime 2) is fixed at 0.15. A value maximizing policy is derived in such a way that in each regime the optimal consumption policy recognizes the possibility of a regime shift. As a result, the optimal consumption wealth ratio tends to be more stable than the current regime specific risky investment policy. These consumption and

Table 5.3: Optimal Consumption-Wealth Ratio under Changes in Volatility

$\sigma_{S,1}$	Baseline	Higher Borrowing Rate: $R = .04 > r = .02$					
		$\epsilon = 0$		$\epsilon = 1$		$\epsilon = 2$	
		$\sigma_{S,2} = .15$		$\sigma_{S,2} = .15$		$\sigma_{S,2} = .15$	
		Regime 1	2	Regime 1	2	Regime 1	2
0.05	0.213	0.108	0.105	0.110	0.106	0.108	0.104
0.10	0.111	0.101	0.099	0.100	0.098	0.098	0.096
0.15	0.093	0.093	0.093	0.089	0.089	0.085	0.085
0.20	0.086	0.089	0.090	0.085	0.086	0.081	0.081
0.25	0.083	0.087	0.088	0.083	0.084	0.078	0.079

Note: Optimal consumption-wealth ratio ($\hat{c}/w$) the baseline parameters and the alternative values with changes in volatility $\sigma_{S,1}$ and ϵ. $\sigma_{S,2}$ is fixed to .15.

investment patterns in optimal policy are qualitatively supported by their empirical time variation observed in Figure 5.1.

Table 5.3 also represents the effect of the preference for wealth on consumption policy. As the coefficient ϵ increases, the investor cares more about her social wealth status, which results in a smaller consumption wealth ratio. The "catching up with the Joneses" motive induces the investor to consume less and raise the saving rate.

These results are robust whether or not we change the other parameter such as the trend, the risk-free rate, and the borrowing rate across regimes. Table 5.4 and Table 5.5 represent the results. Rather than being constant, the drift of equity return and interest rate are set to vary across regimes. We set a pair of drifts $(\mu_{S,1}, \mu_{S,2})$ to $(0.02, 0.05)$, a pair of borrowing rates (R_1, R_2) to $(0.05, 0.07)$, and a risk-free interest rate (r_1, r_2) at $(0.01, 0.03)$.

The results and their implications under changes in both drift and volatility are almost the same as the ones we discussed in Table 5.2 and Table 5.3: the current regime specific investment policy, relatively stable consumption policy, risk averse investment with preference for wealth, and the catching up with the Jones motive in consumption. One notable differences lies in the optimal consumption policy across different regimes. As Table 5.4 shows, the wedges between regime 1 and regime 2 in an optimal consumption ratio are larger than those in Table 5.2. This reflects the fact that the changes in all parameters induce bigger changes in D_i in equation (5.24), through which the associated parameter for optimal consumption policy, K_i is determined.

Table 5.4: Optimal Risky Asset Share with Changes in Trend, Volatility, and Interest Rates

	Baseline	Higher Borrowing Rate: $R = .04 > r = .02$					
		$\epsilon = 0$		$\epsilon = 1$		$\epsilon = 2$	
$\sigma_{S,1}$		$\sigma_{S,2} = .15$		$\sigma_{S,2} = .15$		$\sigma_{S,2} = .15$	
		Regime 1	2	Regime 1	2	Regime 1	2
0.05	6.00	1.00	0.78	0.95	0.67	0.92	0.60
0.10	1.50	0.25	0.78	0.28	0.67	0.29	0.60
0.15	0.67	0.11	0.78	0.14	0.67	0.16	0.60
0.20	0.38	0.06	0.78	0.09	0.67	0.11	0.60
0.25	0.24	0.04	0.78	0.06	0.67	0.08	0.60

Note: Optimal risky asset share ($\hat{\pi}/w$) calculated for $\mu_{S,1} = .02$, $\mu_{S,2} = .10$, $R_1 = .05$, $R_2 = .07$, $r_1 = .01$, and $r_2 = .02$. The others are the alternative values with changes in volatility $\sigma_{S,1}$ and ϵ. $\sigma_{S,2}$ is fixed to .15.

Table 5.5: Optimal Consumption-Wealth Ratio under Changes in Trend, Volatility, and Interest Rates

	Baseline	Higher Borrowing Rate: $R = .04 > r = .02$					
		$\epsilon = 0$		$\epsilon = 1$		$\epsilon = 2$	
$\sigma_{S,1}$		$\sigma_{S,2} = .15$		$\sigma_{S,2} = .15$		$\sigma_{S,2} = .15$	
		Regime 1	2	Regime 1	2	Regime 1	2
0.05	0.213	0.088	0.083	0.086	0.077	0.108	0.081
0.10	0.111	0.086	0.081	0.084	0.075	0.098	0.078
0.15	0.093	0.086	0.080	0.083	0.074	0.085	0.078
0.20	0.086	0.085	0.080	0.083	0.074	0.081	0.077
0.25	0.083	0.085	0.080	0.083	0.073	0.078	0.077

Note: Optimal consumption-wealth ratio ($\hat{c}/w$) calculated for $\mu_{S,1} = .02$, $\mu_{S,2} = .10$, $R_1 = .05$, $R_2 = .07$, $r_1 = .01$, and $r_2 = .02$. The others are the alternative values with changes in volatility $\sigma_{V,1}$ and ϵ. $\sigma_{S,2}$ is fixed to .15.

5.4.3 Effect of Changes in Social Wealth Volatility and Growth

To conclude the numerical analysis, we investigate the effect that the change in the external social wealth has on the investor's optimal policy. Table 5.6 represents the optimal investment policies for the changes in the volatility of social wealth index. As the first column in Table 5.6 shows, we change the volatility of social wealth in regime 1 ($\sigma_{V,1}$) while we fix the volatility in regime 2 to 5 percent ($\sigma_{V,2} = .05$). It is worthwhile to note that the optimal investment policy in the baseline in the second column in Table 5.6 is constant because the standard CRRA model has no preference both for one's own wealth and for external social wealth.

Table 5.6: Optimal Risky Asset Share with Changes in Social Wealth Volatility

	Baseline	Higher Borrowing Rate: $R = .04 > r = .02$					
		$\epsilon = 0$		$\epsilon = 1$		$\epsilon = 2$	
$\sigma_{V,1}$		$\sigma_{V,2} = .05$		$\sigma_{V,2} = .05$		$\sigma_{V,2} = .05$	
		Regime 1	2	Regime 1	2	Regime 1	2
0.00	0.67	0.53	0.58	0.44	0.53	0.38	0.49
0.05	0.67	0.58	0.58	0.53	0.53	0.49	0.49
0.10	0.67	0.68	0.58	0.69	0.53	0.70	0.49
0.15	0.67	0.78	0.58	0.86	0.53	0.92	0.49

Note: Optimal risky asset share ($\hat{\pi}/w$) calculated for the baseline parameters with changes in volatility $\sigma_{V,1}$ and ϵ. $\sigma_{V,2}$ is fixed to .15.

As Table 5.6 shows, a higher volatility of external wealth in regime 1 ($\sigma_{V,1}$) induces the investor to put a higher proportion into risky asset in regime 1. This is due to the positive correlation between equity return and the return on the social wealth index. As Bakshi and Chen (1996) pointed out, adding the risky asset to the investor's portfolio does increase the correlation between her wealth W_t and external wealth V_t, which serves to insure against future uncertain declines in status that can result from rises in external social wealth.

Such hedging demand against the volatility of external social wealth is enhanced as the preference for status ϵ increases. Specifically, the risky investment wealth ratio increases from 0.53 to 0.78 under $\epsilon = 1$, while the ratio increases from 0.38 to 0.92 under $\epsilon = 3$. Consequently, even though the drift and volatility parameters for risky asset return are constant, the optimal investment policy does vary with a wider range when the volatilities

σ_V are different across regimes. In terms of life-cycle investment patterns, these results add insightful view points. When we interpret the social wealth standard as representing the life-cycle wealth profile in a specific group, the results might enhance the age-related risky investment pattern. Since it is natural to assume that the drift and volatility of external wealth standards increase with age, reflecting the income profile of " the Jones", the risky asset share tends to increase with age.

Finally, we consider the effect that the growth rate of social wealth has on the consumption policy. Table 5.7 highlights the results. Given a preference for wealth ϵ, a higher drift in the dynamics of social wealth index (μ_V) induces the investor to consume less and raise the saving rate. These results also might be applied to the analysis of saving behavior in different countries with different growth patterns. A higher growth economy enhances the tendency towards higher saving rate in the economy under a preference for wealth status. As a final note, in terms of life-cycle investment patterns, the combination of the drift (μ_V) and volatility (σ_V) might have a sizable effect on the life-cycle investment pattern.

Table 5.7: Optimal Consumption-Wealth Ratio with Changes in Social Wealth Growth

μ_v	Baseline	Higher Borrowing Rate: $R = .04 > r = .02$					
		$\epsilon = 1$		$\epsilon = 2$		$\epsilon = 3$	
		Regime 1	2	Regime 1	2	Regime 1	2
−.10	0.09	0.13	0.13	0.17	0.17	0.20	0.20
−.05	0.09	0.12	0.12	0.14	0.14	0.17	0.16
0.00	0.09	0.10	0.10	0.11	0.11	0.12	0.12
0.05	0.09	0.09	0.09	0.09	0.09	0.09	0.09
0.10	0.09	0.08	0.08	0.07	0.07	0.05	0.05

Note: Optimal consumption-wealth ratio ($\hat{c}/w$) calculated for the alternative values with changes in trend μ_v and volatility $\sigma_{v,i}$. The volatility pair in 2 regime ($\sigma_{V,1}, \sigma_{V,2}$) is set to (.15,.05).

5.5 Concluding Remarks

Building on the work of Xu and Chen (1998) and of Bakshi and Chen (1996), we develop an analytically tractable model with direct preference for wealth, higher borrowing rate, and regime switches. The model is more general and useful in the sense that it encompasses models with the standard CRRA preference, a borrowing rate equal to a risk-free rate, and a single regime.

We obtain a semi-analytical form and characterize the optimal policy for the consumption and investment decision with regime switches. The analytical form is simple and easily solved for optimal policies numerically, depending on the associated parameters. We also present the closed form solution for the problem with a constant regime as the special case of our general model. Main analytical results show that the optimal investment policy depends only on the current regime, while optimal consumption depends on both the future and the current regime.

Numerical analysis demonstrates that the extended model has favorable features to explain a household's investment behavior, such as the stable consumption to wealth ratio and time varying risky investment observed in time series data. In addition, numerical results also indicate that by combining the evolution of social wealth with return processes, our model can be potentially applied to analyzing various life-cycle risky investment profiles observed in cross-sectional data.

C.1 Derivation of Eq. (5.9)

Suppose $I_t = 1$ at time t. Partitioning by the events in $(t, t+dt)$, we obtain

$$\begin{aligned} J^{(i)}(w,v) = \max_{\{c_s\},\{\pi_s\}} \Big\{ & u(c,w,v)dt + e^{-\delta dt}\Big\{(1-q_i dt) \\ & \times E\left[J^{(i)}(w+dW_t, v+dV_t) \middle| W_t = w, V_t = v, I_t = i\right] \\ & + q_i dt J^{(3-i)}(w,v)\Big\} + o(dt)\Big\}. \end{aligned} \tag{5.34}$$

Applying Ito's lemma to $J^{(i)}(w+dW_t, v+dV_t)$, taking conditional expectation, we obtain from $e^{-\delta dt} = 1 - \delta dt + o(dt)$

$$\begin{aligned} \delta J^{(i)} &= \sup_{c>0}\{u(c,w,v) - cJ_w^{(i)}\} + r_i w J_w^{(i)} + \mu_{V,i} v J_v^{(i)} + \frac{1}{2}\sigma_{V,i}^2 v^2 J_{vv}^{(i)} \\ &\quad + \sup_{\pi}\Big\{(\mu_{S,i} - r_i)\pi J_w^{(i)} - (R_i - r_i)(w-\pi)^- J_w^{(i)} \\ &\quad + \frac{1}{2}\pi^2\sigma_{V,i}^2 J_{ww}^{(i)} + \pi\sigma_i J_{wv}^{(i)}\Big\} - q_i(J^{(i)} - J^{(3-i)}). \end{aligned} \tag{5.35}$$

Since

$$\begin{aligned} r_i w J_w^{(i)} + \mu_{V,i} v J_v^{(i)} + \frac{1}{2}\sigma_{V,i}^2 v^2 J_{vv}^{(i)} + (\mu_{S,i} - r_i)\pi J_w^{(i)} - (R_i - r_i)(w-\pi)^- J_w^{(i)} \\ + \frac{1}{2}\pi^2\sigma_{V,i}^2 J_{ww}^{(i)} + \pi\sigma_i J_{wv}^{(i)} = \begin{cases} f_i(\pi), & \pi \le w \\ g_i(\pi), & w < \pi, \end{cases} \end{aligned} \tag{5.36}$$

we obtain (5.9) by taking supremum for (5.36) with respect to π.

C.2 Proofs of Propositions in Chapter 5

Proof of Proposition 3. The proof consists of the following three steps:

1. (5.27) has unique pair of positive solutions K_1 and K_2.
2. The value function (5.26) satisfies the HJB equation (5.9).
3. The value function (5.26) satisfies the tranversarity condition.

(Step 1) (5.27) can be written as

$$\begin{cases} F_1(K_1) = K_2, \\ F_2(K_2) = K_1 \end{cases} \tag{5.37}$$

where

$$F_i(K_i) = \left(\frac{\delta - D_i}{q_i} + 1\right) K_i - \frac{\theta}{q_i} K_i^{\zeta}, \qquad i = 1, 2. \tag{5.38}$$

Since

$$\begin{aligned} \delta^* &\geq \frac{D_1 - q_1 + D_2 - q_2 + \sqrt{(D_1 - q_1 - D_2 + q_2)^2}}{2} \\ &= \max(D_1 - q_1, D_2 - q_2), \end{aligned} \tag{5.39}$$

$\frac{\delta - D_i}{q_i} + 1 = \{\delta - (D_i - q_i)\}/q_i > 0$.
(i) $\zeta < 0$: It is easy to check that $F_i(K_i)$ is concave and increasing, $\lim_{K_i \to 0} F_i(K_i) = -\infty$, and $F_i(K_i) \approx (\frac{\delta - D_i}{q_i} + 1)K_i$ as $K_i \to \infty$, where $f(x) \approx g(x)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$.
(ii) $0 < \zeta < 1$: It is easy to check that $F_i(K_i)$ is convex, $\lim_{K_i \to 0} F_i(K_i) = 0$, $\lim_{K_i \to 0} F_i'(K_i) = -\infty$, and $F_i(K_i) \approx (\frac{\delta - D_i}{q_i} + 1)K_i$ as $K_i \to \infty$.

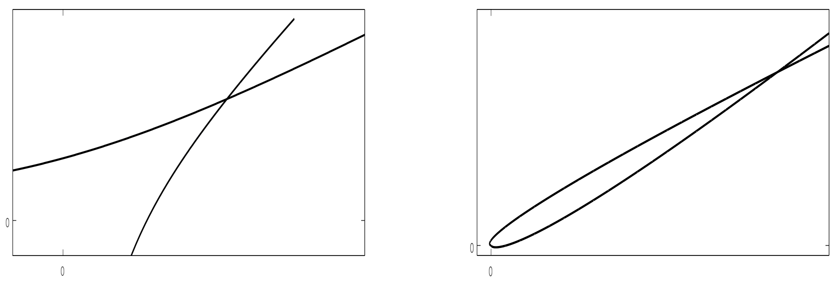

Figure 5.2: Examples of two curves $(K_1, F_1(K_1))$ and $(F_2(K_2), K_2)$. Left panel: $\zeta < 0$. Right panel: $0 < \zeta < 1$.

In both cases, the simultaneous equation (5.37) has unique pair of positive solutions if two curves $(K_1, F_1(K_1))$ and $(F_2(K_2), K_2)$ have unique positive intersection (c.f., Figure 5.2). Since $F_i(K_i) \approx (\frac{\delta - D_i}{q_i})K_i$ in both cases, this condition is equivalent to that their asymptotic straight lines $K_2 = (\frac{\delta - D_1}{q_1})K_1$ and $K_1 = (\frac{\delta - D_2}{q_2})K_2$ intersect each other. Comparing with their slopes, this condition is reduced to

$$\left(\frac{\delta - D_1}{q_1} + 1\right)\left(\frac{\delta - D_2}{q_2} + 1\right) > 1. \tag{5.40}$$

Since δ^* is the larger solution to the quadratic equation

$$\left(\frac{x - D_1}{q_1} + 1\right)\left(\frac{x - D_2}{q_2} + 1\right) = 1, \tag{5.41}$$

(5.40) is satisfied for $\delta > \delta^*$.

(Step 2) Since the simultaneous equation (5.27) has its unique pair of solutions, the value function (5.26) is uniquely determined. It is straightforward to check that the value function (5.26) satisfies the HJB (5.23).

(Step 3) In general, the solution of the HJB provides a candidate of the value function. To ensure that it is indeed the optimal value function, we will use the verification theorem which gives a sufficient condition for optimality. Specifically, if we can show

$$\lim_{t\to\infty} e^{-\delta t} E\left(J^{(I_t)}(W_t, V_t)\middle| W_0 = w, V_0 = v, I_0 = i\right) = 0 \tag{5.42}$$

for any initial state (w, v, i), then the solutions of the HJB equation are the value function, cf., Chang (2004).

To prove (5.42), we prepare some preliminary results. Recall that K_i denotes unique positive solution of (5.37) for $\delta > \delta^*$. Then, (5.37) is rewritten as

$$\begin{cases} K_1 = \frac{1}{\delta - G_1 + q_1}\left\{\frac{1-\gamma-\epsilon}{1-\gamma}K_1^\zeta + q_1 K_2\right\} \\ K_2 = \frac{1}{\delta - G_2 + q_2}\left\{\frac{1-\gamma-\epsilon}{1-\gamma}K_2^\zeta + q_2 K_1\right\} \end{cases} \tag{5.43}$$

where $G_i = D_i - (1-\gamma-\epsilon)K_i^{\zeta-1}$. Let

$$L_i^{(n)}(t) = \frac{1-\gamma-\epsilon}{1-\gamma} E\left(1_{\{N_t \le n\}} K_{I_t}^\zeta e^{\int_0^t G_{I_s} ds}\middle| I_0 = i\right), \quad n = 0, 1, \ldots$$

where 1_A stands for indicator random variable of event A and N_t denotes the number of regime shifts in $(0, t]$. We further define

$$\tilde{L}_i^{(n)} = \int_0^\infty e^{-\delta t} L_i^{(n)}(t)dt, \quad n = 0, 1, \ldots$$

provided that the integral is convergent. It is obvious that $L_i^{(n)}(t)$ and hence $\tilde{L}_i^{(n)}$ is increasing in n.

Lemma 1 $\tilde{L}_i^{(n)}$ exists for $\delta > \delta^*$. Moreover, $\lim_{n\to\infty} \tilde{L}_i^{(n)} \le K_i$.

(Proof) By dividing both sides of (5.37) by $K_i > 0$ and noting that $\theta = -(1-\gamma-\epsilon) + \frac{1-\gamma-\epsilon}{1-\gamma} > -(1-\gamma-\epsilon)$, we obtain

$$\delta = \theta K_i^{\zeta-1} + D_i - q_i + q_i \frac{K_{3-i}}{K_i} > -(1-\gamma-\epsilon)K_i^{\zeta-1} + D_i - q_i = G_i - q_i. \quad (5.44)$$

By considering the first epoch of the regime shift, $L_i^{(n)}(t)$ satisfies

$$\begin{aligned} L_i^{(0)}(t) &= \frac{1-\gamma-\epsilon}{1-\gamma} K_i^{\zeta} e^{G_i t} e^{-q_i t}, \\ L_i^{(n)}(t) &= \frac{1-\gamma-\epsilon}{1-\gamma} K_i^{\zeta} e^{G_i t} e^{-q_i t} \\ &\quad + \int_0^t e^{G_i u} q_i e^{-q_i u} L_{3-i}^{(n-1)}(t-u) du, \quad n = 1, 2, \ldots . \end{aligned} \quad (5.45)$$

Then, $\tilde{L}_i^{(0)}$ is calculated as

$$\tilde{L}_i^{(0)} = \frac{1}{\delta - G_i + q_i} \frac{1-\gamma-\epsilon}{1-\gamma} K_i^{\zeta} \quad (5.46)$$

which is assured to exist from (5.44). By comparing (5.46) with (5.43), $\tilde{L}_i^{(0)} < K_i$ for $i = 1, 2$. To apply an inductive argument, we suppose $\tilde{L}_i^{(n-1)} < K_i$ for $i = 1, 2$ and for some $n \ge 1$. A similar argument as above applied to (5.46) then gives

$$\begin{aligned} \tilde{L}_i^{(n)} &= \frac{1}{\delta - G_i + q_i} \left\{ \frac{1-\gamma-\epsilon}{1-\gamma} K_i^{\zeta} + q_i \tilde{L}_{3-i}^{(n-1)} \right\} \\ &< \frac{1}{\delta - G_i + q_i} \left\{ \frac{1-\gamma-\epsilon}{1-\gamma} K_i^{\zeta} + q_i K_{3-i} \right\} = K_i. \end{aligned}$$

Thus, $\tilde{L}_i^{(n)} < K_i$ for all n and hence $\lim_{n\to\infty} \tilde{L}_i^{(n)} \le K_i$. $\square$

Now we are in a position to prove (5.42). By substituting (5.26) into (5.42), we obtain

$$\begin{aligned} & E\left(J^{(I_t)}(W_t, V_t) \middle| W_0 = w, V_0 = v, I_0 = i \right) \\ &= \frac{w^{1-\gamma-\epsilon} v^{\epsilon}}{1-\gamma-\epsilon} E\left(K_{I_t} W_t^{1-\gamma-\epsilon} V_t^{\epsilon} \middle| W_0 = w, V_0 = v, I_0 = i \right) \\ &= \frac{w^{1-\gamma-\epsilon} v^{\epsilon}}{1-\gamma-\epsilon} E\left(K_{I_t} e^{\int_0^t G_{I_s} ds} \middle| I_0 = i \right). \end{aligned} \quad (5.47)$$

On the other hand, we obtain

$$\lim_{n\to\infty} \tilde{L}_i^{(n)} = \frac{1-\gamma-\epsilon}{1-\gamma} \int_0^\infty e^{-\delta t} E\left(K_{I_t}^{\zeta} e^{\int_0^t G_{I_s} ds} \middle| I_0 = i \right) dt \le K_i \tag{5.48}$$

from Lemma 1 and the monotone convergence theorem. Since K_{I_t} in (5.47) and $K_{I_t}^{\zeta}$ in (5.48) are bounded both from above and below, the existence of the integral in (5.48) implies (5.42). □

Proof of Corollary 1. Note that $K^{(1)}$ and $K^{(2)}$ satisfy

$$\begin{aligned} \delta K^{(1)} &= \theta (K^{(1)})^{\zeta} + D_1 K^{(1)}, \\ \delta K^{(2)} &= \theta (K^{(2)})^{\zeta} + D_2 K^{(2)} \end{aligned}$$

It should be noted that $K^{(1)}$ ($K^{(2)}$, respectively) is the solution of (5.37) when D_2 (D_1) is replaced by D_1 (D_2). On the other hand, $F_i(K_i)$ in (5.38) is a decreasing function of D_i irrespective of $\zeta < 0$ or $0 < \zeta < 1$. It is therefore obvious from Fig.5.2 that both K_1 and K_2 increase as D_i increases. Summarizing these arguments, we conclude $K^{(1)} < K_1, K_2 < K^{(2)}$ if $D_1 < D_2$. □

Chapter 6

Conclusion and Future Issues

The main objective of research in household finance is to develop theoretically well-founded models to analyze an individual's optimal decisions under uncertainty. Better models will improve both investment advice and our understanding of the pricing mechanisms in the financial markets. To satisfy such motives, there have been mainly two approaches: (a) developing a simple analytical model and (b) constructing a large model which takes into account realistic factors as much as possible. One notable advantage in the analytical approach is that the closed form solution allows us to do explicit comparative statics and give a precise characterization of the optimal policy. We follow the former direction for our research. Although some of the model assumptions are often too simple and unrealistic, we can obtain rich insights and implications for realistic problems.

In this thesis, we investigate how investment opportunities associated with debt influences investment under uncertainty. For this purpose we attempt to propose analytically tractable models in household finance, building on three well-established literatures: Merton's consumption and portfolio choice model, option based rational refinancing model, and a regime switching framework. In Chapter 3, we focus on the variations of the ratio of equity investment to financial wealth. We construct a consumption and portfolio choice model in which the debt repayment term decision is incorporated. With an explicit link between the debt-repayment term and consumption-investment policy in hand, we demonstrate the effect that the long term debt repayment has on the portfolio policy both numerically and empirically. The main conclusion from this study is that debt repayment dampens wealth accumulation, which results in a higher risky asset share to financial wealth. The model can produce a hump-shaped investment profile before and after the debt repayment term. While our empirical results almost support the prediction of the model, we also find that the equity share to the individual's net worth decreases monotonically with age. This suggests that any serious study on household investment behavior needs to pay attention to both the total financial wealth and the net worth or home equity.

Optimal mortgage refinancing is a big issue the typical individual with mortgage debt faces over the life-cycle. Chapter 4 provides a new explanation for why the timing of refinancing ought to vary, and in particular why early or late refinancing arises, countering the optimal refinancing timing implied by standard option-based approaches. The main analytical result is that because of the possibility of a regime shift, the optimal refinancing policy takes the form of the different trigger threshold of the interest spread between mortgagor's borrowing rate and the current market mortgage rate for each regime. We evaluate the predictions of the model, based on the estimated parameters for a two-regime model to capture the evolution of the mortgage rates in the US. Our model explains the late refinancing in the 1980s as well as the tendency toward early refinancing in recent periods, both of which have been documented empirically.

In Chapter 5, we return to the individual's consumption and portfolio choice problem. Relating to our main interest (borrowing behavior) and an attractive tool (regime switching) in the previous two chapters, we develop a model with direct preference for wealth, where the individual faces stochastic investment opportunities with a higher borrowing rate and regime switches. The model is more general and useful in the sense that it encompasses the model with the standard CRRA preference, the model with a borrowing rate equal to the deposit rate, and the model with a single regime. With obtained semi-analytical solutions, we demonstrates numerically that our tractable model has favorable features to explain a household's investment behavior, such as the stable consumption to wealth ratio, time varying risky investment observed in time series data, and various life-cycle risky investment profiles observed in cross-sectional data.

Throughout this thesis, we attempt to develop analytically tractable tools and to demonstrate that analytical models in household finance can produce useful insights and implications for realistic problems. Three extensions are pointed out for future research. First, in terms of optimal mortgage refinancing with regime switches, incorporating a more realistic interest rate process, such as Vasicek (1977) or Cox, Ingersoll and Ross (1985), into the model with regime switches is an important research topic. Second, applying such models to develop the pricing model of MBS is a challenging task. Finally, modifying the model developed in chapter 5 to empirical framework for explaining time-series and cross-sectional household behavior and the implication for asset pricing is an attractive research field.

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