Derived equivalences for Gorenstein algebras

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Contents

Introduction

In 1950's, homological algebra was introduced to commutative ring theory and new aspects of the theory were developed. One of the major topics of them is the theory of Gorenstein rings which was begun to study by Grothendieck in the theory of commutative noetherian rings. Gorenstein rings have many interesting homological properties, which are sometimes called "*Gorenstein properties*", and play important roles in algebraic geometry. The properties of Gorenstein rings are deeply studied by Bass (1963). A commutative noetherian ring R is said to be Gorenstein if inj dim ${R_\mathfrak{p}}_{R_\mathfrak{p}} < \infty$ for all prime ideals $\mathfrak{p} \in \operatorname{Spec}(R)$, where $R_\mathfrak{p}$ denotes the localization of R at $\mathfrak{p}$ and inj dim ${R_\mathfrak{p}}_{R_\mathfrak{p}}$ denotes the injective dimension of $R_\mathfrak{p}$ as an $R_\mathfrak{p}$-module. If R is Gorenstein with dim $R = n$ then inj dim $R_R = n$ and if R is Gorenstein with dim $R = 0$ then R is selfinjective. Recently, several people have tried to generalize Gorenstein properties to Noether algebras. Let R be a commutative noetherian ring and A an R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A. Then A is said to be a Noether R-algebra if A is finitely generated as an R-module. Generally speaking, Noether algebras with Gorenstein properties are called "*Gorenstein algebras*". There are several different ways to define Gorenstein algebras. For instance, Goto and Nishida (2002) proposed to call a Noether R-algebra A Gorenstein provided that the Cousin complex of A yields a minimal injective resolution of A_A, which is equivalent to that A is Cohen-Macaulay as an R-module and inj dim ${A_\mathfrak{p}}_{A_\mathfrak{p}} = \dim {A_\mathfrak{p}}_{R_\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp}(A) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid A_\mathfrak{p} \neq 0\}$. Also, in case R is artinian, Auslander and Reiten (1991) proposed to call an Artin R-algebra A Gorenstein provided that inj dim ${}_AA =$ inj dim $A_A < \infty$. In this thesis, we deal with both Gorenstein algebras A in the sense of Goto and Nishida and Noether algebras A with inj dim ${}_AA =$ inj dim $A_A < \infty$.

Bernstein, Gel'fand and Ponomarev (1973) introduced the classical tilting theory to the representation theory of finite dimensional algebras over a field by translating the notion of reflections in root systems into the representation theory of oriented graphs. The classical tilting theory was developed by Brenner and Butler (1980) and completed by Happel and Ringel (1982) as the theory of classical tilting modules. A few years later Happel (1986) and Cline, Parshall and Scott (1986) showed that there is an equivalence of triangulated categories between $\mathcal{D}^\mathrm{b}(\text{mod-}A)$, the derived category of bounded complexes of finitely generated right A-modules over a finite dimensional algebra A over a field, and $\mathcal{D}^{\mathrm{b}}(\text{mod-}B)$ with $B = \mathrm{End}_A(T)$ for any classical tilting right A-module T. After the generalization by Miyashita (1985), the notion of tilting modules was extended to that of tilting complexes by Rickard (1989) and he established the theory of derived equivalences. For a ring A, a cochain complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is said to be a tilting complex if $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet)$ generates $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as a triangulated category, where $\mathcal{P}_A$ denotes the category of finitely generated projective right A-modules, $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ denotes the homotopy category of bounded complexes over $\mathcal{P}_A$, Mod-A denotes the category of right A-modules, $\mathcal{K}(\text{Mod-}A)$ denotes the homotopy category of bounded complexes over Mod-A, $(-)[i]$ denotes the shift functor and $\mathrm{add}(P^\bullet)$ denotes the full additive subcategory of $\mathcal{K}(\text{Mod-}A)$ whose objects are direct summands of finite direct sums of copies of $P^\bullet$. Then A and $B = \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$ are derived equivalent. In this thesis, we study the Gorenstein properties of Noether algebras by using the theory of derived equivalences.

In Chapter 1, we introduce a notion of Gorenstein algebras as a generalization of selfinjective Artin algebras. Let R be a commutative Gorenstein ring and A a Noether R-algebra. We call A Gorenstein provided that A has Gorenstein dimension zero as an R-module and that DA is a projective generator in the category of right A-modules, where $D = \mathrm{Hom}_R(-, R)$. Assume A is a Gorenstein R-algebra. We see that A satisfies the Auslander condition and has selfinjective dimension at most dim R on both sides. It follows that A is a Gorenstein algebra in the sense of Auslander and Reiten if $\sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}(A)\} < \infty$. In case A is commutative, A is a Gorenstein ring. Also, in case $\dim R = 0$, A is a selfinjective Artin algebra. Furthermore, for any prime ideal $\mathfrak{p}$ of R with $A_\mathfrak{p} \neq 0$ we will see that $A_\mathfrak{p}$ is maximal Cohen-Macaulay as an $R_\mathfrak{p}$-module and has selfinjective dimension equal to dim $R_\mathfrak{p}$ on both sides. It follows that A is a Gorenstein algebra in the sense of Goto and Nishida (Proposition 1.3.7). Next we study derived equivalences for Gorenstein algebras. Let $P^\bullet$ be a tilting complex over A and $B = \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. We show that B is a Gorenstein R-algebra if and only if $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, where $\nu = D \circ \mathrm{Hom}_A(-, A)$ (Theorem 1.4.5). Furthermore, we provide an example of A and $P^\bullet$ such that B does not have Gorenstein dimension zero as an R-module (Example 1.4.7).

In Chapter 2, we deal with Noether algebras A which are Gorenstein in the sense of Auslander and Reiten, i.e., Noether algebras A such that inj dim ${}_AA =$ inj dim $A_A < \infty$. Note that A itself is a dualizing complex for A if and only if inj dim ${}_AA =$ inj dim $A_A < \infty$. Let R be a commutative noetherian ring and A a Noether R-algebra. We set $A^{\mathrm{e}} = A^{\mathrm{op}} \otimes_R A$, where A^{op} denotes the opposite ring of A. Take a minimal injective resolution $R \to I^\bullet$ in Mod-R and set $V^\bullet = \mathrm{Hom}^\bullet_R(A, I^\bullet) \in \mathcal{K}^+(\text{Mod-}A^{\mathrm{e}})$. In this chapter, we are mainly concerned with the case where $V^\bullet$ is a dualizing complex for A. We see that $V^\bullet$ is a dualizing complex for A if and only if $R_\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p} \in \mathrm{Supp}(A)$ and $\sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}(A)\} < \infty$ (Propositions 2.3.7 and 2.3.8). Assume $V^\bullet$ is a dualizing complex for A. Then we show that the following statements are equivalent: (1) inj dim ${}_AA =$ inj dim $A_A < \infty$; (2)

there exists a quasi-isomorphism $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ a tilting complex such that $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$; (3) there exists a quasi-isomorphism $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\text{op}})$ with $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\text{op}}})$ a tilting complex such that $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A^{\text{op}})}(Q^\bullet)^{\text{op}}$; and (4) there exist quasi-isomorphisms $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\text{op}})$ with $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\text{op}}})$ (Theorem 2.3.9). Namely, A itself is a dualizing complex for A if and only if $V^\bullet$ is quasi-isomorphic to tilting complexes in both sides. Assume further that A itself is a dualizing complex for A. Then we show that the functor $- \otimes^{\mathbf{L}}_A V^\bullet$ induces a self-equivalence of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ (Theorem 2.4.7).

In Chapter 3, we study derived equivalences for selfinjective Artin algebras. Let A be an Artin algebra. Rickard raised a question whether a complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ is a tilting complex or not if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of $K_0(A)$, the Grothendieck group of A. Our first aim of this chapter is to show that if A is a representation-finite selfinjective Artin algebra then every $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, where ν is the Nakayama functor, is a direct summand of a tilting complex (Theorem 3.3.6). Our second aim of this chapter is to show that for any derived equivalent representation-finite selfinjective Artin algebras A, B there exists a sequence of selfinjective Artin algebras $A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1 (Theorem 3.3.7).

In Chapter 4, we deal with Frobenius extensions. Let A be a ring and $e \in A$ an idempotent. Assume A contains a subring R such that $xe = ex$ for all $x \in R$, Ae_R is finitely generated and eA_A is embedded in $\mathrm{Hom}_R(Ae, R_R)_A$ as a submodule. Then there exists a tilting complex of the form

$$T^\bullet : \cdots \to 0 \to T^{-1} \to T^0 \to 0 \to \cdots$$

such that $T^0 \in \mathrm{add}((1-e)A_A)$, $T^{-1} \in \mathrm{add}(eA_A)$ and $eA[1] \in \mathrm{add}(T^\bullet)$. This type of tilting complex plays an important role in the theory of derived equivalences. Our aim in this chapter is to provide a way to construct extensions A of a given ring R containing such an idempotent. To do so, we need the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku which we modify as follows. Let A be a ring containing a ring R as a subring. Then A is said to be a Frobenius extension of R if A_R and ${}_RA$ are finitely generated projective and $A_A \cong \mathrm{Hom}_R(A, R_R)_A$ and ${}_AA \cong {}_A\mathrm{Hom}_R(A, {}_RR)$. We see that if R is Auslander-Gorenstein then so is A. Next, for any integer $n \geq 1$, any permutation π of $I = \{1, \cdots, n\}$ and any ring R, we provide a way to construct a Frobenius extension A of R. Then $1 = \sum_{i \in I} e_i$ with the e_i orthogonal idempotents in A and for any nonempty π-stable subset J of I we get a desired idempotent $e = \sum_{j \in J} e_j$. Finally, we show that if R contains a regular element and if $i \neq \pi(i)$ then A is derived equivalent to a generalized triangular matrix ring

$$\begin{pmatrix} e_iAe_i & \mathrm{Ext}^1_A(A/Ae_iA, e_iA) \\ 0 & A/Ae_iA \end{pmatrix}.$$

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Chapter 1

Derived equivalences and Gorenstein algebras

In this chapter, extending the notion of selfinjective Artin algebras to Noether algebras, we introduce a notion of Gorenstein algebras. Our main aim is to provide a necessary and sufficient condition for a tilting complex over a Gorenstein algebra to have a Gorenstein algebra as the endomorphism algebra.

Let R be a commutative noetherian ring and A a Noether R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \longrightarrow A$ whose image is contained in the center of A and A is finitely generated as an R-module. To define the Gorensteinness for A, we assume the base ring R is a Gorenstein ring (see [11]). Then we call A a Gorenstein R-algebra provided that A has Gorenstein dimension zero as an R-module (see [7]) and that DA is a projective generator in the category of right A-modules, where $D = \mathrm{Hom}_R(-, R)$. Assume A is a Gorenstein R-algebra. We will see in Section 3 that A satisfies the Auslander condition (see [12]) and has selfinjective dimension at most $\dim R$ on both sides, where $\dim R$ denotes the Krull dimension of R. In particular, in case A is commutative, A is a Gorenstein ring. Also, in case $\dim R = 0$, A is a selfinjective Artin algebra. Furthermore, for any prime ideal $\mathfrak{p}$ of R with $A_\mathfrak{p} \neq 0$ we will see that $A_\mathfrak{p}$ is maximal Cohen-Macaulay as an $R_\mathfrak{p}$-module and has selfinjective dimension equal to $\dim R_\mathfrak{p}$ on both sides. It follows that A is a Gorenstein algebra in the sense of [18] in which the theory of Gorenstein algebras is studied in detail. So we refer to [18] for the relationship of the notion of Gorenstein algebras to the theory of commutative Gorenstein rings. Next, let $P^\bullet$ be a tilting complex (see [39]) over A and denote by B the endomorphism algebra of $P^\bullet$ in the homotopy category. We will show in Section 4 that B is a Gorenstein R-algebra if and only if $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, where $\nu = D \circ \mathrm{Hom}_A(-, A)$, and that if $A \cong DA$ as A-bimodules then B is a Gorenstein R-algebra with $B \cong DB$ as B-bimodules. Furthermore, we will provide an example of A and $P^\bullet$ such

This chapter is based on my joint paper with M. Hoshino [3].

that B does not have Gorenstein dimension zero as an R-module. On the other hand, we will show in Section 5 that if $P^\bullet$ is associated with a certain sequence of idempotents in A then the condition $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$ is always satisfied. There is another notion of Gorenstein algebras. Consider the case where R is an artinian Gorenstein ring. Then an R-algebra A is sometimes called a Gorenstein algebra if $\mathrm{inj\ dim}\ {}_AA = \mathrm{inj\ dim}\ A_A < \infty$ (see e.g. [8]). It follows by [34, Proposition 1.6] that an R-algebra A is a Gorenstein algebra in this sense if and only if $D({}_AA)$ is a tilting module. We will extend this fact to the case where R is a Gorenstein ring with $\dim R < \infty$.

For a ring A, we denote by Mod-A the category of right A-modules and mod-A the full subcategory of Mod-A consisting of finitely presented modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module X considered is a right (resp., left) A-module. In particular, we denote by $\mathrm{inj\ dim}\ X_A$ (resp., $\mathrm{inj\ dim}\ {}_AX$) the injective dimension of a right (resp., left) A-module X. A similar notation is used for projective and flat dimensions. In this chapter, complexes are cochain complexes of modules and as usual modules are considered as complexes concentrated in degree zero. For a complex $X^\bullet$ and an integer $n \in \mathbb{Z}$, we denote by $\mathrm{B}^n(X^\bullet)$, $\mathrm{Z}^n(X^\bullet)$, $\mathrm{B}'^n(X^\bullet)$, $\mathrm{Z}'^n(X^\bullet)$ and $\mathrm{H}^n(X^\bullet)$ the n-th boundary, the n-th cycle, the n-th coboundary, the n-th cocycle and the n-th cohomology of $X^\bullet$, respectively. We denote by $\mathcal{K}(\text{Mod-}A)$ (resp., $\mathcal{D}(\text{Mod-}A)$) the homotopy (resp., derived) category of complexes of right A-modules and by $\mathcal{K}^+(\text{Mod-}A)$, $\mathcal{K}^-(\text{Mod-}A)$, $\mathcal{K}^{\mathrm{b}}(\text{Mod-}A)$ (resp., $\mathcal{D}^+(\text{Mod-}A)$, $\mathcal{D}^-(\text{Mod-}A)$, $\mathcal{D}^{\mathrm{b}}(\text{Mod-}A)$) the full triangulated subcategories of $\mathcal{K}(\text{Mod-}A)$ (resp., $\mathcal{D}(\text{Mod-}A)$) consisting of bounded below complexes, bounded above complexes and bounded complexes, respectively. We denote by $\mathcal{P}_A$ the full subcategory of mod-A consisting of finitely generated projective modules and by $\mathcal{K}^*(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}^*(\text{Mod-}A)$ consisting of complexes whose terms belong to $\mathcal{P}_A$, where $* = +, -, \mathrm{b}$ or nothing. We use the notation $\mathrm{Hom}^\bullet(-,-)$ (resp., $- \otimes^\bullet -$) to denote the single complex associated with the double hom (resp., tensor) complex. Finally, for an object X in an additive category $\mathfrak{A}$ we denote by $\mathrm{add}(X)$ the full additive subcategory of $\mathfrak{A}$ whose objects are direct summands of finite direct sums of copies of X and by $\overset{n}{\bigoplus} X$ the direct sum of n copies of X.

We refer to [13], [22], [43] for basic results in the theory of derived categories and to [39] for definitions and basic properties of derived equivalences and tilting complexes. Also, we refer to [15] for standard homological algebra in module categories and to [33] for standard commutative ring theory.

1.1 Preliminaries

Throughout this chapter, R is a commutative ring and A is an R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A. We assume further that R is a noetherian ring and A is a Noether R-algebra, i.e., A is finitely generated as an R-module. Note that A is a left and right noetherian ring. In particular, mod-A is abelian and consists of all finitely generated right A-modules. We set $D = \mathrm{Hom}_R(-, R)$. Note that for any $X \in \text{Mod-}A$ we have a functorial isomorphism in Mod-A^{op}

$$DX \xrightarrow{\sim} \mathrm{Hom}_A(X, DA), h \mapsto (x \mapsto (a \mapsto h(xa))).$$

For R-algebras A, B we identify an $(A^{\mathrm{op}} \otimes_R B)$-module X with an A-B-bimodule X such that $rx = xr$ for all $r \in R$ and $x \in X$. Also, for an R-algebra A we set $A^{\mathrm{e}} = A^{\mathrm{op}} \otimes_R A$. We identify $(A^{\mathrm{op}})^{\mathrm{op}}$ with A and $(A^{\mathrm{e}})^{\mathrm{op}}$ with A^{e}.

In this section, we recall several definitions and basic facts which we need in later sections.

Definition 1.1.1. A module $X \in \text{Mod-}R$ is said to be reflexive if the canonical homomorphism

$$\varepsilon_X : X \to D^2X, x \mapsto (h \mapsto h(x))$$

is an isomorphism, where $D^2X = D(DX)$.

Definition 1.1.2 ([7])**.** A module $X \in \text{mod-}R$ is said to have Gorenstein dimension zero if X is reflexive, $\mathrm{Ext}^i_R(X, R) = 0$ for $i > 0$ and $\mathrm{Ext}^i_R(DX, R) = 0$ for $i > 0$. We denote by $\mathcal{G}_R$ the full additive subcategory of mod-R consisting of modules which have Gorenstein dimension zero.

Lemma 1.1.3 ([7, Lemma 3.10])**.** *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in* mod-R*. Then the following hold.*

(1) If $Y, Z \in \mathcal{G}_R$, then $X \in \mathcal{G}_R$.

(2) Assume $\mathrm{Ext}^1_R(Z, R) = 0$. If $X, Y \in \mathcal{G}_R$, then $Z \in \mathcal{G}_R$.

Proof. See the proof of [7, Lemma 3.10]. □

Lemma 1.1.4. *For any $X^\bullet \in \mathcal{K}(\text{Mod-}R)$ we have a functorial homomorphism*

$$\xi_{X^\bullet} : \mathrm{H}^0(DX^\bullet) \to D\mathrm{H}^0(X^\bullet)$$

and the following hold.

(1) If $\mathrm{B}^0(DX^\bullet) \xrightarrow{\sim} D\mathrm{B}'^0(X^\bullet)$ canonically, then $\xi_{X^\bullet}$ is monic.

(2) If $\mathrm{B}^0(DX^\bullet) \xrightarrow{\sim} D\mathrm{B}'^0(X^\bullet)$ canonically and $\mathrm{Ext}^1_R(\mathrm{B}'^0(X^\bullet), R) = 0$, then $\xi_{X^\bullet}$ is an isomorphism.

Proof. We have functorial commutative diagrams in Mod-R with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{B}^0(DX^\bullet) \arrow[r] \arrow[d, "\eta_{X^\bullet}"'] & DX^0 \arrow[r] \arrow[d, equal] & \mathrm{Z}'^0(DX^\bullet) \arrow[r] \arrow[d, "\zeta_{X^\bullet}"] & 0 \\
0 \arrow[r] & D\mathrm{B}'^0(X^\bullet) \arrow[r] & DX^0 \arrow[r] & D\mathrm{Z}^0(X^\bullet),
\end{tikzcd}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^0(DX^\bullet) & \longrightarrow & \mathrm{Z}'^0(DX^\bullet) & \longrightarrow & DX^{-1} \\ & & {\scriptstyle \xi_{X^\bullet}}\downarrow & & \downarrow{\scriptstyle \zeta_{X^\bullet}} & & \| \\ 0 & \longrightarrow & D\mathrm{H}^0(X^\bullet) & \longrightarrow & D\mathrm{Z}^0(X^\bullet) & \longrightarrow & DX^{-1}. \end{array}$$

Assume $\eta_{X^\bullet}$ is an isomorphism. Then $\zeta_{X^\bullet}$ is monic and so is $\xi_{X^\bullet}$. Furthermore, if $\mathrm{Ext}^1_R(\mathrm{B}'^0(X^\bullet), R) = 0$, then $DX^0 \to D\mathrm{Z}^0(X^\bullet)$ is epic, so that $\zeta_{X^\bullet}$ and hence $\xi_{X^\bullet}$ are isomorphisms. □

Recall that rings A, B are said to be derived equivalent if $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$, $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ are equivalent as triangulated categories (see [39] for details). Since A is a Noether R-algebra, every ring B derived equivalent to A is also a Noether R-algebra ([39, Proposition 9.4]).

Lemma 1.1.5. *Let A, B be derived equivalent R-algebras. Let $F : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be an equivalence of triangulated categories and $F^* : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ a quasi-inverse of F. Set $P^\bullet = F(B) \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet = \mathrm{Hom}^\bullet_B(F^*(A), B) \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B^{\mathrm{op}}})$. Then for any $i \in \mathbb{Z}$ we have an isomorphism in* Mod-$(B^{\mathrm{op}} \otimes_R A)$

$$\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(A, P^\bullet[i]) \cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}B^{\mathrm{op}})}(B, Q^\bullet[i])$$

and an isomorphism in Mod-$(A^{\mathrm{op}} \otimes_R B)$

$$\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, A[i]) \cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}B^{\mathrm{op}})}(Q^\bullet, B[i]).$$

Proof. Set

$$\begin{aligned} G &= F \circ \mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(-, B) \ : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B^{\mathrm{op}}}) \to \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A), \\ G^* &= \mathrm{Hom}^\bullet_B(-, B) \circ F^* : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \to \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B^{\mathrm{op}}}). \end{aligned}$$

Then for any $i \in \mathbb{Z}$ we have a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(X^\bullet, G(Y^\bullet)[i]) \cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}B^{\mathrm{op}})}(Y^\bullet, G^*(X^\bullet)[i])$$

for $X^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Y^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B^{\mathrm{op}}})$. Since $G(B) \cong P^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ and $G^*(A) \cong Q^\bullet$ in $\mathcal{K}(\text{Mod-}B^{\mathrm{op}})$, and since $G^*(P^\bullet) \cong B$ in $\mathcal{K}(\text{Mod-}B^{\mathrm{op}})$ and $G(Q^\bullet) \cong A$ in $\mathcal{K}(\text{Mod-}A)$, the assertions follow. □

In several places below, our argument will depend on the term length of a complex. So we truncate redundant terms of complexes. To do so, we need the following.

Remark 1.1.6. For any $P^\bullet \in \mathcal{K}(\mathcal{P}_A)$ the following hold.

(1) We have a functorial isomorphism of complexes

$$P^\bullet \xrightarrow{\sim} \mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(\mathrm{Hom}^\bullet_A(P^\bullet, A), A).$$

(2) If $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ and $\mathrm{H}^i(P^\bullet) = 0$ for all $i \in \mathbb{Z}$, then $P^\bullet = 0$ in $\mathcal{K}(\text{Mod-}A)$.

(3) If $P^\bullet \in \mathcal{K}^+(\mathcal{P}_A)$ and $\mathrm{H}^i(\mathrm{Hom}^\bullet_A(P^\bullet, A)) = 0$ for all $i \in \mathbb{Z}$, then $P^\bullet = 0$ in $\mathcal{K}(\text{Mod-}A)$.

Now, for any complex $X^\bullet$ and $n \in \mathbb{Z}$ we define the following truncations:

$$\begin{aligned}
\sigma_{>n}(X^\bullet) &: \cdots \to 0 \to \mathrm{B}'^n(X^\bullet) \to X^{n+1} \to X^{n+2} \to \cdots, \\
\sigma_{\le n}(X^\bullet) &: \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Z}^n(X^\bullet) \to 0 \to \cdots, \\
\sigma'_{\ge n}(X^\bullet) &: \cdots \to 0 \to \mathrm{Z}'^n(X^\bullet) \to X^{n+1} \to X^{n+2} \to \cdots, \\
\sigma'_{<n}(X^\bullet) &: \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{B}^n(X^\bullet) \to 0 \to \cdots.
\end{aligned}$$

Remark 1.1.7. For any $P^\bullet \in \mathcal{K}(\mathcal{P}_A)$ and $n \in \mathbb{Z}$ the following hold.

(1) If $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ and $\mathrm{H}^i(P^\bullet) = 0$ for $i > n$, then $\sigma_{\le n}(P^\bullet) \in \mathcal{K}^-(\mathcal{P}_A)$ and $P^\bullet \cong \sigma_{\le n}(P^\bullet)$ in $\mathcal{K}(\text{Mod-}A)$.

(2) If $P^\bullet \in \mathcal{K}^+(\mathcal{P}_A)$ and $\mathrm{H}^{-i}(\mathrm{Hom}^\bullet_A(P^\bullet, A)) = 0$ for $i < n$, then $\sigma'_{\ge n}(P^\bullet) \in \mathcal{K}^+(\mathcal{P}_A)$ and $P^\bullet \cong \sigma'_{\ge n}(P^\bullet)$ in $\mathcal{K}(\text{Mod-}A)$.

Proof. (1) It follows by the assumption that $\sigma_{>n}(P^\bullet) = 0$ in $\mathcal{K}(\text{Mod-}A)$ and $\mathrm{B}'^n(P^\bullet) \in \mathcal{P}_A$. Since the exact sequence $0 \to \mathrm{Z}^n(P^\bullet) \to P^n \to \mathrm{B}'^n(P^\bullet) \to 0$ in Mod-A splits, $\sigma_{\le n}(P^\bullet) \in \mathcal{K}^-(\mathcal{P}_A)$ and $P^\bullet \cong \sigma_{\le n}(P^\bullet) \oplus \sigma_{>n}(P^\bullet)$ as complexes, so that $P^\bullet \cong \sigma_{\le n}(P^\bullet)$ in $\mathcal{K}(\text{Mod-}A)$.

(2) Set $Q^\bullet = \mathrm{Hom}^\bullet_A(P^\bullet, A) \in \mathcal{K}^-(\mathcal{P}_{A^{\mathrm{op}}})$. Since $\mathrm{H}^i(Q^\bullet) = 0$ for $i > -n$, by (1) $\sigma_{\le -n}(Q^\bullet) \in \mathcal{K}^-(\mathcal{P}_{A^{\mathrm{op}}})$ and $Q^\bullet \cong \sigma_{\le -n}(Q^\bullet)$ in $\mathcal{K}(\text{Mod-}A^{\mathrm{op}})$. Thus we have isomorphisms in $\mathcal{K}(\text{Mod-}A)$

$$\begin{aligned}
P^\bullet &\cong \mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(Q^\bullet, A) \\
&\cong \mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(\sigma_{\le -n}(Q^\bullet), A) \\
&\cong \sigma'_{\ge n}(\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(Q^\bullet, A)) \\
&\cong \sigma'_{\ge n}(P^\bullet).
\end{aligned}$$

□

Definition 1.1.8. For any $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ with $P^\bullet \ne 0$ in $\mathcal{K}(\text{Mod-}A)$ we set

$$a(P^\bullet) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}^i(P^\bullet) \ne 0\}$$

and for any $P^\bullet \in \mathcal{K}^+(\mathcal{P}_A)$ with $P^\bullet \ne 0$ in $\mathcal{K}(\text{Mod-}A)$ we set

$$b(P^\bullet) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}^{-i}(\mathrm{Hom}^\bullet_A(P^\bullet, A)) \ne 0\}.$$

Then for any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $P^\bullet \ne 0$ in $\mathcal{K}(\text{Mod-}A)$ we set $l(P^\bullet) = a(P^\bullet) - b(P^\bullet)$.

Recall that an idempotent $e \in A$ is said to be primitive if eA is an indecomposable A-module and to be local if $eAe \cong \mathrm{End}_A(eA)$ is a local ring. Then a ring A is said to be semiperfect if $1 = e_1 + \cdots + e_n$ in A with the e_i orthogonal local idempotents (cf. [10]).

Lemma 1.1.9. *Assume R is a complete local ring. Then A is semiperfect and the Krull-Schmidt theorem holds in* mod-A*, i.e., for any nonzero $X \in$* mod-A *the following hold.*

(1) X decomposes into a direct sum of indecomposable submodules.

(2) X is indecomposable if and only if $\mathrm{End}_A(X)$ *is local.*

Proof. This is well known but for the benefit of the reader we include a proof. Let $\mathfrak{m}$ be the maximal ideal of R and I an injective envelope of $R/\mathfrak{m}$ in Mod-R. Since A is right noetherian, $A = e_1A \oplus \cdots \oplus e_nA$ with the e_i orthogonal primitive idempotents. Furthermore, every $\mathrm{Hom}_R(e_iA, I) \in \text{Mod-}A^{\mathrm{op}}$ is indecomposable injective and hence $e_iAe_i \cong \mathrm{End}_A(e_iA) \cong \mathrm{End}_{A^{\mathrm{op}}}(\mathrm{Hom}_R(e_iA, I))^{\mathrm{op}}$ is local. Next, for any nonzero $X \in$ mod-A, $\mathrm{End}_A(X)$ is a Noether R-algebra and hence is semiperfect. The last assertion follows. □

1.2 Nakayama functor

In the following, we set $\nu = D \circ \mathrm{Hom}_A(-, A)$ which we call the Nakayama functor for A. Note that for any $P \in \mathcal{P}_A$ we have a functorial isomorphism in Mod-A

$$P \otimes_A DA \xrightarrow{\sim} \nu P, x \otimes h \mapsto (g \mapsto h(g(x))).$$

Lemma 1.2.1. *For any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathcal{K}(\text{Mod-}A)$ we have a bifunctorial isomorphism of complexes*

$$D\mathrm{Hom}^\bullet_A(P^\bullet, Q^\bullet) \cong \mathrm{Hom}^\bullet_A(Q^\bullet, \nu P^\bullet).$$

Proof. For any $P \in \mathcal{P}_A$ and $Q \in$ Mod-A, we have a bifunctorial isomorphism

$$Q \otimes_A \mathrm{Hom}_A(P, A) \xrightarrow{\sim} \mathrm{Hom}_A(P, Q), x \otimes h \mapsto (a \mapsto xh(a))$$

and hence bifunctorial isomorphisms

$$\begin{aligned} D\mathrm{Hom}_A(P, Q) &\cong D(Q \otimes_A \mathrm{Hom}_A(P, A)) \\ &\cong \mathrm{Hom}_A(Q, \nu P). \end{aligned}$$

It is obvious that the bifunctorial isomorphism

$$D\mathrm{Hom}_A(P, Q) \cong \mathrm{Hom}_A(Q, \nu P)$$

for $P \in \mathcal{P}_A$ and $Q \in$ Mod-A can be extended to a bifunctorial isomorphism of complexes

$$D\mathrm{Hom}^\bullet_A(P^\bullet, Q^\bullet) \cong \mathrm{Hom}^\bullet_A(Q^\bullet, \nu P^\bullet)$$

for $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathcal{K}(\text{Mod-}A)$. □

Lemma 1.2.2. *For any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathcal{K}(\text{Mod-}A)$ we have a bifunctorial homomorphism*

$$\xi_{P^\bullet,Q^\bullet} : \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, \nu P^\bullet) \to D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, Q^\bullet).$$

Furthermore, in case $Q^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ and $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, Q^\bullet[i]) = 0$ for $i > 0$, the following hold.

(1) $\xi_{P^\bullet,Q^\bullet}$ is monic if $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \leq i < a(Q^\bullet) - b(P^\bullet)$.

(2) $\xi_{P^\bullet,Q^\bullet}$ is an isomorphism if $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \leq i \leq a(Q^\bullet) - b(P^\bullet)$.

Proof. Set $X^\bullet = \mathrm{Hom}^\bullet_A(P^\bullet, Q^\bullet) \in \mathcal{K}(\text{Mod-}R)$. Then $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, Q^\bullet) \cong \mathrm{H}^0(X^\bullet)$ and by Lemma 1.2.1 $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, \nu P^\bullet) \cong \mathrm{H}^0(DX^\bullet)$. Thus the functorial homomorphism $\xi_{X^\bullet} : \mathrm{H}^0(DX^\bullet) \to D\mathrm{H}^0(X^\bullet)$ in Lemma 1.1.4 yields a desired bifunctorial homomorphism. Next, assume $Q^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ and assume $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, Q^\bullet[i]) = 0$ for $i > 0$. Set $l = a(Q^\bullet) - b(P^\bullet)$. By Remark 1.1.7, we may assume $X^i = 0$ for $i > l$. In case $l \leq 0$, we have $\mathrm{B}'^0(X^\bullet) = 0$ and $\mathrm{B}^0(DX^\bullet) = 0$. Assume $l \geq 1$. Then, since $\mathrm{H}^i(X^\bullet) = 0$ for $i > 0$, we have an exact sequence

$$0 \to \mathrm{B}'^0(X^\bullet) \to X^1 \to \cdots \to X^l \to 0$$

with $X^i \in \mathrm{add}(A_R)$ for all $1 \leq i \leq l$. Thus, if $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \leq i < l$, then $\mathrm{B}^0(DX^\bullet) \xrightarrow{\sim} D\mathrm{B}'^0(X^\bullet)$ canonically. Furthermore, if $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \leq i \leq l$, then $\mathrm{Ext}^1_R(\mathrm{B}'^0(X^\bullet), R) = 0$. The last assertions now follow by Lemma 1.1.4. □

Corollary 1.2.3. *Assume $\mathrm{Ext}^i_A(A, R) = 0$ for $i > 0$. Then for any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$, we have $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet[i]) = 0$ for $i < 0$.*

Proof. For any $i < 0$, since $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[-i + j]) = 0$ for $j > 0$, by applying Lemma 1.2.2(2) to $Q^\bullet = P^\bullet[-i]$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet[i]) &\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet[-i], \nu P^\bullet)\\ &\cong D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[-i])\\ &= 0.\end{aligned}$$

□

In the following, for a complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ we always define $\mathrm{add}(P^\bullet)$ as a full subcategory of $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$. Note however that the canonical functor $\mathcal{K}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}A)$ induces an equivalence between $\mathrm{add}(P^\bullet)$ defined in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $\mathrm{add}(P^\bullet)$ defined in $\mathcal{D}(\text{Mod-}A)$ (cf. [26, Remark 1.7]).

Definition 1.2.4 ([39]). A complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is said to be a tilting complex if the following conditions are satisfied:

(1) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$; and

(2) $\mathrm{add}(P^\bullet)$ generates $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as a triangulated category, i.e., a full triangulated subcategory of $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ coincides with $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ if it contains $\mathrm{add}(P^\bullet)$ and is closed under isomorphisms.

Remark 1.2.5 *([39, Proposition 5.4]).* Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$. Then $P^\bullet$ is a tilting complex if and only if for any $X^\bullet \in \mathcal{D}^-(\mathrm{Mod}\text{-}A)$ with $\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, X^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$ we have $X^\bullet = 0$ in $\mathcal{D}(\mathrm{Mod}\text{-}A)$.

Definition 1.2.6. For any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ we denote by $\mathcal{S}(P^\bullet)$ the full subcategory of $\mathcal{D}^-(\mathrm{Mod}\text{-}A)$ consisting of complexes $X^\bullet$ with $\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, X^\bullet[i]) = 0$ for $i \neq 0$.

Proposition 1.2.7 ([39]). *Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex and $B = \mathrm{End}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$. Then there exists an equivalence of triangulated categories*

$$F^* : \mathcal{D}^-(\mathrm{Mod}\text{-}A) \xrightarrow{\sim} \mathcal{D}^-(\mathrm{Mod}\text{-}B)$$

such that $F^*(X^\bullet) \cong \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, X^\bullet)$ in $\mathcal{D}(\mathrm{Mod}\text{-}B)$ for all $X^\bullet \in \mathcal{S}(P^\bullet)$. In particular, we have an equivalence

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, -) : \mathcal{S}(P^\bullet) \xrightarrow{\sim} \mathrm{Mod}\text{-}B$$

Proof. See [39, Section 4] for the first assertion. Then, since $F^*(P^\bullet) \cong B$ in $\mathcal{D}(\mathrm{Mod}\text{-}B)$, F^* induces an equivalence $\mathcal{S}(P^\bullet) \xrightarrow{\sim} \mathcal{S}(B)$. Note also that we have an equivalence $\mathrm{Mod}\text{-}B \xrightarrow{\sim} \mathcal{S}(B)$. Thus the last assertion follows (cf. [27, Theorem 1.3(3)]). □

In the following, we use the notation A_A (resp., ${}_AA$) to stress that A is considered as a right (resp., left) A-module. Then the notation $D(A_A)$ (resp., $D({}_AA)$) is used to stress that DA is considered as a left (resp., right) A-module. Note that $\nu(A_A) \cong D({}_AA)$ and $\mathcal{P}_A = \mathrm{add}(A_A)$.

Lemma 1.2.8. *Assume A is reflexive as an R-module and $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$. Then we have an equivalence $\nu : \mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_A$. In particular, for any tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$, $\nu P^\bullet$ is also a tilting complex and the following are equivalent.*

(1) $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ and $P^\bullet \in \mathcal{S}(\nu P^\bullet)$.

(2) $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$.

Proof. We have an anti-equivalence $\mathrm{Hom}_A(-, A) : \mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_{A^{\mathrm{op}}}$. Also, since A is reflexive as an R-module, we have an anti-equivalence $D : \mathcal{P}_{A^{\mathrm{op}}} \xrightarrow{\sim} \mathrm{add}(D({}_AA))$. Thus, since $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$, we have an equivalence $\nu : \mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_A$ which is extended to an equivalence of triangulated categories $\nu : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$, so that $\nu P^\bullet$ is a tilting complex.

(1) $\Rightarrow$ (2). We have $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet \oplus \nu P^\bullet, (P^\bullet \oplus \nu P^\bullet)[i]) = 0$ for $i \neq 0$ and hence by [26, Lemma 1.8] $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$.

(2) $\Rightarrow$ (1). Obvious. □

Lemma 1.2.9. *Assume $A \cong DA$ in* Mod-A^{e}. *Then the following hold.*

(1) For any $P^\bullet \in \mathcal{K}(\mathcal{P}_A)$ we have a functorial isomorphism of complexes $P^\bullet \cong \nu P^\bullet$.

(2) $A \in \mathcal{G}_R$ as an R-module if and only if $\mathrm{Ext}_R^i(A, R) = 0$ *for $i > 0$.*

Proof. (1) Fix an isomorphism $A \xrightarrow{\sim} DA$ in Mod-A^{e}. Then we have functorial isomorphisms of complexes $P^\bullet \cong P^\bullet \otimes_A^\bullet A \cong P^\bullet \otimes_A^\bullet DA \cong \nu P^\bullet$.

(2) For any $X, Y \in$ Mod-A^{e} we have a bifunctorial isomorphism

$$\theta_{X,Y} : \mathrm{Hom}_{A^{\mathrm{e}}}(X, DY) \xrightarrow{\sim} \mathrm{Hom}_{A^{\mathrm{e}}}(Y, DX), h \mapsto Dh \circ \varepsilon_Y.$$

We claim that $\theta_{A,A} = \mathrm{id}_{\mathrm{Hom}_{A^{\mathrm{e}}}(A,DA)}$. Let $h \in \mathrm{Hom}_{A^{\mathrm{e}}}(A, DA)$ and $a, b \in A$. Then $h(a)(b) = (h(1)a)(b) = h(1)(ab)$ and $h(b)(a) = (bh(1))(a) = h(1)(ab)$, so that $(\theta_{A,A}(h)(a))(b) = \varepsilon_A(a)(h(b)) = h(b)(a) = h(a)(b)$. It follows that $\theta_{A,A}(h) = h$. Since $Dh \circ \varepsilon_A = h$, if h is an isomorphism, so is ε_A. Thus A is reflexive as an R-module and the assertion follows. □

Proposition 1.2.10. *Assume $A \cong DA$ in* Mod-A^{e} *and $A \in \mathcal{G}_R$ as an R-module. Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with* $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ *for $i \neq 0$ and* $B = \mathrm{End}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$. *Then $B \cong DB$ in* Mod-B^{e}.

Proof. By Lemmas 1.2.2(2), 1.2.9(1) we have isomorphisms in Mod-B^{e}

$$\begin{aligned} DB &= D\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet) \\ &\cong \mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, \nu P^\bullet) \\ &\cong \mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet) \\ &= B. \end{aligned}$$

□

1.3 Gorenstein algebras

In this section, we introduce the notion of Gorenstein R-algebras over a Gorenstein ring R. We refer to [11] for the definition and basic properties of commutative Gorenstein rings.

We denote by $\dim R$ the Krull dimension of R, by $\mathrm{Spec}(R)$ the set of prime ideals in R and by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p} \in \mathrm{Spec}(R)$. Also, for a module $X \in$ Mod-R we denote by $\mathrm{Supp}(X)$ the subset of $\mathrm{Spec}(R)$ consisting of $\mathfrak{p} \in \mathrm{Spec}(R)$ with $X_\mathfrak{p} \neq 0$. Note that we do not exclude the case where $\mathrm{Supp}(A) \neq \mathrm{Spec}(R)$, i.e., the kernel of the structure ring homomorphism $R \to A$ may not be nilpotent.

Definition 1.3.1. Assume R is a Gorenstein ring. Then A is said to be a Gorenstein R-algebra if $A \in \mathcal{G}_R$ as an R-module and $\mathrm{add}(D({}_A A)) = \mathcal{P}_A$.

In the rest of this section, we provide several basic properties of Gorenstein R-algebras. Especially, we will see that our Gorenstein R-algebras are Gorenstein algebras in the sense of [18]. However, unless otherwise stated, R is assumed to be an arbitrary commutative noetherian ring.

Remark 1.3.2. Assume A is reflexive as an R-module. Then the following hold.

(1) $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$ if and only if $\mathrm{add}(D(A_A)) = \mathcal{P}_{A^{\mathrm{op}}}$.

(2) In case R is a complete local ring, $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$ if either $A_A \in \mathrm{add}(D({}_AA))$ or $D({}_AA) \in \mathcal{P}_A$.

Proof. (1) Obvious.

(2) It follows by Lemma 1.1.9 that $A = e_1A \oplus \cdots \oplus e_nA$ with the e_i orthogonal local idempotents and every indecomposable module in $\mathcal{P}_A$ is isomorphic to some e_iA. In particular, $\mathcal{P}_A$ contains only a finite number of nonisomorphic indecomposable modules. Also, as remarked in the proof of Lemma 1.2.8, we have an equivalence $\nu : \mathcal{P}_A \xrightarrow{\sim} \mathrm{add}(D({}_AA))$. Thus $\mathcal{P}_A$ and $\mathrm{add}(D({}_AA))$ contain the same number of nonisomorphic indecomposable modules and the assertion follows. □

Lemma 1.3.3. *The following hold.*

(1) If $I \in \mathrm{Mod}\text{-}R$ is injective, so is $\mathrm{Hom}_R(A, I) \in \mathrm{Mod}\text{-}A$.

(2) Let $\mathfrak{p} \in \mathrm{Supp}(A)$ and $X \in \mathrm{Mod}\text{-}A_{\mathfrak{p}}$. Then $X \in \mathrm{Mod}\text{-}A_{\mathfrak{p}}$ is flat if and only if so is $X \in \mathrm{Mod}\text{-}A$.

Proof. (1) Obvious.

(2) The "only if" part follows by the flatness of $A_{\mathfrak{p}}$ as an A-module and the "if" part follows by the fact that $A_{\mathfrak{p}} \otimes_A A_{\mathfrak{p}} \xrightarrow{\sim} A_{\mathfrak{p}}$ canonically. □

Lemma 1.3.4. *Assume $\mathrm{Ext}^i_R(A, R) = 0$ for $i > 0$. Then the following hold.*

(1) For an injective resolution $R \to I^\bullet$ in $\mathrm{Mod}\text{-}R$, we have an injective resolution $DA \to \mathrm{Hom}^\bullet_R(A, I^\bullet)$ in $\mathrm{Mod}\text{-}A$. In particular, we have $\mathrm{inj\,dim}\, D({}_AA) \leq \mathrm{inj\,dim}\, R_R$.

(2) For any $X \in \mathrm{Mod}\text{-}A$, we have $\mathrm{Ext}^i_A(X, DA) \cong \mathrm{Ext}^i_R(X, R)$ for all $i \geq 0$.

(3) If R is a Gorenstein ring, then for any $X \in \mathrm{mod}\text{-}A$, $X \in \mathcal{G}_R$ as an R-module if and only if $\mathrm{Ext}^i_A(X, DA) = 0$ for $i > 0$.

(4) If R is a Gorenstein ring with $\dim R = \dim R_{\mathfrak{p}}$ for all maximal ideals $\mathfrak{p} \in \mathrm{Spec}(R)$, then $\mathrm{inj\,dim}\, D({}_AA) = \dim R$.

Proof. (1) follows by Lemma 1.3.3(1).

(2) Take an injective resolution $R \to I^\bullet$ in $\mathrm{Mod}\text{-}R$. Then by (1) for any $i \geq 0$ we have

$$\begin{aligned}\mathrm{Ext}^i_A(X, DA) &\cong \mathrm{H}^i(\mathrm{Hom}^\bullet_A(X, \mathrm{Hom}^\bullet_R(A, I^\bullet))) \\ &\cong \mathrm{H}^i(\mathrm{Hom}^\bullet_R(X, I^\bullet)) \\ &\cong \mathrm{Ext}^i_R(X, R).\end{aligned}$$

(3) The "only if" part follows by (2). Assume $\mathrm{Ext}_A^i(X, DA) = 0$ for $i > 0$. Then by (2) $\mathrm{Ext}_R^i(X, R) = 0$ for $i > 0$. Take a projective resolution $P^\bullet \to X$ in mod-R and set $Q^\bullet = \mathrm{Hom}_R^\bullet(P^\bullet, R) \in \mathcal{K}^+(\mathcal{P}_R)$. We have only to show that $\mathrm{Ext}_R^i(\mathrm{Z}'^1(Q^\bullet), R) = 0$ for $i > 0$ (see [7, Proposition 3.8]). Note that $\mathrm{H}^i(Q^\bullet) = 0$ for $i > 0$. Thus for any $i > 0$ and $\mathfrak{p} \in \mathrm{Spec}(R)$ we have

$$\begin{aligned}\mathrm{Ext}_R^i(\mathrm{Z}'^1(Q^\bullet), R)_\mathfrak{p} &\cong \mathrm{Ext}_R^{i+j}(\mathrm{Z}'^{1+j}(Q^\bullet), R)_\mathfrak{p} \\ &\cong \mathrm{Ext}_{R_\mathfrak{p}}^{i+j}(\mathrm{Z}'^{1+j}(Q^\bullet)_\mathfrak{p}, R_\mathfrak{p}) \\ &= 0\end{aligned}$$

for $j \geq \dim R_\mathfrak{p}$. Thus $\mathrm{Ext}_R^i(\mathrm{Z}'^1(Q^\bullet), R) = 0$ for $i > 0$.

(4) Take a maximal ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ with $R/\mathfrak{p} \otimes_R A \neq 0$. Let $d = \dim R_\mathfrak{p} = \dim R$. Note that $d < \infty$. Then, since $R/\mathfrak{p} \otimes_R A$ is a finite direct sum of copies of $R/\mathfrak{p}$ in Mod-R, and since $\mathrm{Ext}_R^d(R/\mathfrak{p}, R) \neq 0$, we have $\mathrm{Ext}_R^d(R/\mathfrak{p} \otimes_R A, R) \neq 0$ and hence by (2) $\mathrm{Ext}_A^d(R/\mathfrak{p} \otimes_R A, DA) \neq 0$. The assertion follows by (1). $\square$

Definition 1.3.5 (cf. [12]). A left and right noetherian ring A is said to satisfy the Auslander condition if it admits an injective resolution $A \to E^\bullet$ in Mod-A such that flat dim $E^n \leq n$ for all $n \geq 0$.

Proposition 1.3.6. *Assume R is a Gorenstein ring, $A \in \mathcal{G}_R$ as an R-module and ${}_AA \in \mathrm{add}(D(A_A))$. Then the following hold.*

(1) inj dim ${}_{A_\mathfrak{p}}A_\mathfrak{p} \leq \dim R_\mathfrak{p}$ *for all* $\mathfrak{p} \in \mathrm{Supp}(A)$.

(2) For any $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ with $P^\bullet \neq 0$ in $\mathcal{K}(\mathrm{Mod}\text{-}A)$, $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, A[i]) \neq 0$ for some $i \in \mathbb{Z}$.

(3) A satisfies the Auslander condition.

Proof. (1) Note that $\mathrm{Ext}_{R_\mathfrak{p}}^i(A_\mathfrak{p}, R_\mathfrak{p}) \cong \mathrm{Ext}_R^i(A, R)_\mathfrak{p} = 0$ for $i > 0$ and $D(A_A)_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})$ in Mod-$A_\mathfrak{p}^{\mathrm{op}}$. Thus we can apply Lemma 1.3.4(1) to $A_\mathfrak{p}^{\mathrm{op}}$ to conclude that inj dim $D(A_A)_\mathfrak{p} \leq \dim R_\mathfrak{p}$ as a left $A_\mathfrak{p}$-module. Then, since ${}_AA \in \mathrm{add}(D(A_A))$, we have ${}_{A_\mathfrak{p}}A_\mathfrak{p} \in \mathrm{add}(D(A_A)_\mathfrak{p})$ and hence inj dim ${}_{A_\mathfrak{p}}A_\mathfrak{p} \leq \dim R_\mathfrak{p}$.

(2) Let $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$. Set $Q^\bullet = \mathrm{Hom}_A^\bullet(P^\bullet, A) \in \mathcal{K}^+(\mathcal{P}_{A^{\mathrm{op}}})$ and assume $\mathrm{H}^i(Q^\bullet) \cong \mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, A[i]) = 0$ for all $i \in \mathbb{Z}$. We claim that $P^\bullet = 0$ in $\mathcal{K}(\mathrm{Mod}\text{-}A)$. It suffices to show that $\mathrm{H}^i(P^\bullet)_\mathfrak{p} = 0$ for all $i \in \mathbb{Z}$ and $\mathfrak{p} \in \mathrm{Spec}(R)$. Let $\mathfrak{p} \in \mathrm{Spec}(R)$. For any $X \in$ mod-A^{op} we have a functorial isomorphism

$$\mathrm{Hom}_{A^{\mathrm{op}}}(X, A)_\mathfrak{p} \xrightarrow{\sim} \mathrm{Hom}_{A_\mathfrak{p}^{\mathrm{op}}}(X_\mathfrak{p}, A_\mathfrak{p}).$$

Thus for any $i \in \mathbb{Z}$ we have

$$\begin{aligned}\mathrm{H}^i(P^\bullet)_\mathfrak{p} &\cong \mathrm{H}^i(\mathrm{Hom}_{A^{\mathrm{op}}}^\bullet(Q^\bullet, A))_\mathfrak{p} \\ &\cong \mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A^{\mathrm{op}})}(Q^\bullet, A[i])_\mathfrak{p} \\ &\cong \mathrm{Ext}_{A^{\mathrm{op}}}^j(\mathrm{Z}'^{-i+j}(Q^\bullet), A)_\mathfrak{p} \\ &\cong \mathrm{Ext}_{A_\mathfrak{p}^{\mathrm{op}}}^j(\mathrm{Z}'^{-i+j}(Q^\bullet)_\mathfrak{p}, A_\mathfrak{p})\end{aligned}$$

for all $j > 0$. It follows by (1) that $\mathrm{H}^i(P^\bullet)_\mathfrak{p} = 0$.

(3) By Lemma 1.3.4(1), it suffices to show that flat dim $\mathrm{Hom}_R(A, E(R/\mathfrak{p}))_A \leq$ dim $R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, where $E(R/\mathfrak{p})$ denotes an injective envelope of $R/\mathfrak{p}$ in Mod-R. Note that $E(R/\mathfrak{p}) \in$ Mod-$R_\mathfrak{p}$ and hence $\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, E(R/\mathfrak{p})) \cong \mathrm{Hom}_R(A, E(R/\mathfrak{p}))$ in Mod-$A_\mathfrak{p}$. Thus we may assume $\mathfrak{p} \in \mathrm{Supp}(A)$ and by Lemma 1.3.3(2) we have

$$\text{flat dim } \mathrm{Hom}_R(A, E(R/\mathfrak{p}))_A = \text{flat dim } \mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, E(R/\mathfrak{p}))_{A_\mathfrak{p}}.$$

On the other hand, since by (1) inj dim ${}_{A_\mathfrak{p}}A_\mathfrak{p} \leq$ dim $R_\mathfrak{p}$, for any $i >$ dim $R_\mathfrak{p}$ and $X \in$ mod-$A_\mathfrak{p}^{\mathrm{op}}$ we have

$$\begin{aligned} \mathrm{Tor}_i^{A_\mathfrak{p}}(\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, E(R/\mathfrak{p})), X) &\cong \mathrm{Hom}_{R_\mathfrak{p}}(\mathrm{Ext}^i_{A_\mathfrak{p}^{\mathrm{op}}}(X, A_\mathfrak{p}), E(R/\mathfrak{p})) \\ &= 0 \end{aligned}$$

and hence flat dim $\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, E(R/\mathfrak{p}))_{A_\mathfrak{p}} \leq$ dim $R_\mathfrak{p}$. □

Proposition 1.3.7. *Assume R is a Gorenstein ring and A is a Gorenstein R-algebra. Then for any $\mathfrak{p} \in \mathrm{Supp}(A)$ the following hold.*

(1) $A_\mathfrak{p}$ is a Gorenstein $R_\mathfrak{p}$-algebra.

(2) $A_\mathfrak{p}$ is maximal Cohen-Macaulay as an $R_\mathfrak{p}$-module.

(3) inj dim ${}_{A_\mathfrak{p}}A_\mathfrak{p} =$ inj dim $A_{\mathfrak{p}A_\mathfrak{p}} =$ dim $R_\mathfrak{p}$.

Proof. (1) Note that $D({}_AA)_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})$ in Mod-$A_\mathfrak{p}$. Thus we have $\mathrm{add}(\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})_{A_\mathfrak{p}}) = \mathcal{P}_{A_\mathfrak{p}}$. Also, $\mathrm{Ext}^i_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p}) \cong \mathrm{Ext}^i_R(A, R)_\mathfrak{p} = 0$ for $i > 0$. Thus by Lemma 1.3.4(3) $A_\mathfrak{p} \in \mathcal{G}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module.

(2) Note that by (1) $A_\mathfrak{p} \in \mathcal{G}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module. Take a projective resolution $P^\bullet \to \mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})$ in mod-$R_\mathfrak{p}$ and set $Q^\bullet = \mathrm{Hom}^\bullet_{R_\mathfrak{p}}(P^\bullet, R_\mathfrak{p}) \in \mathcal{K}^+(\mathcal{P}_{R_\mathfrak{p}})$. Then we have an exact sequence in mod-$R_\mathfrak{p}$

$$0 \to A_\mathfrak{p} \to Q^0 \to Q^1 \to \cdots$$

and the assertion follows.

(3) By Lemma 1.3.4(4) inj dim $\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})_{A_\mathfrak{p}} =$ dim $R_\mathfrak{p}$. Thus, since by (1) $\mathrm{add}(\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})_{A_\mathfrak{p}}) = \mathcal{P}_{A_\mathfrak{p}}$, inj dim $A_{\mathfrak{p}A_\mathfrak{p}} =$ dim $R_\mathfrak{p}$. By symmetry, we also have inj dim ${}_{A_\mathfrak{p}}A_\mathfrak{p} =$ dim $R_\mathfrak{p}$. □

Assume R is a Gorenstein ring and A is a Gorenstein R-algebra. It then follows by (2), (3) of Proposition 1.3.7 that A is a Gorenstein algebra in the sense of [18]. So we refer to [18] for further properties enjoyed by A and for the relationship of the notion of Gorenstein algebras to the theory of commutative Gorenstein rings. Also, in case R is a semilocal ring with dim $R =$ dim $R_\mathfrak{p}$ for all maximal ideals $\mathfrak{p} \in \mathrm{Spec}(R)$ and $A \cong DA$ in Mod-A^{e}, it follows by Proposition 1.3.7(2) that A is a Gorenstein R-order in the sense of [6].

There is another notion of Gorenstein algebras. Consider the case where

R is an artinian Gorenstein ring. Then an R-algebra A is sometimes called a Gorenstein algebra if inj dim ${}_AA =$ inj dim $A_A < \infty$ (see e.g. [8]). It follows by [34, Proposition 1.6] that an R-algebra A is a Gorenstein algebra in this sense if and only if $D({}_AA)$ is a tilting module. In the following, we will extend this fact to the case where R is a Gorenstein ring with dim $R < \infty$.

Definition 1.3.8. A module $T \in \text{Mod-}A$ is said to be a tilting module if there exists a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $P^\bullet \cong T$ in $\mathcal{D}(\text{Mod-}A)$, i.e., $\mathrm{H}^i(P^\bullet) = 0$ for $i \neq 0$ and $\mathrm{H}^0(P^\bullet) \cong T$ in Mod-A.

Proposition 1.3.9 (cf. [34]). *A module $T \in \text{Mod-}A$ is a tilting module if and only if the following conditions are satisfied:*

(1) $\mathrm{Ext}^i_A(T,T) = 0$ *for* $i > 0$;

(2) there exists an exact sequence $0 \to P^{-l} \to \cdots \to P^0 \to T \to 0$ *in* Mod-A *with* $P^{-i} \in \mathcal{P}_A$ *for all* $0 \leq i \leq l$; *and*

(3) there exists an exact sequence $0 \to A \to T^0 \to \cdots \to T^m \to 0$ *in* Mod-A *with* $T^i \in \mathrm{add}(T)$ *for all* $0 \leq i \leq m$.

Proof. This is well known but for the benefit of the reader we include a proof.

"If" part. By the condition (2) we have a projective resolution $P^\bullet \to T$ in Mod-A with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$. Then $P^\bullet \cong T$ in $\mathcal{D}(\text{Mod-}A)$ and by the condition (1) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$. Finally, for any $X^\bullet \in \mathcal{D}^-(\text{Mod-}A)$ with $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, X^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$, by the condition (3) we have $\mathrm{H}^i(X^\bullet) \cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(A, X^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$ and hence $X^\bullet = 0$ in $\mathcal{D}(\text{Mod-}A)$. Thus by Remark 1.2.5 $P^\bullet$ is a tilting complex.

"Only if" part. According to Remark 1.1.7, we have a projective resolution $P^\bullet \to T$ in Mod-A with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ a tilting complex. Thus the conditions (1), (2) are satisfied. Let $B = \mathrm{End}_A(T)$. Then $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet) \cong B$ and there exists an equivalence of triangulated categories $F : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $F(B) \cong P^\bullet$. Let $F^* : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ be a quasi-inverse of F. Then $Q^\bullet = \mathrm{Hom}^\bullet_B(F^*(A), B) \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B^{\mathrm{op}}})$ is a tilting complex with $\mathrm{End}_{\mathcal{K}(\text{Mod-}B^{\mathrm{op}})}(Q^\bullet) \cong A^{\mathrm{op}}$. Also, by Lemma 1.1.5 $Q^\bullet$ is a projective resolution of T in Mod-B^{op}. Thus $\mathrm{End}_{B^{\mathrm{op}}}(T) \cong A^{\mathrm{op}}$ and we have a right resolution $A \to \mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(Q^\bullet, T)$ in Mod-A. Since every $\mathrm{Hom}_{B^{\mathrm{op}}}(Q^i, T)$ belongs to $\mathrm{add}(T_A)$, the condition (3) is satisfied. □

Proposition 1.3.10. *Assume R is a Gorenstein ring with* dim $R < \infty$ *and* $A \in \mathcal{G}_R$ *as an R-module. Then the following hold.*

(1) proj dim $D({}_AA) < \infty$ *if and only if* inj dim ${}_AA < \infty$.

(2) $D({}_AA)$ *is a tilting module if and only if* inj dim ${}_AA =$ inj dim $A_A < \infty$.

(3) If $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$, *then* inj dim ${}_AA =$ inj dim $A_A \leq$ dim R.

Proof. (1) "If" part. For any injective $I \in \text{Mod-}R$ and any $X \in \text{mod-}A^{\text{op}}$ we have

$$\text{Tor}_i^A(\text{Hom}_R(A, I), X) \cong \text{Hom}_R(\text{Ext}^i_{A^{\text{op}}}(X, A), I)$$

for all $i \geq 0$ and hence flat dim $\text{Hom}_R(A, I)_A < \infty$. Then by Lemma 1.3.4(1) flat dim $D({}_AA) < \infty$. Finally, since $D({}_AA) \in \text{mod-}A$, flat dim $D({}_AA) =$ proj dim $D({}_AA)$.

"Only if" part. Take a projective resolution $P^\bullet \to DA$ in Mod-A with $P^\bullet \in \mathcal{K}^{\text{b}}(\mathcal{P}_A)$. Then we have a right resolution $A \to DP^\bullet$ in Mod-A^{op}. Since by applying Lemma 1.3.4(1) to A^{op} we have inj dim $D(A_A) < \infty$, and since every term of $DP^\bullet$ belongs to $\text{add}(D(A_A))$, it follows that inj dim ${}_AA < \infty$.

(2) "If" part. By applying (1) to both A and A^{op} we have proj dim $D({}_AA) < \infty$ and proj dim $D(A_A) < \infty$. Also, by applying Lemma 1.3.4(2) to both A and A^{op} we have $\text{Ext}^i_A(DA, DA) = \text{Ext}^i_{A^{\text{op}}}(DA, DA) = 0$ for $i > 0$. Since $A \xrightarrow{\sim} \text{End}_A(DA)$ and $A \xrightarrow{\sim} \text{End}_{A^{\text{op}}}(DA)^{\text{op}}$ canonically, the assertion follows by [34, Proposition 1.6].

"Only if" part. Since $A \xrightarrow{\sim} \text{End}_A(DA)$ canonically, it follows by [34, Theorem 1.5] that $D(A_A)$ is also a tilting module. Thus by applying (1) to both A and A^{op} we have inj dim ${}_AA < \infty$ and inj dim $A_A < \infty$. The assertion follows by [44, Lemma A].

(3) By Lemma 1.3.4(1) inj dim $D({}_AA) \leq \dim R$ and, since $A_A \in \text{add}(D({}_AA))$, inj dim $A_A \leq \dim R$. By symmetry, we also have inj dim ${}_AA \leq \dim R$. The assertion follows by [44, Lemma A]. □

1.4 Derived equivalences in Gorenstein algebras

In this section, for a tilting complex $P^\bullet$ over a Gorenstein R-algebra A we ask when $B = \text{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$ is also a Gorenstein R-algebra. This question does not seem to depend on the base ring R. So, unless otherwise stated, we assume R is an arbitrary commutative noetherian ring.

We fix a complex $P^\bullet \in \mathcal{K}^{\text{b}}(\mathcal{P}_A)$ such that $P^\bullet \neq 0$ in $\mathcal{K}(\text{Mod-}A)$ and $\text{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$. Set $B = \text{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$ and $X^\bullet = \text{Hom}^\bullet_A(P^\bullet, P^\bullet) \in \mathcal{K}^{\text{b}}(\text{Mod-}R)$. Note that $X^i \in \text{add}(A_R)$ for all $i \in \mathbb{Z}$. Since $\text{H}^i(X^\bullet) = 0$ for $i \neq 0$, we have exact sequences in mod-R of the form

$$(*) \qquad 0 \to \text{Z}^0(X^\bullet) \to X^0 \to \cdots \to X^l \to 0,$$

$$(**) \quad 0 \to X^{-l} \to \cdots \to X^{-1} \to \text{Z}^0(X^\bullet) \to B \to 0.$$

Lemma 1.4.1. *The following hold.*

(1) Assume $\text{Ext}^i_R(A, R) = 0$ *for* $i > 0$. *Then* $\text{Ext}^i_R(B, R) = 0$ *for* $i > 0$ *if and only if* $\nu P^\bullet \in \mathcal{S}(P^\bullet)$.

(2) Assume $A \in \mathcal{G}_R$ *as an* R*-module. Then* $B \in \mathcal{G}_R$ *as an* R*-module if and only if* $\nu P^\bullet \in \mathcal{S}(P^\bullet)$.

(3) Assume $A \in \mathcal{P}_R$ as an R-module. Then $B \in \mathcal{P}_R$ as an R-module if and only if $\nu P^\bullet \in \mathcal{S}(P^\bullet)$.

Proof. The "only if" parts of (2), (3) follow by (1).

(1) Apply D to $(*)$. Then $DX^0 \to D\mathrm{Z}^0(X^\bullet)$ is epic and $\mathrm{Ext}^i_R(\mathrm{Z}^0(X^\bullet), R) = 0$ for $i > 0$. Next, apply D to $(**)$. Then

$$\begin{aligned}\mathrm{Ext}^1_R(B, R) &\cong \mathrm{Cok}(D\mathrm{Z}^0(X^\bullet) \to D\mathrm{B}^0(X^\bullet)) \\ &\cong \mathrm{Cok}(DX^0 \to D\mathrm{B}^0(X^\bullet)) \\ &\cong \mathrm{H}^1(DX^\bullet)\end{aligned}$$

and $\mathrm{Ext}^i_R(B, R) \cong \mathrm{Ext}^{i-1}_R(\mathrm{B}^0(X^\bullet), R) \cong \mathrm{H}^i(DX^\bullet)$ for $i > 1$. Since by Lemma 1.2.1

$$\begin{aligned}\mathrm{H}^i(DX^\bullet) &\cong \mathrm{H}^i(\mathrm{Hom}^\bullet_A(P^\bullet, \nu P^\bullet)) \\ &\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet[i])\end{aligned}$$

for all $i \in \mathbb{Z}$, and since by Corollary 1.2.3 $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet[i]) = 0$ for $i < 0$, the assertion follows.

(2) "If" part. Note that $X^i \in \mathcal{G}_R$ for all $i \in \mathbb{Z}$. Applying Lemma 1.1.3(1) successively to $(*)$, we conclude that $\mathrm{Z}^0(X^\bullet) \in \mathcal{G}_R$. Next, since by (1) $\mathrm{Ext}^i_R(B, R) = 0$ for $i > 0$, by applying Lemma 1.1.3(2) successively to $(**)$, we conclude that $B \in \mathcal{G}_R$ as an R-module.

(3) "If" part. By $(*)$ we have $\mathrm{Z}^0(X^\bullet) \in \mathcal{P}_R$. Since by (1) $\mathrm{Ext}^i_R(B, R) = 0$ for $i > 0$, it follows by $(**)$ that $B \in \mathcal{P}_R$ as an R-module. □

Lemma 1.4.2. *For any $\mathfrak{p} \in \mathrm{Supp}(A)$ with $A_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module the following are equivalent.*

(1) $B_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module.

(2) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet[i])_\mathfrak{p} = 0$ for $i \neq 0$, this is the case if $\nu P^\bullet \in \mathcal{S}(P^\bullet)$.

Proof. For any $X \in \text{mod-}A$ and $Y \in \text{Mod-}A$ we have a bifunctorial isomorphism

$$\mathrm{Hom}_A(X, Y)_\mathfrak{p} \xrightarrow{\sim} \mathrm{Hom}_{A_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p}).$$

Also, for any $X \in \text{mod-}A$ we have functorial isomorphisms in $\text{Mod-}A_\mathfrak{p}$

$$\begin{aligned}(\nu X)_\mathfrak{p} &\cong \mathrm{Hom}_{R_\mathfrak{p}}(\mathrm{Hom}_A(X, A)_\mathfrak{p}, R_\mathfrak{p}) \\ &\cong \mathrm{Hom}_{R_\mathfrak{p}}(\mathrm{Hom}_{A_\mathfrak{p}}(X_\mathfrak{p}, A_\mathfrak{p}), R_\mathfrak{p}).\end{aligned}$$

Thus we can apply Lemma 1.4.1(3) to $P^\bullet \otimes^\bullet_R R_\mathfrak{p} \in \mathcal{K}^\mathrm{b}(\mathcal{P}_{A_\mathfrak{p}})$ (cf. [41, Theorem 2.1]). □

Theorem 1.4.3. *Assume $A \cong DA$ in $\text{Mod-}A^\mathrm{e}$ and $A \in \mathcal{G}_R$ as an R-module. Then the following hold.*

(1) $B \cong DB$ in $\text{Mod-}B^\mathrm{e}$ and $B \in \mathcal{G}_R$ as an R-module.

(2) If $A \in \mathcal{P}_R$ as an R-module, then $B \in \mathcal{P}_R$ as an R-module.

(3) For any $\mathfrak{p} \in \operatorname{Supp}(A)$, if $A_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module, then $B_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module.

Proof. By Proposition 1.2.10 $B \cong DB$ in Mod-B^{e}. Also, by Lemma 1.2.9(1) $\nu P^\bullet \in \mathcal{S}(P^\bullet)$. The assertions follow by Lemmas 1.4.1(2), 1.4.1(3) and 1.4.2, respectively. □

Throughout the rest of this section, we assume $P^\bullet$ is a tilting complex. Then A, B are derived equivalent and hence there exists a tilting complex $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ such that $A \cong \operatorname{End}_{\mathcal{K}(\text{Mod-}B)}(Q^\bullet)$.

Remark 1.4.4. We have $\operatorname{Supp}(A) = \operatorname{Supp}(B)$.

Proof. It follows by $(*)$, $(**)$ that for any $\mathfrak{p} \in \operatorname{Spec}(R)$ with $A_\mathfrak{p} = 0$ we have $B_\mathfrak{p} = 0$. By symmetry, the assertion follows. □

Theorem 1.4.5. *Assume $A \in \mathcal{G}_R$ as an R-module and $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$. Then the following are equivalent.*

(1) $B \in \mathcal{G}_R$ as an R-module and $\operatorname{add}(D({}_BB)) = \mathcal{P}_B$.

(2) $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ and $P^\bullet \in \mathcal{S}(\nu P^\bullet)$.

(3) $\operatorname{add}(P^\bullet) = \operatorname{add}(\nu P^\bullet)$.

Proof. By Proposition 1.2.7 we have an equivalence

$$\operatorname{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, -) : \mathcal{S}(P^\bullet) \xrightarrow{\sim} \text{Mod-}B.$$

Also, by Lemma 1.2.2(2) $\operatorname{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet) \cong DB$ in Mod-B. The assertion follows by Lemmas 1.2.8, 1.4.1(2). □

According to Lemma 1.4.1(3), we can replace $\mathcal{G}_R$ by $\mathcal{P}_R$ in Theorem 1.4.5.

Corollary 1.4.6. *Assume $A \in \mathcal{P}_R$ as an R-module and $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$. Then the following are equivalent.*

(1) $B \in \mathcal{P}_R$ as an R-module and $\operatorname{add}(D({}_BB)) = \mathcal{P}_B$.

(2) $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ and $P^\bullet \in \mathcal{S}(\nu P^\bullet)$.

(3) $\operatorname{add}(P^\bullet) = \operatorname{add}(\nu P^\bullet)$.

Example 1.4.7. Assume R contains a regular element c which is not a unit. Let

$$A = \begin{pmatrix} R & R \\ cR & R \end{pmatrix}$$

be an R-algebra which is free of rank 4 as an R-module. We construct a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $\nu P^\bullet \notin \mathcal{S}(P^\bullet)$. Set

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $\nu(e_1A) \cong e_2A$ and $\nu(e_2A) \cong e_1A$. In particular, $D({}_AA) \cong A_A$, so that A is a Gorenstein R-algebra if R is a Gorenstein ring. Set $P_1^\bullet = e_1A[1]$ and let $P_2^\bullet$ be the mapping cone of $h : e_1A \to e_2A, x \mapsto ax$. Then Cok $h \cong R/cR$ in Mod-R and $\mathrm{Hom}_R(\mathrm{Cok}\ h, e_1A) = 0$. Thus $\mathrm{Hom}_A(\mathrm{Cok}\ h, e_1A) = 0$ and by [25, Proposition 1.2] $P^\bullet = P_1^\bullet \oplus P_2^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is a tilting complex. On the other hand, $\nu P_2^\bullet$ is isomorphic to the mapping cone of $e_2A \to e_1A, x \mapsto bx$, and hence $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P_1^\bullet, \nu P_2^\bullet[1]) \neq 0$. Thus $\nu P^\bullet \notin \mathcal{S}(P^\bullet)$ and by Lemma 1.4.1(1) $\mathrm{Ext}^1_R(B, R) \neq 0$, where $B = \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. More precisely, we have an R-algebra isomorphism

$$B \cong \begin{pmatrix} R & R/cR \\ 0 & R/cR \end{pmatrix}.$$

At present, we do not have any example of tilting complexes $P^\bullet$ over a Gorenstein R-algebra A such that $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ and $\mathrm{add}(P^\bullet) \neq \mathrm{add}(\nu P^\bullet)$. In case R is an artinian Gorenstein ring, it follows by the exactness of D that for any tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ we have $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ (cf. [25, Lemma 3.1]).

Proposition 1.4.8. *Assume $A, B \in \mathcal{G}_R$ as R-modules. Then the following hold.*

(1) $A \in \mathcal{P}_R$ as an R-module if and only if $B \in \mathcal{P}_R$ as an R-module.

(2) For any $\mathfrak{p} \in \mathrm{Supp}(A)$, $A_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module if and only if $B_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module.

(3) If $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$, then $D({}_BB)$ is a tilting module.

Proof. (1) follows by (2), (3) of Lemma 1.4.1 and (2) follows by Lemmas 1.4.1(2), 1.4.2.

(3) By Lemma 1.2.8 $\nu P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is a tilting complex and by Lemma 1.4.1(2) $\nu P^\bullet \in \mathcal{S}(P^\bullet)$. Let $F^* : \mathcal{D}^-(\text{Mod-}A) \xrightarrow{\sim} \mathcal{D}^-(\text{Mod-}B)$ be the equivalence of triangulated categories stated in Proposition 1.2.7. Then we have $F^*(\nu P^\bullet) \cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet)$ in $\mathcal{D}(\text{Mod-}B)$. Since by Lemma 1.2.2(2) we have $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet) \cong DB$ in Mod-B, the assertion follows. □

Proposition 1.4.9. *Assume R is a Gorenstein ring with* $\dim R < \infty$ *and $A, B \in \mathcal{G}_R$ as R-modules. Then $D({}_AA)$ is a tilting module if and only if so is $D({}_BB)$.*

Proof. By [30, Proposition 1.7(2)], inj dim $A_A < \infty$ if and only if inj dim $B_B < \infty$. Note also that A^{op}, B^{op} are derived equivalent ([39, Proposition 9.1]). Thus inj dim ${}_AA < \infty$ if and only if inj dim ${}_BB < \infty$. According to [44, Lemma A], the assertion follows by Proposition 1.3.10(2). □

1.5 Suitable tilting complexes

In this section, R is an arbitrary commutative noetherian ring. Following [26], we provide a way to construct tilting complexes $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $\mathrm{add}(T^\bullet) = \mathrm{add}(\nu T^\bullet)$.

We start by formulating the argument in [14, Lemma of 1.2] as follows.

Lemma 1.5.1. *Let $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex. Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $P^\bullet \neq 0$ in $\mathcal{K}(\text{Mod-}A)$ and with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and form a distinguished triangle in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$*

$$Q^\bullet \to \bigoplus^n P^\bullet \xrightarrow{f} T^\bullet \to$$

such that $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, f)$ is epic. Then $Q^\bullet \oplus P^\bullet$ is a tilting complex if the following conditions are satisfied:

(1) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, T^\bullet[i]) = 0$ unless $-1 \leq i \leq 0$;

(2) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, P^\bullet[i]) = 0$ for $i > 1$;

(3) $P^\bullet \in \mathrm{add}(\nu P^\bullet)$; and

(4) $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \leq i < a(Q^\bullet) - b(P^\bullet) - 1$.

Proof. Note first that such a homomorphism f exists. Since $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, T^\bullet) \cong \mathrm{H}^0(\mathrm{Hom}^\bullet_A(P^\bullet, T^\bullet)) \in \text{mod-}R$, it follows that $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, T^\bullet)$ is finitely generated over $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. Let $f_1, \cdots, f_n \in \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, T^\bullet)$ be generators over $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$ and set

$$f = (f_1, \cdots, f_n) : \bigoplus^n P^\bullet \to T^\bullet.$$

Then $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, f)$ is epic.

Obviously, $\mathrm{add}(Q^\bullet \oplus P^\bullet)$ generates $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as a triangulated category.

Claim. The following hold.

(1) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, Q^\bullet[i]) = 0$ for $i \neq 0$.

(2) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$.

(3) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, Q^\bullet[i]) = 0$ for $i > 1$.

(4) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, T^\bullet[i]) = 0$ for $i < -1$.

Proof. (1), (3) and (4) follow by the construction.

(2) Let $i > 0$. By the construction, $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, P^\bullet[i]) = 0$. Next, since

$$\begin{aligned} a(Q^\bullet[i]) - b(P^\bullet) &= a(Q^\bullet) - i - b(P^\bullet) \\ &\leq a(Q^\bullet) - b(P^\bullet) - 1, \end{aligned}$$

by (1) and Lemma 1.2.2(1) we have $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet[i], \nu P^\bullet) = 0$. It then follows that $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet[i], P^\bullet) = 0$.

Now, by (1), (3) of Claim we have $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, Q^\bullet[i]) = 0$ for $i > 0$ and by (2), (4) of Claim we have $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, Q^\bullet[i]) = 0$ for $i < 0$. □

Throughout the rest of this section, we fix a sequence of idempotents $e_0, e_1, \cdots$ in A such that $\mathrm{add}(e_0A_A) = \mathcal{P}_A$ and $e_{i+1} \in e_iAe_i$ for all $i \geq 0$. We will construct inductively a sequence of complexes $T_0^\bullet, T_1^\bullet, \cdots$ in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as follows. Set $T_0^\bullet = e_0A$. Let $k \geq 1$ and assume $T_0^\bullet, T_1^\bullet, \cdots, T_{k-1}^\bullet$ have been constructed. Then we form a distinguished triangle in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$

$$Q_k^\bullet \to \bigoplus^{n_k} e_kA \xrightarrow{f_k} T_{k-1}^\bullet \to$$

such that $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(e_kA, f_k)$ is epic and set $T_k^\bullet = Q_k^\bullet \oplus e_kA$.

Lemma 1.5.2. *For any $l \geq 0$ the following hold.*

(1) $T_l^i = 0$ *unless* $0 \leq i \leq l$.

(2) $T_l^i \in \mathrm{add}(e_{l-i}A_A)$ *for* $0 \leq i \leq l$.

(3) $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(e_lA, T_l^\bullet[i]) = 0$ *for* $i > 0$.

(4) $\mathrm{add}(T_l^\bullet)$ *generates* $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ *as a triangulated category.*

Proof. By induction on $l \geq 0$. □

Lemma 1.5.3. *For any $l \geq 1$ the following hold.*

(1) $\mathrm{H}^j(T_l^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_{l-i}A)$ *for* $0 \leq i < j \leq l$.

(2) If $D(e_iA_A) \in \mathrm{add}(_AAe_i)$ *for* $1 \leq i \leq l$, *then* $\mathrm{H}^j(\nu T_l^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_{l-i}A)$ *for* $0 \leq i < j \leq l$.

Proof. (1) We have $\mathrm{H}^j(T_l^\bullet) = \mathrm{H}^j(Q_l^\bullet) \cong \mathrm{H}^{j-1}(T_{l-1}^\bullet) \cong \cdots \cong \mathrm{H}^1(T_{l-j+1}^\bullet)$. Also, by Lemma 1.5.2(3)

$$\begin{aligned}\mathrm{H}^1(T_{l-j+1}^\bullet) \otimes_A Ae_{l-j+1} &\cong \mathrm{H}^1(T_{l-j+1}^\bullet \otimes_A^\bullet Ae_{l-j+1}) \\ &\cong \mathrm{H}^1(\mathrm{Hom}_A^\bullet(e_{l-j+1}A, T_{l-j+1}^\bullet)) \\ &\cong \mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(e_{l-j+1}A, T_{l-j+1}^\bullet[1]) \\ &= 0.\end{aligned}$$

Thus, since $l - i \geq l - j + 1$, it follows that $\mathrm{H}^j(T_l^\bullet) \otimes_A Ae_{l-i} = 0$.

(2) Since by (1) $\mathrm{H}^j(T_l^\bullet \otimes_A^\bullet Ae_{l-i}) \cong \mathrm{H}^j(T_l^\bullet) \otimes_A Ae_{l-i} = 0$, we have

$$\begin{aligned}\mathrm{H}^j(\nu T_l^\bullet) \otimes_A Ae_{l-i} &\cong \mathrm{H}^j(\nu T_l^\bullet \otimes_A^\bullet Ae_{l-i}) \\ &\cong \mathrm{H}^j(T_l^\bullet \otimes_A^\bullet DA \otimes_A^\bullet Ae_{l-i}) \\ &\cong \mathrm{H}^j(T_l^\bullet \otimes_A^\bullet D(e_{l-i}A)) \\ &= 0.\end{aligned}$$

□

Lemma 1.5.4 ([26, Remark 2.3]). *Let $l \geq 0$. For any* $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$, $\mathrm{add}(T^\bullet)$ *is uniquely determined if the following conditions are satisfied:*

(1) $T^i = 0$ *unless* $0 \le i \le l$*;*

(2) $T^i \in \mathrm{add}(e_{l-i}A_A)$ *for* $0 \le i \le l$*;*

(3) $\mathrm{H}^j(T^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_{l-i}A)$ *for* $0 \le i < j \le l$*; and*

(4) $\mathrm{add}(T^\bullet)$ *generates* $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ *as a triangulated category.*

Proof. We can apply [26, Remark 2.3] to $P^\bullet = T^\bullet[l]$. □

Theorem 1.5.5. *Let* $l \ge 1$ *and assume* $\mathrm{Ext}^i_R(A, R) = 0$ *for* $1 \le i < l-1$*. Then the following hold.*

(1) If $e_iA_A \in \mathrm{add}(D({}_AAe_i))$ *for* $1 \le i \le l$*, then* $T^\bullet_l$ *is a tilting complex.*

(2) If A *is reflexive as an* R*-module and* $\mathrm{add}(e_iA_A) = \mathrm{add}(D({}_AAe_i))$ *for* $0 \le i \le l$*, then* $\mathrm{add}(T^\bullet_l) = \mathrm{add}(\nu T^\bullet_l)$*.*

Proof. (1) It is obvious that $T^\bullet_0$ is a tilting complex. Thus by Lemmas 1.5.1, 1.5.2 we can make use of induction to prove that $T^\bullet_k$ is a tilting complex for $0 \le k \le l$.

(2) By (1) $T^\bullet_l$ is a tilting complex. Then, since $\mathrm{add}(e_0A_A) = \mathcal{P}_A$, we have $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$ and hence by Lemma 1.2.8 $\nu T^\bullet_l$ is also a tilting complex. Thus by Lemmas 1.5.2, 1.5.3 both $T^\bullet_l$ and $\nu T^\bullet_l$ satisfy the conditions (1)–(4) of Lemma 1.5.4 and hence $\mathrm{add}(T^\bullet_l) = \mathrm{add}(\nu T^\bullet_l)$. □

Chapter 2

Noether algebras of finite selfinjective dimension

Let R be a commutative noetherian ring and A a Noether R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. Note that Noether algebras are left and right noetherian rings. We set $A^{\mathrm{e}} = A^{\mathrm{op}} \otimes_R A$, where A^{op} denotes the opposite ring of A. Then a complex $V^\bullet \in \mathcal{K}^+(\text{Mod-}A^{\mathrm{e}})$ is said to be a dualizing complex for A if the following conditions are satisfied: (1) $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A)_{\mathrm{fid}}$ and $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A^{\mathrm{op}})_{\mathrm{fid}}$; (2) $V^\bullet \in \mathcal{D}_{\text{mod-}A}(\text{Mod-}A)$ and $V^\bullet \in \mathcal{D}_{\text{mod-}A^{\mathrm{op}}}(\text{Mod-}A^{\mathrm{op}})$; (3) $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(V^\bullet, V^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A^{\mathrm{op}})}(V^\bullet, V^\bullet[i]) = 0$ for $i \neq 0$; and (4) there exists a ring isomorphism $A \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}(\text{Mod-}A)}(V^\bullet)$ given by left multiplication and there exists a ring isomorphism $A \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}(\text{Mod-}A^{\mathrm{op}})}(V^\bullet)^{\mathrm{op}}$ given by right multiplication (see [22] for details). Note that A itself is a dualizing complex for A if and only if inj dim ${}_AA =$ inj dim $A_A < \infty$ (cf. [44, Lemma A]). Now, take a minimal injective resolution $R \to I^\bullet$ in Mod-R and set $V^\bullet = \mathrm{Hom}^\bullet_R(A, I^\bullet)$. In this chapter, we are mainly concerned with the case where $V^\bullet$ is a dualizing complex for A. We will see that $V^\bullet$ is a dualizing complex for A if and only if the following conditions are satisfied: (A1) $R_\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p} \in \mathrm{Supp}(A)$; and (A2) $\sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}(A)\} < \infty$, where $\mathrm{Supp}(A) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid A_\mathfrak{p} \neq 0\}$ and $\dim R_\mathfrak{p}$ denotes the Krull dimension of $R_\mathfrak{p}$ (Propositions 2.3.7 and 2.3.8). Assume $V^\bullet$ is a dualizing complex for A. Then we will show in Section 3 that the following statements are equivalent: (1) inj dim ${}_AA =$ inj dim $A_A < \infty$; (2) there exists a quasi-isomorphism $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ a tilting complex such that $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$; (3) there exists a quasi-isomorphism $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\mathrm{op}})$ with $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\mathrm{op}}})$ a tilting complex such that $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A^{\mathrm{op}})}(Q^\bullet)^{\mathrm{op}}$; and (4) there exist quasi-isomorphisms

This chapter is based on my paper [1].

$P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\mathrm{op}})$ with $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\mathrm{op}}})$ (Theorem 2.3.9). Namely, A itself is a dualizing complex for A if and only if $V^\bullet$ is quasi-isomorphic to tilting complexes in both sides. Assume further that A itself is a dualizing complex for A. Then we will show in Section 4 that the functor $- \otimes^{\mathbf{L}}_A V^\bullet$ induces a self-equivalence of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ (Theorem 2.4.7).

In case the base ring R is a field, $V^\bullet = \mathrm{Hom}_R(A, R)$ and our results mentioned above have been established by several authors (see e.g. [16, Theorem 2.1], [19, Chapter III, Theorem 2.10] and [34, Proposition 1.6]).

Let A be a left and right noetherian ring. We denote by Mod-A the category of right A-modules and mod-A the full subcategory of Mod-A consisting of finitely generated modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. We denote by Proj-A (resp., Inj-A, Flat-A) the full subcategory of Mod-A consisting of projective (resp., injective, flat) modules and by $\mathcal{P}_A$ the full subcategory of mod-A consisting of finitely generated projective modules. Sometimes, we use the notation M_A (resp., ${}_AM$) to stress that the module M considered is a right (resp., left) A-module. For an object X in an additive category $\mathcal{B}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of finite direct sums of copies of X. For an additive category $\mathcal{B}$, we denote by $\mathcal{K}(\mathcal{B})$ (resp., $\mathcal{K}^-(\mathcal{B})$, $\mathcal{K}^+(\mathcal{B})$, $\mathcal{K}^{\mathrm{b}}(\mathcal{B})$) the homotopy category of complexes (resp., bounded above complexes, bounded below complexes, bounded complexes) over $\mathcal{B}$. As usual, we consider objects of $\mathcal{B}$ as complexes over $\mathcal{B}$ concentrated in degree zero. For an abelian category $\mathcal{A}$, we denote by $\mathcal{D}(\mathcal{A})$ (resp., $\mathcal{D}^-(\mathcal{A})$, $\mathcal{D}^+(\mathcal{A})$, $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$) the derived category of complexes (resp., complexes which have bounded above homology, complexes which have bounded below homology, complexes which have bounded homology) over $\mathcal{A}$. We always consider $\mathcal{K}^*(\mathcal{B})$ (resp., $\mathcal{D}^*(\mathcal{A})$) as a full triangulated subcategory of $\mathcal{K}(\mathcal{B})$ (resp., $\mathcal{D}(\mathcal{A})$), where $* = -, +$ or b. For a cochain complex $X^\bullet$ over an abelian category $\mathcal{A}$, we denote by $\mathrm{Z}^n(X^\bullet)$, $\mathrm{Z}'^n(X^\bullet)$ and $\mathrm{H}^n(X^\bullet)$ the n-th cycle, the n-th cocycle and the n-th homology of $X^\bullet$, respectively. Finally, we use the notation $\mathrm{Hom}^\bullet(-,-)$ (resp., $- \otimes^\bullet -$) to denote the single complex associated with the double hom (resp., tensor) complex and $\mathrm{Ext}^n(-,-)$ (resp., $\mathrm{Tor}^n(-,-)$) to denote the n-th hyper Ext (resp., the n-th hyper Tor), i.e., $\mathrm{Ext}^n(-,-) = \mathrm{H}^n(\mathbf{R}\mathrm{Hom}^\bullet(-,-))$ (resp., $\mathrm{Tor}^n(-,-) = \mathrm{H}^{-n}(- \otimes^{\mathbf{L}} -)$).

We refer to [13], [22] and [43] for basic results in the theory of derived categories and to [39] for definitions and basic properties of tilting complexes and derived equivalences. Also, we refer to [15] for standard homological algebra in module categories and to [33] for standard commutative ring theory.

2.1 Preliminaries

In this section, we recall several definitions and basic facts which we need in later sections. Throughout this section, A is a left and right noetherian ring.

Lemma 2.1.1. *Assume* inj dim $A_A < \infty$. *Then the following hold.*

(1) flat dim ${}_AE \le$ inj dim A_A *for all* $E \in$ Inj-A^{op}.

(2) proj dim $M_A \le$ inj dim A_A *for all* $M \in$ Mod-A *with* proj dim $M_A < \infty$.

(3) proj dim $M_A \le$ inj dim A_A *for all* $M \in$ Mod-A *with* flat dim $M_A < \infty$.

Proof. This is well known but for the benefit of the reader we include a proof. Let inj dim $A_A = d < \infty$.

(1) For any $X \in$ mod-A and $i > d$ we have

$$\mathrm{Tor}_i^A(X, E) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Ext}_A^i(X, A), E) = 0$$

(see [15, Chapter VI, Proposition 5.3]). Thus flat dim ${}_AE \le d$.

(2) Let $0 \to P^{-n} \to \cdots \to P^0 \to M \to 0$ be an exact sequence in Mod-A with the P^k projective. Assume $n > d$. Then, since inj dim $P_A^{-n} \le d$, we have $\mathrm{Ext}_A^n(M, P^{-n}) = 0$ and the inclusion $P^{-n} \to P^{-n+1}$ splits.

(3) Let $X \in$ mod-A and $P^\bullet \to X$ a projective resolution in mod-A. For any $K \in$ Flat-A and $i > d$ we have

$$\begin{aligned}\mathrm{Ext}_A^i(X, K) &\cong \mathrm{H}^i(\mathrm{Hom}_A^\bullet(P^\bullet, K)) \\ &\cong \mathrm{H}^i(K \otimes_A^\bullet \mathrm{Hom}_A^\bullet(P^\bullet, A)) \\ &\cong K \otimes_A \mathrm{H}^i(\mathrm{Hom}_A^\bullet(P^\bullet, A)) \\ &\cong K \otimes_A \mathrm{Ext}_A^i(X, A) \\ &= 0\end{aligned}$$

and inj dim $K_A \le d$. Next, let $F \in$ Flat-A and $Q^\bullet \to F$ a projective resolution in Mod-A. Set $K = \mathrm{Z}'^{-d-1}(Q^\bullet)$. Since F and the Q^k are flat, so is K. Thus inj dim $K_A \le d$ and $\mathrm{Ext}_A^{d+1}(F, K) = 0$, so that the inclusion $K \to Q^{-d}$ splits and proj dim $F_A \le d$. Finally, let $F^\bullet \to M$ be a bounded flat resolution in Mod-A. Since proj dim $F_A^k \le d$ for all $k \in \mathbb{Z}$, we have proj dim $M_A < \infty$ and hence proj dim $M_A \le d$. □

For later use, we replace A by A^{op} in Lemma 2.1.1.

Lemma 2.1.2. *Assume* inj dim ${}_AA < \infty$. *Then the following hold.*

(1) flat dim $E_A \le$ inj dim ${}_AA$ *for all* $E \in$ Inj-A.

(2) proj dim ${}_AM \le$ inj dim ${}_AA$ *for all* $M \in$ Mod-A^{op} *with* proj dim ${}_AM < \infty$.

(3) proj dim ${}_AM \le$ inj dim ${}_AA$ *for all* $M \in$ Mod-A^{op} *with* flat dim ${}_AM < \infty$.

Definition 2.1.3 ([22]). A complex $X^\bullet \in \mathcal{D}^+(\mathrm{Mod}\text{-}A)$ is said to have finite injective dimension if $\mathrm{Ext}_A^i(-, X^\bullet)$ vanishes on mod-A for $i \gg 0$. We denote by $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}A)_{\mathrm{fid}}$ the full triangulated subcategory of $\mathcal{D}^+(\mathrm{Mod}\text{-}A)$ consisting of $X^\bullet \in \mathcal{D}^+(\mathrm{Mod}\text{-}A)$ which have finite injective dimension.

Remark 2.1.4 *([22]).* For a complex $X^\bullet \in \mathcal{D}^+(\text{Mod-}A)$ the following are equivalent.

(1) $X^\bullet$ has finite injective dimension.

(2) There exists a quasi-isomorphism $X^\bullet \to E^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $E^\bullet \in \mathcal{K}^{\mathrm{b}}(\text{Inj-}A)$. In particular, $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A)$.

Definition 2.1.5 **([22]).** A complex $X^\bullet \in \mathcal{D}(\text{Mod-}A)$ is said to have finite Tor dimension if $\mathrm{Tor}_i^A(X^\bullet, -)$ vanishes on mod-A^{op} for $i \gg 0$. For $* = \mathrm{b}$ or nothing, we denote by $\mathcal{D}^*(\text{Mod-}A)_{\mathrm{fTd}}$ the full triangulated subcategory of $\mathcal{D}^*(\text{Mod-}A)$ consisting of $X^\bullet \in \mathcal{D}^*(\text{Mod-}A)$ which have finite Tor dimension.

Definition 2.1.6 **([13]).** We denote by $\mathcal{K}(\text{Proj-}A)_{\mathrm{L}}$ the full triangulated subcategory of $\mathcal{K}(\text{Proj-}A)$ consisting of $X^\bullet \in \mathcal{K}(\text{Proj-}A)$ such that $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(X^\bullet, -)$ vanishes on acyclic complexes. Then $\mathcal{K}(\text{Proj-}A)_{\mathrm{L}} \xrightarrow{\sim} \mathcal{D}(\text{Mod-}A)$ canonically as triangulated categories.

Definition 2.1.7 **([22]).** For any $X^\bullet \in \mathcal{K}(\text{Mod-}A)$ and $k \in \mathbb{Z}$, we define the following truncated complexes

$$\sigma_{\leq k}(X^\bullet) : \cdots \to X^{k-2} \to X^{k-1} \to \mathrm{Z}^k(X^\bullet) \to 0 \to \cdots,$$

$$\tau_{\leq k}(X^\bullet) : \cdots \to X^{k-2} \to X^{k-1} \to X^k \to 0 \to \cdots.$$

Lemma 2.1.8 **([22]).** *For a complex $X^\bullet \in \mathcal{K}(\text{Mod-}A)$ with $\mathrm{H}^i(X^\bullet) \in \text{mod-}A$ for all $i \in \mathbb{Z}$ and $\mathrm{H}^i(X^\bullet) = 0$ for $i \gg 0$, there exists a quasi-isomorphism $P^\bullet \to X^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$.*

Proof. Let $n \in \mathbb{Z}$ and assume $\mathrm{H}^i(X^\bullet) = 0$ for $i > n$. Then we have a quasi-isomorphism $\sigma_{\leq n}(X^\bullet) \to X^\bullet$. Then we may assume $X^\bullet \in \mathcal{K}^-(\text{Mod-}A)$. □

2.2 Gorenstein dimension

Throughout this section, R is a commutative noetherian ring and $\mu : R \to I^\bullet$ is a minimal injective resolution in Mod-R. We will provide an equivalent condition for a complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}R)$ to have finite Gorenstein dimension in the sense of [30] (see Proposition 2.2.8 below).

Definition 2.2.1. A module $M \in \text{Mod-}R$ is said to be reflexive if the canonical homomorphism

$$\varepsilon_M : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R)R), a \mapsto (h \mapsto h(a))$$

is an isomorphism.

For any $X^\bullet \in \mathcal{K}^-(\text{Mod-}R)$ we have a functorial homomorphism in $\mathcal{K}(\text{Mod-}R)$

$$\varepsilon_{X^\bullet} : X^\bullet \to \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X^\bullet, R), R)$$

such that $\varepsilon^k_{X^\bullet} = \varepsilon_{X^k}$ for all $k \in \mathbb{Z}$. Next, for each $k \in \mathbb{Z}$ we have a functorial homomorphism

$$\eta^k_M : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, I^k), I^k), a \mapsto (g \mapsto g(a))$$

for $M \in \text{Mod-}R$, where I^k is the k-th term of $I^\bullet$. Then for any $X^\bullet \in \mathcal{K}^-(\text{Mod-}R)$ we have a functorial homomorphism in $\mathcal{K}(\text{Mod-}R)$

$$\eta_{X^\bullet} : X^\bullet \to \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X^\bullet, I^\bullet), I^\bullet)$$

whose i-th term

$$\eta^i_{X^\bullet} : X^i \to \prod_{j-k+l=i} \mathrm{Hom}_R(\mathrm{Hom}_R(X^j, I^k), I^l)$$

is given by $\eta^i_{X^\bullet}(x)_{j,k,l} = \eta^k_{X^i}(x)$ if $k = l$ and $\eta^i_{X^\bullet}(x)_{j,k,l} = 0$ otherwise for all $x \in X^i$ and $j, k, l \in \mathbb{Z}$ with $j - k + l = i$.

Lemma 2.2.2. *For any $X^\bullet \in \mathcal{K}^-(\text{Mod-}R)$ we have a commutative diagram in $\mathcal{K}(\text{Mod-}R)$*

$$\begin{array}{ccc}
X^\bullet & \xrightarrow{\varepsilon_{X^\bullet}} & \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X^\bullet, R), R) \\
{\scriptstyle \eta_{X^\bullet}}\downarrow & & \downarrow{\scriptstyle \mu_\dagger} \\
\mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X^\bullet, I^\bullet), I^\bullet) & \xrightarrow[\mu_\ddagger]{} & \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X^\bullet, R), I^\bullet),
\end{array}$$

where $\mu_\dagger = \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X^\bullet, R), \mu)$ and $\mu_\ddagger = \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(X^\bullet, \mu), I^\bullet)$.

Proof. For each $n \in \mathbb{Z}$ we will check the commutativity of the following diagram

$$\begin{array}{ccc}
X^n & \xrightarrow{\varepsilon_{X^n}} & \mathrm{Hom}_R(\mathrm{Hom}_R(X^n, R), R) \\
{\scriptstyle \eta^n_{X^\bullet}}\downarrow & & \downarrow{\scriptstyle \mu^n_\dagger} \\
\prod\limits_{p-q+r=n} \mathrm{Hom}_R(\mathrm{Hom}_R(X^p, I^q), I^r) & \xrightarrow[\mu^n_\ddagger]{} & \prod\limits_{p+r=n} \mathrm{Hom}_R(\mathrm{Hom}_R(X^p, R), I^r).
\end{array}$$

Let $x \in X^n$. Let $\mu^n_\dagger(\varepsilon_{X^n}(x)) = (\varphi_{p,r})_{p+r=n}$. It is easy to see that $\varphi_{n,0}(h) = \mu(h(x))$ for $h \in \mathrm{Hom}_R(X^n, R)$ and $\varphi_{p,r} = 0$ unless $r = 0$. Next, let $\eta^n_{X^\bullet}(x) = (g_{p,q,r})_{p-q+r=n}$ and $\mu^n_\ddagger(\eta^n_{X^\bullet}(x)) = (\psi_{p,r})_{p+r=n}$. Then $g_{n,q,q}(f) = f(x)$ for $f \in \mathrm{Hom}_R(X^n, I^q)$ and $g_{p,q,r} = 0$ unless $q = r$, so that $\psi_{n,0}(h) = g_{n,0,0}(\mu h) = \mu(h(x))$ for $h \in \mathrm{Hom}_R(X^n, R)$ and $\psi_{p,r} = 0$ unless $r = 0$. Thus $\mu^n_\dagger(\varepsilon_{X^n}(x)) = \mu^n_\ddagger(\eta^n_{X^\bullet}(x))$. $\square$

For any $X^\bullet, Y^\bullet \in \mathcal{D}(\text{Mod-}R)$ we have a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(X^\bullet, \mathbf{R}\mathrm{Hom}^\bullet_R(Y^\bullet, R)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(Y^\bullet, \mathbf{R}\mathrm{Hom}^\bullet_R(X^\bullet, R)),$$

which we denote by $\theta_{X^\bullet, Y^\bullet}$. We set $\xi_{X^\bullet} = \theta^{-1}_{X^\bullet, \mathbf{R}\mathrm{Hom}^\bullet_R(X^\bullet, R)}(\mathrm{id}_{\mathbf{R}\mathrm{Hom}^\bullet_R(X^\bullet, R)})$ for $X^\bullet \in \mathcal{D}(\text{Mod-}R)$.

Lemma 2.2.3. *Let $X^\bullet \in \mathcal{D}^-(\text{Mod-}R)$ and take a quasi-isomorphism $Y^\bullet \to \text{Hom}^\bullet_R(X^\bullet, R)$ with $Y^\bullet \in \mathcal{K}(\text{Proj-}R)_{\text{L}}$. Then we have the following commutative diagrams in $\mathcal{D}(\text{Mod-}R)$*

$$\begin{tikzcd}
\text{Hom}^\bullet_R(\text{Hom}^\bullet_R(X^\bullet, I^\bullet), I^\bullet) \arrow[d, "\wr"] & X^\bullet \arrow[l, "\eta_{X^\bullet}"'] \arrow[d, "\varepsilon_{X^\bullet}"] \\
\text{Hom}^\bullet_R(\text{Hom}^\bullet_R(X^\bullet, R), I^\bullet) \arrow[d, "\wr"] & \text{Hom}^\bullet_R(\text{Hom}^\bullet_R(X^\bullet, R), R) \arrow[l] \arrow[d] \\
\text{Hom}^\bullet_R(Y^\bullet, I^\bullet) & \text{Hom}^\bullet_R(Y^\bullet, R), \arrow[l, "\sim"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
X^\bullet \arrow[r, "\xi_{X^\bullet}"] \arrow[d, "\varepsilon_{X^\bullet}"] & \mathbf{R}\text{Hom}^\bullet_R(\mathbf{R}\text{Hom}^\bullet_R(X^\bullet, R), R) \arrow[d, "\wr"] \\
\text{Hom}^\bullet_R(\text{Hom}^\bullet_R(X^\bullet, R), R) \arrow[r] \arrow[d] & \mathbf{R}\text{Hom}^\bullet_R(\text{Hom}^\bullet_R(X^\bullet, R), R) \arrow[d, "\wr"] \\
\text{Hom}^\bullet_R(Y^\bullet, R) \arrow[r, "\sim"] & \mathbf{R}\text{Hom}^\bullet_R(Y^\bullet, R).
\end{tikzcd}$$

In particular, $\eta_{X^\bullet}$ is a quasi-isomorphism if and only if $\xi_{X^\bullet}$ is an isomorphism.

Proof. We may assume $X^\bullet \in \mathcal{K}^-(\text{Proj-}R)$. By Lemma 2.2.2 the top square of the first diagram is commutative and by [30, Lemma 2.5] the top square of the second diagram is commutative . It is easy to see that the bottom squares are commutative. □

Definition 2.2.4 ([7]). A module $M \in \text{mod-}R$ is said to have Gorenstein dimension zero if M is reflexive, $\text{Ext}^i_R(M, R) = 0$ for $i > 0$ and $\text{Ext}^i_R(M^*, R) = 0$ for $i > 0$, where $(-)^* = \text{Hom}_R(-, R)$. We denote by $\mathcal{G}_R$ the full additive subcategory of mod-R consisting of modules which have Gorenstein dimension zero. Note that $\mathcal{P}_R \subset \mathcal{G}_R$. Next, a module $M \in \text{mod-}R$ is said to have finite Gorenstein dimension if M has a left resolution $P^\bullet \to M$ with $P^\bullet \in \mathcal{K}^{\text{b}}(\mathcal{G}_R)$.

Remark 2.2.5 *([7]).* For any $M \in \text{mod-}R$ the following are equivalent.

(1) M has Gorenstein dimension zero.

(2) $\text{Ext}^i_R(M, R) = 0$ for $i > 0$ and $\text{Ext}^i_R(\text{Z}'^1(Q^\bullet), R) = 0$ for $i > 0$, where $Q^\bullet = \text{Hom}^\bullet_R(P^\bullet, R)$ with $P^\bullet \to M$ a projective resolution in mod-R.

Definition 2.2.6. A complex $X^\bullet \in \mathcal{D}^{\text{b}}(\text{mod-}R)$ is said to have finite Gorenstein dimension if $X^\bullet \cong Y^\bullet$ in $\mathcal{D}(\text{Mod-}R)$ for some $Y^\bullet \in \mathcal{K}^{\text{b}}(\mathcal{G}_R)$.

Remark 2.2.7. For any $M \in \text{mod-}R$ the following are equivalent.

(1) M has finite Gorenstein dimension as a module.

(2) M has finite Gorenstein dimension as a complex.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. Conversely, let $Y^\bullet \cong M$ in $\mathcal{D}(\text{Mod-}R)$ with $Y^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{G}_R)$. Since $\mathrm{H}^i(Y^\bullet) = 0$ for $i > 0$, it follows by [7, Lemma 3.10] that $\mathrm{Z}^0(Y^\bullet) \in \mathcal{G}_R$. Thus we have a left resolution $\sigma_{\leq 0}(Y^\bullet) \to M$ with $\sigma_{\leq 0}(Y^\bullet) \in \mathcal{K}^{\mathrm{b}}(\mathcal{G}_R)$. □

Proposition 2.2.8 (cf. [30]). *For any $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}R)$ the following are equivalent.*

(1) $X^\bullet$ has finite Gorenstein dimension.

(2) $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(X^\bullet, R[i]) = 0$ for $i \gg 0$ and $\eta_{X^\bullet}$ is a quasi-isomorphism.

Proof. By Lemma 2.2.3 and [30, Proposition 2.10]. □

2.3 Main result

Throughout this section, R is a commutative noetherian ring and A is a Noether R-algebra. We assume that A has finite Gorenstein dimension as an R-module. Let $R \to I^\bullet$ be a minimal injective resolution in Mod-R and set $V^\bullet = \mathrm{Hom}^\bullet_R(A, I^\bullet) \in \mathcal{K}^+(\text{Mod-}A^{\mathrm{e}})$, where $A^{\mathrm{e}} = A^{\mathrm{op}} \otimes_R A$. Note that since $\mathrm{H}^i(V^\bullet) \cong \mathrm{Ext}^i_R(A, R) = 0$ for $i \gg 0$ we have $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A^{\mathrm{e}})$. Also, $V^\bullet \in \mathcal{K}^+(\text{Inj-}A)$ and $V^\bullet \in \mathcal{K}^+(\text{Inj-}A^{\mathrm{op}})$ because $\mathrm{Hom}_R({}_AA, E) \in \text{Inj-}A$ and $\mathrm{Hom}_R(A_A, E) \in \text{Inj-}A^{\mathrm{op}}$ for all $E \in \text{Inj-}R$.

Lemma 2.3.1. *The following hold.*

(1) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(V^\bullet, V^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A^{\mathrm{op}})}(V^\bullet, V^\bullet[i]) = 0$ for $i \neq 0$.

(2) There exist R-algebra isomorphisms $A \xrightarrow{\sim} \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(V^\bullet)$ given by left multiplication and $A \xrightarrow{\sim} \mathrm{End}_{\mathcal{K}(\text{Mod-}A^{\mathrm{op}})}(V^\bullet)^{\mathrm{op}}$ given by right multiplication.

Proof. Since A_R has finite Gorenstein dimension, by Proposition 2.2.8 we have a quasi-isomorphism

$$\eta_A : A \to \mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(A, I^\bullet), I^\bullet).$$

Also, by adjointness we have isomorphisms in $\mathcal{K}(\text{Mod-}A)$

$$\begin{aligned}\mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_R(A, I^\bullet), I^\bullet) &\cong \mathrm{Hom}^\bullet_A(\mathrm{Hom}^\bullet_R(A, I^\bullet), \mathrm{Hom}^\bullet_R(A, I^\bullet)) \\ &= \mathrm{Hom}^\bullet_A(V^\bullet, V^\bullet).\end{aligned}$$

Thus we have a quasi-isomorphism

$$\delta_A : A \to \mathrm{Hom}^\bullet_A(V^\bullet, V^\bullet)$$

and hence $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(V^\bullet, V^\bullet[i]) \cong \mathrm{H}^i(\mathrm{Hom}_A^\bullet(V^\bullet, V^\bullet)) = 0$ for $i \neq 0$ and $A \cong \mathrm{H}^0(\mathrm{Hom}_A^\bullet(V^\bullet, V^\bullet)) \cong \mathrm{End}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(V^\bullet)$ in Mod-R. Next, δ_A is given by the composite of the following maps

$$A \to \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(\mathrm{Hom}_R(A, I^k), I^k),\, a \mapsto ((f_k \mapsto f_k(a)))_{k\in\mathbb{Z}},$$

$$\prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(\mathrm{Hom}_R(A, I^k), I^k) \xrightarrow{\sim} \prod_{k\in\mathbb{Z}} \mathrm{Hom}_A(V^k, V^k)$$

which sends $((f_k \mapsto f_k(a)))_{k\in\mathbb{Z}}$ to $((f_k \mapsto af_k))_{k\in\mathbb{Z}}$. It follows that δ_A induces an R-algebra isomorphism $A \xrightarrow{\sim} \mathrm{End}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(V^\bullet)$ which is given by left multiplication. By symmetry, the assertion follows. □

Lemma 2.3.2. *Assume* inj dim ${}_AA$ = inj dim $A_A < \infty$. *Then there exist quasi-isomorphisms* $P^\bullet \to V^\bullet$ *in* $\mathcal{K}(\mathrm{Mod}\text{-}A)$ *with* $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ *and* $Q^\bullet \to V^\bullet$ *in* $\mathcal{K}(\mathrm{Mod}\text{-}A^{\mathrm{op}})$ *with* $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\mathrm{op}}})$.

Proof. We claim first that $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}A)_{\mathrm{fTd}}$. Let $M \in$ mod-A^{op} and $i >$ inj dim ${}_AA$. By Lemma 2.1.2(1) flat dim $V_A^k \leq$ inj dim ${}_AA$ for all $k \geq 0$. Thus, since for any $k \geq 1$ we have a distinguished triangle in $\mathcal{D}(\mathrm{Mod}\text{-}A)$

$$\tau_{\leq k-1}(V^\bullet) \to \tau_{\leq k}(V^\bullet) \to V^k[-k] \to,$$

we can make use of induction on $k \geq 0$ to conclude $\mathrm{Tor}_i^A(\tau_{\leq k}(V^\bullet), M) = 0$ for all $k \geq 0$. Then, since $M \in$ mod-A^{op}, we have

$$\begin{aligned}
\mathrm{Tor}_i^A(\prod_{k\geq 0} \tau_{\leq k}(V^\bullet), M) &\cong \mathrm{H}^{-i}((\prod_{k\geq 0} \tau_{\leq k}(V^\bullet)) \otimes_A^{\mathbf{L}} M) \\
&\cong \mathrm{H}^{-i}(\prod_{k\geq 0} (\tau_{\leq k}(V^\bullet) \otimes_A^{\mathbf{L}} M)) \\
&\cong \prod_{k\geq 0} \mathrm{H}^{-i}(\tau_{\leq k}(V^\bullet) \otimes_A^{\mathbf{L}} M) \\
&\cong \prod_{k\geq 0} \mathrm{Tor}_i^A(\tau_{\leq k}(V^\bullet), M) \\
&= 0.
\end{aligned}$$

Thus $\prod_{k\geq 0} \tau_{\leq k}(V^\bullet) \in \mathcal{D}(\mathrm{Mod}\text{-}A)_{\mathrm{fTd}}$. Now, since $\varprojlim \tau_{\leq k}(V^\bullet) \cong V^\bullet$ as complexes, we have an exact sequence of complexes

$$0 \to V^\bullet \to \prod_{k\geq 0} \tau_{\leq k}(V^\bullet) \xrightarrow{1-\mathrm{shift}} \prod_{k\geq 0} \tau_{\leq k}(V^\bullet) \to 0$$

which yields a distinguished triangle in $\mathcal{D}(\mathrm{Mod}\text{-}A)$ (cf. [13])

$$V^\bullet \to \prod_{k\geq 0} \tau_{\leq k}(V^\bullet) \xrightarrow{1-\mathrm{shift}} \prod_{k\geq 0} \tau_{\leq k}(V^\bullet) \to .$$

It follows that $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A)_{\text{fTd}}$.

Next, since $\mathrm{H}^i(V^\bullet) \cong \mathrm{Ext}^i_R(A, R) \in \text{mod-}R$ for all $i \geq 0$, there exists a quasi-isomorphism $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$. Since $\mathrm{H}^i(P^\bullet) \cong \mathrm{H}^i(V^\bullet)$=0 for $i < 0$, we have a projective resolution $\tau_{\leq 0}(P^\bullet) \to \mathrm{Z}'^0(P^\bullet)$ in mod-A. Then for any $M \in \text{mod-}A^{\text{op}}$ and $i \gg 0$ we have

$$\begin{aligned}
\mathrm{Tor}^A_i(\mathrm{Z}'^0(P^\bullet), M) &\cong \mathrm{H}^{-i}(\tau_{\leq 0}(P^\bullet) \otimes^\bullet_A M) \\
&\cong \mathrm{H}^{-i}(P^\bullet \otimes^\bullet_A M) \\
&\cong \mathrm{H}^{-i}(V^\bullet \otimes^{\mathrm{L}}_A M) \\
&\cong \mathrm{Tor}^A_i(V^\bullet, M) \\
&= 0.
\end{aligned}$$

Thus flat dim $\mathrm{Z}'^0(P^\bullet) < \infty$ and, since $\mathrm{Z}'^0(P^\bullet) \in \text{mod-}A$, proj dim $\mathrm{Z}'^0(P^\bullet) < \infty$. So, by truncating redundant terms, we may assume $\tau_{\leq 0}(P^\bullet)$ and hence $P^\bullet$ belong to $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$. By symmetry, the last assertion follows. □

Definition 2.3.3. Let $\mathcal{K}$ be a triangulated category. A subcategory $\mathcal{B}$ of $\mathcal{K}$ is said to generate $\mathcal{K}$ as a triangulated category if a full triangulated subcategory of $\mathcal{K}$ which contains $\mathcal{B}$ and is closed under isomorphisms coincides with $\mathcal{K}$. For a subcategory $\mathcal{B}$ of $\mathcal{K}$ we denote by $\langle \mathcal{B} \rangle$ the full triangulated subcategory of $\mathcal{K}$ generated by $\mathcal{B}$.

Proposition 2.3.4. *Assume that there exist quasi-isomorphisms $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\text{op}})$ with $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\text{op}}})$. Then the following hold.*

(1) $P^\bullet$ is a tilting complex with $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$.

(2) $Q^\bullet$ is a tilting complex with $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A^{\text{op}})}(Q^\bullet)^{\text{op}}$.

Proof. (1) Since $V^\bullet \in \mathcal{K}^+(\text{Inj-}A)$, by Lemma 2.3.1 we have

$$\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) \cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(V^\bullet, V^\bullet[i]) = 0$$

for $i \neq 0$ and $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$, and since $V^\bullet \in \mathcal{K}^+(\text{Inj-}A^{\text{op}})$, by Lemma 2.3.1 we have quasi-isomorphisms in $\mathcal{K}(\text{Mod-}A)$

$$A \to \mathrm{Hom}^\bullet_{A^{\text{op}}}(V^\bullet, V^\bullet) \to \mathrm{Hom}^\bullet_{A^{\text{op}}}(Q^\bullet, V^\bullet).$$

Also, since $\mathrm{Hom}^\bullet_{A^{\text{op}}}(Q^k, V^\bullet) \in \langle \mathrm{add}(V^\bullet) \rangle$ for all $k \in \mathbb{Z}$, $\mathrm{Hom}^\bullet_{A^{\text{op}}}(Q^\bullet, V^\bullet) \in \langle \mathrm{add}(V^\bullet) \rangle$. Thus $A \in \langle \mathrm{add}(V^\bullet) \rangle$ in $\mathcal{D}(\text{Mod-}A)$. Let $M^\bullet \in \mathcal{D}^-(\text{Mod-}A)$ and assume $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, M^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$. Then, since $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(V^\bullet, M^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$, and since $A \in \langle \mathrm{add}(V^\bullet) \rangle$, we have

$$\mathrm{H}^i(M^\bullet) \cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(A, M^\bullet[i]) = 0$$

for all $i \in \mathbb{Z}$ and $M^\bullet = 0$ in $\mathcal{D}(\text{Mod-}A)$. It now follows by [39, Proposition 5.4] that $P^\bullet$ is a tilting complex.

(2) By symmetry. □

In the following, we are mainly concerned with the case where $V^\bullet$ is a dualizing complex for A (see [22] for details).

Definition 2.3.5. Let A be a left and right noetherian ring. For $* = -, \mathrm{b}$ or nothing, we denote by $\mathcal{D}^*_{\text{mod-}A}(\text{Mod-}A)$ the full triangulated subcategory of $\mathcal{D}^*(\text{Mod-}A)$ consisting of $X^\bullet \in \mathcal{D}^*(\text{Mod-}A)$ with $\mathrm{H}^i(X^\bullet) \in \text{mod-}A$ for all $i \in \mathbb{Z}$.

Definition 2.3.6 ([22]). Let A be a left and right noetherian ring and $V^\bullet$ a bounded below complex of A-A-bimodules with bounded homology. Then $V^\bullet$ is said to be a dualizing complex for A if the following conditions are satisfied:

(1) $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A)_{\text{fid}}$ and $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A^{\text{op}})_{\text{fid}}$;

(2) $V^\bullet \in \mathcal{D}_{\text{mod-}A}(\text{Mod-}A)$ and $V^\bullet \in \mathcal{D}_{\text{mod-}A^{\text{op}}}(\text{Mod-}A^{\text{op}})$;

(3) $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(V^\bullet, V^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A^{\text{op}})}(V^\bullet, V^\bullet[i]) = 0$ for $i \neq 0$; and

(4) there exist ring isomorphisms $A \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}(\text{Mod-}A)}(V^\bullet)$ given by left multiplication and $A \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}(\text{Mod-}A^{\text{op}})}(V^\bullet)^{\text{op}}$ given by right multiplication.

We denote by $\mathrm{Spec}(R)$ the set of prime ideals of R. For any $\mathfrak{p} \in \mathrm{Spec}(R)$ we denote by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p}$ and by $\dim R_\mathfrak{p}$ the Krull dimension of $R_\mathfrak{p}$. We set $\mathrm{Supp}(A) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid A_\mathfrak{p} \neq 0\}$.

Proposition 2.3.7. *Assume* $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A)_{\text{fid}}$. *Then the following conditions are satisfied:*

(A1) $R_\mathfrak{p}$ *is a Gorenstein ring for all* $\mathfrak{p} \in \mathrm{Supp}(A)$*; and*

(A2) $\sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}(A)\} < \infty$.

Proof. There exists an integer $d \in \mathbb{Z}$ such that $\mathrm{Ext}^i_A(-, V^\bullet)$ vanishes on mod-A for $i > d$. Thus for any $i > d$ and $M \in \text{mod-}A$, since

$$\begin{aligned} \mathrm{Ext}^i_R(M, R) &\cong \mathrm{Ext}^i_R(M, I^\bullet) \\ &\cong \mathrm{Ext}^i_A(M, V^\bullet) \\ &= 0, \end{aligned}$$

we have $\mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) \cong \mathrm{Ext}^i_R(M, R)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \mathrm{Supp}(A)$. Let $\mathfrak{p} \in \mathrm{Supp}(A)$ and $M = A \otimes_R R/\mathfrak{p}$. Then $M_\mathfrak{p} \cong A_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a finite dimensional vector space over a filed $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and $M_\mathfrak{p}$ is a finite direct sum of copies of $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ in Mod-$R_\mathfrak{p}$. It follows that $\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, R_\mathfrak{p}) = 0$ for $i > d$ and $R_\mathfrak{p}$ is a Gorenstein ring with $\dim R_\mathfrak{p} \leq d$. □

Throughout the rest of this section, we assume the conditions (A1), (A2) in the proposition above are satisfied.

Proposition 2.3.8. *The following hold.*

(1) A has finite Gorenstein dimension as an R-module.

(2) $V^\bullet$ is a dualizing complex for A.

Proof. Set $d = \sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}(A)\}$.

(1) Let $P^\bullet \to A$ be a projective resolution in mod-R and set $X = \mathrm{Z}'^{-d}(P^\bullet)$. Set $Q^\bullet = \operatorname{Hom}^\bullet_R(P^\bullet, R)$ and $Y = \mathrm{Z}'^{d+1}(Q^\bullet)$. According to Remark 2.2.5, it suffices to show that $\operatorname{Ext}^i_R(X,R)_\mathfrak{p} = \operatorname{Ext}^i_R(Y,R)_\mathfrak{p} = 0$ for all $i \geq 1$ and $\mathfrak{p} \in \operatorname{Supp}(A)$. Let $\mathfrak{p} \in \operatorname{Supp}(A)$. Since inj dim $R_\mathfrak{p} \leq d$, we have

$$\begin{aligned}\operatorname{Ext}^i_R(X,R)_\mathfrak{p} &\cong \operatorname{Ext}^i_{R_\mathfrak{p}}(X_\mathfrak{p}, R_\mathfrak{p})\\ &\cong \operatorname{Ext}^{d+i}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})\\ &= 0,\end{aligned}$$

for all $i \geq 1$. Then $\mathrm{H}^i(Q^\bullet \otimes^\bullet_R R_\mathfrak{p}) = 0$ for all $i \geq d+1$ and hence

$$\begin{aligned}\operatorname{Ext}^i_R(Y,R)_\mathfrak{p} &\cong \operatorname{Ext}^i_{R_\mathfrak{p}}(Y_\mathfrak{p}, R_\mathfrak{p})\\ &\cong \operatorname{Ext}^{d+i}_{R_\mathfrak{p}}(\mathrm{Z}'^{2d+1}(Q^\bullet)_\mathfrak{p}, R_\mathfrak{p})\\ &= 0,\end{aligned}$$

for all $i \geq 1$.

(2) Let $M \in$ mod-A. We claim that $\operatorname{Ext}^i_A(M, V^\bullet) = 0$ for $i > d$. Note that $\operatorname{Ext}^i_A(M, V^\bullet) \cong \operatorname{Ext}^i_R(M, I^\bullet) \cong \operatorname{Ext}^i_R(M, R)$ for all $i \geq 0$. For any $i > d$ and $\mathfrak{p} \in \operatorname{Supp}(A)$, since inj dim $R_\mathfrak{p} \leq d$, we have

$$\begin{aligned}\operatorname{Ext}^i_A(M, V^\bullet)_\mathfrak{p} &\cong \operatorname{Ext}^i_R(M,R)_\mathfrak{p}\\ &\cong \operatorname{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p})\\ &= 0\end{aligned}$$

and hence $\operatorname{Ext}^i_A(M, V^\bullet) = 0$. Thus $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A)_{\text{fid}}$. By symmetry, we also have $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A^{\mathrm{op}})_{\text{fid}}$. Next, we have $\mathrm{H}^i(V^\bullet) \cong \operatorname{Ext}^i_R(A,R) \in$ mod-A for all $i \in \mathbb{Z}$. Finally, by Lemma 2.3.1, $V^\bullet$ is a dualizing complex for A □

Note that A itself is a dualizing complex for A if and only if inj dim ${}_AA =$ inj dim $A_A < \infty$.

Theorem 2.3.9. *The following are equivalent.*

(1) inj dim ${}_AA =$ inj dim $A_A < \infty$.

(2) There exists a quasi-isomorphism $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ a tilting complex such that $A \cong \operatorname{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$.

(3) There exists a quasi-isomorphism $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\mathrm{op}})$ with $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\mathrm{op}}})$ a tilting complex such that $A \cong \operatorname{End}_{\mathcal{K}(\text{Mod-}A^{\mathrm{op}})}(Q^\bullet)^{\mathrm{op}}$.

(4) There exist quasi-isomorphisms $P^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A)$ with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\mathrm{op}})$ with $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\mathrm{op}}})$.

Proof. We have proved (1) $\Rightarrow$ (4) in Lemma 2.3.2 and (4) $\Rightarrow$ (2) and (3) in Proposition 2.3.4.

(2) $\Rightarrow$ (4). Since $P^\bullet \cong V^\bullet$ in $\mathcal{D}(\text{Mod-}A)$, $P^\bullet \in \mathcal{D}^{\text{b}}(\text{Mod-}A)_{\text{fid}}$ and hence $A \in \mathcal{K}^{\text{b}}(\mathcal{P}_A) = \langle \text{add}(P^\bullet) \rangle \subset \mathcal{D}^{\text{b}}(\text{Mod-}A)_{\text{fid}}$. Thus inj dim $A_A < \infty$. Next, by Proposition 2.3.8 we have a quasi-isomorphism $V^\bullet \to E^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\text{op}})$ with $E^\bullet \in \mathcal{K}^{\text{b}}(\text{Inj-}A^{\text{op}})$. Since by Lemma 2.1.1(1) flat dim $E^k \leq$ inj dim A_A for all $k \in \mathbb{Z}$, we have $E^\bullet \in \mathcal{D}^{\text{b}}(\text{Mod-}A^{\text{op}})_{\text{fTd}}$ and hence $V^\bullet \in \mathcal{D}^{\text{b}}(\text{Mod-}A^{\text{op}})_{\text{fTd}}$. Since $\text{H}^i(V^\bullet) \in \text{mod-}A^{\text{op}}$ for all $i \in \mathbb{Z}$, there exists a quasi-isomorphism $Q^\bullet \to V^\bullet$ in $\mathcal{K}(\text{Mod-}A^{\text{op}})$ with $Q^\bullet \in \mathcal{K}^-(\mathcal{P}_{A^{\text{op}}})$. Note that since $\text{H}^i(Q^\bullet) = 0$ for all $i < 0$ we have a projective resolution $\tau_{\leq 0}(Q^\bullet) \to \text{Z}'^0(Q^\bullet)$ in mod-A^{op} and that $Q^\bullet \in \mathcal{D}^{\text{b}}(\text{Mod-}A^{\text{op}})_{\text{fTd}}$. Thus for any $M \in \text{mod-}A$ we have

$$\begin{aligned}\text{Tor}_i^A(M, \text{Z}'^0(Q^\bullet)) &\cong \text{H}^{-i}(M \otimes_A^\bullet \tau_{\leq 0}(Q^\bullet)) \\ &\cong \text{H}^{-i}(M \otimes_A^\bullet Q^\bullet) \\ &= 0\end{aligned}$$

for $i \gg 0$ and flat dim $\text{Z}'^0(Q^\bullet) < \infty$. Now, since $\text{Z}'^0(Q^\bullet) \in \text{mod-}A^{\text{op}}$, we have proj dim $\text{Z}'^0(Q^\bullet) < \infty$. So, by truncating redundant terms, we may assume $\tau_{\leq 0}(Q^\bullet)$ and hence $Q^\bullet$ belong to $\mathcal{K}^{\text{b}}(\mathcal{P}_{A^{\text{op}}})$.

(3) $\Rightarrow$ (4). By symmetry.

(4) $\Rightarrow$ (1). Note that we have proved (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (4). We have proved inj dim $A_A < \infty$ in the proof of (2) $\Rightarrow$ (4). By symmetry, we also have inj dim ${}_AA < \infty$. It follows by [44, Lemma A] that inj dim ${}_AA =$ inj dim $A_A < \infty$. □

2.4 Self-equivalence

In this section, we show that if both A and $V^\bullet$ are dualizing complexes for A then $- \otimes_A^{\mathbf{L}} V^\bullet$ induces a self-equivalence of $\mathcal{D}^{\text{b}}(\text{mod-}A)$.

Remark 2.4.1 *(cf. [22, Chapter I, Proposition 4.8]).* Let A be a left and right noetherian ring. For $* = -$ or b, there exists an equivalence of triangulated categories

$$\mathcal{D}^*(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^*_{\text{mod-}A}(\text{Mod-}A).$$

In the following, we need the notion of way-outness of exact functors. We refer to [22] for the definition and basic properties of way-out functors.

Lemma 2.4.2 ([22]). *Let A, B be left and right noetherian rings and $F : \mathcal{D}_{\text{mod-}A}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}B)$ an exact functor. Then the following hold.*

(1) Assume $F(P) \in \mathcal{D}_{\text{mod-}B}(\text{Mod-}B)$ for all $P \in \mathcal{P}_A$ and F is way-out left. Then $F(X^\bullet) \in \mathcal{D}_{\text{mod-}B}(\text{Mod-}B)$ for all $X^\bullet \in \mathcal{D}^+_{\text{mod-}A}(\text{Mod-}A)$.

(2) Assume $F(M) \in \mathcal{D}_{\text{mod-}B}(\text{Mod-}B)$ for all $M \in \text{mod-}A$ and F is way-out in both directions. Then $F(X^\bullet) \in \mathcal{D}_{\text{mod-}B}(\text{Mod-}B)$ for all $X^\bullet \in \mathcal{D}_{\text{mod-}A}(\text{Mod-}A)$.

Proof. See [22, Chapter I, Proposition 7.3]. □

Lemma 2.4.3 (**[22]**). *Let R be a commutative ring and A an R-algebra. For a complex $X^\bullet \in \mathcal{D}(\text{Mod-}A)$, the following hold.*

(1) $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A)_{\mathrm{fid}}$ if and only if $\mathbf{R}\mathrm{Hom}^\bullet{}_A(-, X^\bullet) : \mathcal{D}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}R)$ is way-out in both directions.

(2) $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{Mod-}A)_{\mathrm{fTd}}$ if and only if $X^\bullet \otimes^{\mathbf{L}}_R - : \mathcal{D}(\text{Mod-}A^{\mathrm{op}}) \to \mathcal{D}(\text{Mod-}R)$ is way-out in both directions.

Proof. (1) is a special case of [22, Chapter I, Proposition 7.6] and (2) is a slight modification of [22, Chapter II, Proposition 4.2] □

Throughout the rest of this section, R is a commutative noetherian ring and A is a Noether R-algebra. Take a minimal injective resolution $R \to I^\bullet$ in Mod-R and set $V^\bullet = \mathrm{Hom}^\bullet_R(A, I^\bullet) \in \mathcal{K}^+(\text{Mod-}A^{\mathrm{e}})$. We assume the conditions (A1), (A2) in Proposition 2.3.7 are satisfied. Then by Proposition 2.3.8 A has finite Gorenstein dimension as an R-module and $V^\bullet$ is a dualizing complex for A.

Lemma 2.4.4. *There exists a functorial isomorphism*

$$X^\bullet \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(\mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, V^\bullet), V^\bullet)$$

for all $X^\bullet \in \mathcal{D}_{\text{mod-}A}(\text{Mod-}A)$.

Proof. By Lemma 2.4.3,

$$\mathbf{R}\mathrm{Hom}^\bullet_A(-, V^\bullet) \circ \mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(-, V^\bullet) : \mathcal{D}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}A)$$

is way-out in both directions. Thus, since $A \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(\mathbf{R}\mathrm{Hom}^\bullet_A(A, V^\bullet), V^\bullet)$, the assertion follows by [22, Chapter I, Proposition 7.1]). □

Lemma 2.4.5. *For $* = \mathrm{b}$ or nothing, we have an anti-equivalence*

$$\mathbf{R}\mathrm{Hom}^\bullet_A(-, V^\bullet) : \mathcal{D}^*_{\text{mod-}A}(\text{Mod-}A) \xrightarrow{\sim} \mathcal{D}^*_{\text{mod-}A^{\mathrm{op}}}(\text{Mod-}A^{\mathrm{op}})$$

whose quasi-inverse is given by

$$\mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(-, V^\bullet) : \mathcal{D}^*_{\text{mod-}A^{\mathrm{op}}}(\text{Mod-}A^{\mathrm{op}}) \xrightarrow{\sim} \mathcal{D}^*_{\text{mod-}A}(\text{Mod-}A).$$

Proof. (1) By Lemma 2.4.3 $\mathbf{R}\mathrm{Hom}^\bullet_A(-, V^\bullet) : \mathcal{D}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}A^{\mathrm{op}})$ is way-out in both directions. Since $\mathbf{R}\mathrm{Hom}^\bullet_A(A, V^\bullet) \cong V^\bullet \in \mathcal{D}_{\text{mod-}A^{\mathrm{op}}}(\text{Mod-}A^{\mathrm{op}})$, we have $\mathbf{R}\mathrm{Hom}^\bullet_A(P, V^\bullet) \in \mathcal{D}_{\text{mod-}A^{\mathrm{op}}}(\text{Mod-}A^{\mathrm{op}})$ for all $P \in \mathcal{P}_A$. Therefore, by Lemma 2.4.2, we have $\mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, V^\bullet) \in \mathcal{D}_{\text{mod-}A^{\mathrm{op}}}(\text{Mod-}A^{\mathrm{op}})$ for all $X^\bullet \in \mathcal{D}_{\text{mod-}A}(\text{Mod-}A)$. Thus we have an exact functor

$$\mathbf{R}\mathrm{Hom}^\bullet_A(-, V^\bullet) : \mathcal{D}_{\text{mod-}A}(\text{Mod-}A) \to \mathcal{D}_{\text{mod-}A^{\mathrm{op}}}(\text{Mod-}A^{\mathrm{op}}).$$

By symmetry, we have an exact functor

$$\mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(-, V^\bullet) : \mathcal{D}_{\text{mod-}A^{\mathrm{op}}}(\text{Mod-}A^{\mathrm{op}}) \to \mathcal{D}_{\text{mod-}A}(\text{Mod-}A).$$

The assertion for $* =$ nothing follows by Lemma 2.4.4. Then, since $\mathbf{R}\mathrm{Hom}^\bullet_A(-, V^\bullet)$ and $\mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(-, V^\bullet)$ are way-out in both direction, the assertion for $* = \mathrm{b}$ follows automatically. □

Lemma 2.4.6. *Assume* inj dim ${}_AA =$ inj dim $A_A < \infty$. *Then for* $* =$ b *or nothing we have an anti-equivalence*

$$\mathbf{R}\mathrm{Hom}^\bullet_A(-, A) : \mathcal{D}^*_{\text{mod-}A}(\text{Mod-}A) \xrightarrow{\sim} \mathcal{D}^*_{\text{mod-}A^{\text{op}}}(\text{Mod-}A^{\text{op}})$$

whose quasi-inverse is given by

$$\mathbf{R}\mathrm{Hom}^\bullet_{A^{\text{op}}}(-, A) : \mathcal{D}^*_{\text{mod-}A^{\text{op}}}(\text{Mod-}A^{\text{op}}) \xrightarrow{\sim} \mathcal{D}^*_{\text{mod-}A}(\text{Mod-}A).$$

Proof. Since A is a dualizing complex, we can replace $V^\bullet$ by A in the proof of Lemma 2.4.5. □

Theorem 2.4.7. *Assume* inj dim ${}_AA =$ inj dim $A_A < \infty$. *Then for any* $X^\bullet \in \mathcal{D}^{\text{b}}(\text{mod-}A)$ *we have a functorial isomorphism*

$$X^\bullet \otimes^{\mathbf{L}}_A V^\bullet \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_{A^{\text{op}}}(\mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, A), V^\bullet).$$

In particular, we have an equivalence

$$- \otimes^{\mathbf{L}}_A V^\bullet : \mathcal{D}^{\text{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\text{b}}(\text{mod-}A).$$

Proof. By Lemmas 2.4.5, 2.4.6 and Remark 2.4.1, we have anti-equivalences

$$\mathbf{R}\mathrm{Hom}^\bullet_{A^{\text{op}}}(-, V^\bullet) : \mathcal{D}^{\text{b}}(\text{mod-}A^{\text{op}}) \xrightarrow{\sim} \mathcal{D}^{\text{b}}(\text{mod-}A),$$

$$\mathbf{R}\mathrm{Hom}^\bullet_A(-, A) : \mathcal{D}^{\text{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\text{b}}(\text{mod-}A^{\text{op}}).$$

Next, for any $X^\bullet \in \mathcal{D}^{\text{b}}(\text{mod-}A)$, we have functorial isomorphisms

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^\bullet_{A^{\text{op}}}(X^\bullet \otimes^{\mathbf{L}}_A V^\bullet, V^\bullet) &\cong \mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, \mathbf{R}\mathrm{Hom}^\bullet_{A^{\text{op}}}(V^\bullet, V^\bullet)) \\ &\cong \mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, A)\end{aligned}$$

in $\mathcal{D}(\text{Mod-}A)$. Note that we have seen in the proof of Lemma 2.3.2 that $V^\bullet \in \mathcal{D}^{\text{b}}(\text{Mod-}A^{\text{op}})_{\text{fTd}}$. Thus by Lemma 2.4.3(2) $- \otimes^{\mathbf{L}}_A V^\bullet$ is way-out in both directions. For any $X^\bullet \in \mathcal{D}^{\text{b}}(\text{mod-}A)$, by Lemma 2.4.2 we have $X^\bullet \otimes^{\mathbf{L}}_A V^\bullet \in \mathcal{D}_{\text{mod-}A}(\text{Mod-}A)$ and hence by Lemma 2.4.4 we have functorial isomorphisms

$$\begin{aligned}X^\bullet \otimes^{\mathbf{L}}_A V^\bullet &\cong \mathbf{R}\mathrm{Hom}^\bullet_{A^{\text{op}}}(\mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet \otimes^{\mathbf{L}}_A V^\bullet, V^\bullet), V^\bullet) \\ &\cong \mathbf{R}\mathrm{Hom}^\bullet_{A^{\text{op}}}(\mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, \mathbf{R}\mathrm{Hom}^\bullet_{A^{\text{op}}}(V^\bullet, V^\bullet)), V^\bullet) \\ &\cong \mathbf{R}\mathrm{Hom}^\bullet_{A^{\text{op}}}(\mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, A), V^\bullet)\end{aligned}$$

in $\mathcal{D}(\text{Mod-}A)$. □

Chapter 3

Derived equivalences for selfinjective algebras

Let A be an Artin algebra. Rickard [39, Proposition 9.3] showed that for any tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of $K_0(A)$, the Grothendieck group of A, which generalizes earlier results [20, Proposition 3.2] and [34, Theorem 1.19]. He raised a question whether a complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ is a tilting complex or not if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of $K_0(A)$ (see also [34]). In case $P^\bullet$ is a projective resolution of a module $T \in \text{mod-}A$ with proj dim $T_A \leq 1$, Bongartz [14, Lemma of 2.1] has settled the question affirmatively. More precisely, he showed that every $T \in \text{mod-}A$ with proj dim $T_A \leq 1$ and $\mathrm{Ext}^1_A(T,T) = 0$ is a direct summand of a classical tilting module, i.e., a tilting module of projective dimension ≤ 1. Unfortunately, this is not true in general (see [39, Section 8]). Our first aim of this chapter is to show that if A is a representation-finite selfinjective Artin algebra then every $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, where ν is the Nakayama functor, is a direct summand of a tilting complex (Theorem 3.3.6).

Rickard [40, Theorem 4.2] showed that the Brauer tree algebras over a field with the same numerical invariants are derived equivalent to each other. Subsequently, Okuyama pointed out that for any Brauer tree algebras A, B with the same numerical invariants there exists a sequence of Brauer tree algebras $A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1. These facts can be formulated as follows. For any tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ associated with a certain sequence of idempotents in a ring A, there exists a sequence of rings $A = B_0, B_1, \cdots, B_m = \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$ such that, for any $0 \leq i < m$, B_{i+1} is

This chapter is based on my joint paper with M. Hoshino [2].

the endomorphism ring of a tilting complex for B_i of length ≤ 1 determined by an idempotent (see [26, Proposition 3.2]). We refer to [21], [32] for other examples of derived equivalences which are iterations of derived equivalences induced by tilting complexes of length ≤ 1. Our second aim of this chapter is to show that for any derived equivalent representation-finite selfinjective Artin algebras A, B there exists a sequence of selfinjective Artin algebras $A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1 (Theorem 3.3.7).

For a ring A, we denote by Mod-A the category of right A-modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module X considered is a right (resp., left) A-module. For an object X in an additive category $\mathcal{B}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of finite direct sums of copies of X and by $X^{(n)}$ the direct sum of n copies of X. For a cochain complex $X^\bullet$ over an abelian category $\mathcal{A}$, we denote by $\mathrm{Z}^n(X^\bullet)$, $\mathrm{Z}'^n(X^\bullet)$ and $\mathrm{H}^n(X^\bullet)$ the n-th cycle, the n-th cocycle and the n-th cohomology of $X^\bullet$, respectively. For an additive category $\mathcal{B}$, we denote by $\mathcal{K}(\mathcal{B})$ (resp., $\mathcal{K}^+(\mathcal{B})$, $\mathcal{K}^-(\mathcal{B})$, $\mathcal{K}^{\mathrm{b}}(\mathcal{B})$) the homotopy category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) over $\mathcal{B}$. As usual, we consider objects of $\mathcal{B}$ as complexes over $\mathcal{B}$ concentrated in degree zero. For an abelian category $\mathcal{A}$, we denote by $\mathcal{D}(\mathcal{A})$ (resp., $\mathcal{D}^+(\mathcal{A})$, $\mathcal{D}^-(\mathcal{A})$, $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$) the derived category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) over $\mathcal{A}$. We always consider $\mathcal{K}^*(\mathcal{B})$ (resp., $\mathcal{D}^*(\mathcal{A})$) as a full triangulated subcategory of $\mathcal{K}(\mathcal{B})$ (resp., $\mathcal{D}(\mathcal{A})$), where $* = +$, $-$ or b. We denote by $\mathrm{Hom}^\bullet(-,-)$ the associated single complex of the double hom complex.

We refer to [13], [22], [43] for basic results in the theory of derived categories and to [39], [41] for definitions and basic properties of derived equivalences and tilting complexes.

3.1 Preliminaries

Throughout this chapter, R is a commutative artinian ring with the Jacobson radical $\mathfrak{m}$ and A is an Artin R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and is finitely generated as an R-module.

For any Artin R-algebra A, we denote by mod-A the full subcategory of Mod-A consisting of finitely generated modules and by $\mathcal{P}_A$ (resp., $\mathcal{I}_A$) the full subcategory of mod-A consisting of projective (resp., injective) modules. Also, we set $D = \mathrm{Hom}_R(-, E(R/\mathfrak{m}))$, where $E(R/\mathfrak{m})$ is an injective envelope of $R/\mathfrak{m}$ in Mod-R, and $\nu = D \circ \mathrm{Hom}_A(-, A)$, which is called the Nakayama functor.

Remark 3.1.1. The Krull-Schmidt theorem holds in mod-A, i.e., for any nonzero module $X \in$ mod-A the following hold.

(1) X decomposes into a direct sum of indecomposable submodules.

(2) X is indecomposable if and only if $\mathrm{End}_A(X)$ is local.

Remark 3.1.2. The following hold.

(1) $X \xrightarrow{\sim} D^2X, x \mapsto (h \mapsto h(x))$, for all $X \in \text{mod-}R$.

(2) $D : \text{mod-}A \to \text{mod-}A^{\text{op}}$ is an anti-equivalence and induces anti-equivalences $\mathcal{P}_A \xrightarrow{\sim} \mathcal{I}_{A^{\text{op}}}$ and $\mathcal{I}_A \xrightarrow{\sim} \mathcal{P}_{A^{\text{op}}}$.

(3) $\nu : \text{mod-}A \to \text{mod-}A$ induces an equivalence $\mathcal{P}_A \xrightarrow{\sim} \mathcal{I}_A$.

Lemma 3.1.3. *For any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ the following are equivalent.*

(1) $P^\bullet \in \mathrm{add}(\nu P^\bullet)$.

(2) $\nu P^\bullet \in \mathrm{add}(P^\bullet)$.

(3) $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$.

Proof. Note that every idempotent splits in $\mathcal{K}(\text{Mod-}A)$ (see [13, Proposition 3.2]). Thus, since we have an isomorphism of Artin R-algebras

$$\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet) \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(\nu P^\bullet),$$

it follows that $P^\bullet$ and $\nu P^\bullet$ have the same number of nonisomorphic indecomposable direct summands. $\square$

Recall that A is said to be selfinjective if the equivalent conditions of Lemma 3.1.3 are satisfied for $P^\bullet = A$.

Remark 3.1.4. If A is selfinjective, then $\nu : \text{mod-}A \to \text{mod-}A$ is an equivalence and induces an equivalence $\mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_A$.

Lemma 3.1.5 ([25, Lemma 3.1]). *For any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $X^\bullet \in \mathcal{K}(\text{Mod-}A)$ we have a bifunctorial isomorphism*

$$\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(X^\bullet, \nu P^\bullet) \cong D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, X^\bullet).$$

Definition 3.1.6. For any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ we denote by $\mathcal{S}(P^\bullet)$ the full subcategory of $\mathcal{D}^-(\text{Mod-}A)$ consisting of complexes $X^\bullet$ with $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, X^\bullet[i]) = 0$ for $i \neq 0$.

Lemma 3.1.7. *Assume A is selfinjective. Then for any tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ the following are equivalent.*

(1) $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$ *is selfinjective.*

(2) $P^\bullet \in \mathcal{S}(\nu P^\bullet)$.

(3) $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$.

Proof. Set $B = \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. Note that by Lemma 3.1.5 $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ and $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet) \cong D({}_B B)$.

(1) $\Leftrightarrow$ (3). Note first that we have an equivalence (see [39, Section 4])

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, -) : \mathcal{S}(P^\bullet) \xrightarrow{\sim} \text{Mod-}B.$$

We may consider $\mathrm{add}(P^\bullet)$ and $\mathrm{add}(\nu P^\bullet)$ as full subcategories of $\mathcal{S}(P^\bullet)$ via the canonical functor $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \to \mathcal{D}^-(\text{Mod-}A)$. Then $\mathrm{add}(P^\bullet)$ and $\mathrm{add}(\nu P^\bullet)$ are closed under direct summands because every idempotent splits in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ (see [13, Proposition 3.4]). Thus the equivalence above induces equivalences $\mathrm{add}(P^\bullet) \xrightarrow{\sim} \mathcal{P}_B$ and $\mathrm{add}(\nu P^\bullet) \xrightarrow{\sim} \mathcal{I}_B$.

(2) $\Rightarrow$ (3). We have $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet \oplus \nu P^\bullet, (P^\bullet \oplus \nu P^\bullet)[i]) = 0$ for $i \neq 0$ and hence by [26, Lemma 1.8] $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$.

(3) $\Rightarrow$ (2). Obvious. □

In case A, B are finite dimensional selfinjective algebras over a field and $F : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ is an equivalence of triangulated categories, it was pointed out in [5, Section 2] that for any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ there exists an object-wise isomorphism $F(\nu P^\bullet) \cong \nu F(P^\bullet)$. We need to extend this fact to the case of Artin algebras.

Lemma 3.1.8. *Let A, B be derived equivalent selfinjective Artin R-algebras and $F : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ an equivalence of triangulated categories. Then for any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ we have a functorial isomorphism $\nu F(P^\bullet) \cong F(\nu P^\bullet)$.*

Proof. Let $G : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a quasi-inverse of F. Then for any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$, by Lemma 3.1.5 we have bifunctorial isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}(\text{Mod-}B)}(Q^\bullet, \nu F(P^\bullet)) &\cong D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}B)}(F(P^\bullet), Q^\bullet) \\ &\cong D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, G(Q^\bullet)) \\ &\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(G(Q^\bullet), \nu P^\bullet) \\ &\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}B)}(Q^\bullet, F(\nu P^\bullet)).\end{aligned}$$

The assertion follows by Yoneda lemma. □

Definition 3.1.9. For any nonzero $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ we set

$$a(P^\bullet) = \max\{i \in \mathbb{Z} \mid \mathrm{H}^i(P^\bullet) \neq 0\},$$

and for any nonzero $P^\bullet \in \mathcal{K}^+(\mathcal{P}_A)$ we set

$$b(P^\bullet) = \min\{i \in \mathbb{Z} \mid \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet[i], A) \neq 0\}.$$

Then for any nonzero $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ we set $l(P^\bullet) = a(P^\bullet) - b(P^\bullet)$ and call it the length of $P^\bullet$. For the sake of convenience, we set $l(P^\bullet) = 0$ for $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $P^\bullet \cong 0$.

Remark 3.1.10 *([6])*. For any complex $X^\bullet$ and $n \in \mathbb{Z}$ we define truncations

$$\sigma_{\le n}(X^\bullet) : \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Z}^n(X^\bullet) \to 0 \to \cdots,$$
$$\sigma'_{\ge n}(X^\bullet) : \cdots \to 0 \to \mathrm{Z}'^n(X^\bullet) \to X^{n+1} \to X^{n+2} \to \cdots.$$

Then $P^\bullet \cong \sigma_{\le a}(P^\bullet)$ for any nonzero $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$, where $a = a(P^\bullet)$, and $P^\bullet \cong \sigma'_{\ge b}(P^\bullet)$ for any nonzero $P^\bullet \in \mathcal{K}^+(\mathcal{P}_A)$, where $b = b(P^\bullet)$.

3.2 Torsion theories

We need to recall several definitions and basic results on torsion theories.

Definition 3.2.1 ([3]). A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories $\mathcal{T}$, $\mathcal{F}$ in an abelian category $\mathcal{A}$ is said to be a torsion theory for $\mathcal{A}$ if the following conditions are satisfied:

(1) $\mathcal{T} \cap \mathcal{F} = \{0\}$;

(2) $\mathcal{T}$ is closed under factor objects;

(3) $\mathcal{F}$ is closed under subobjects; and

(4) for any $X \in \mathcal{A}$ there exists an exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$.

Definition 3.2.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$. Then we denote by ${}^\perp\mathcal{C}$ (resp., $\mathcal{C}^\perp$) the full subcategory of $\mathcal{A}$ consisting of objects X with $\mathrm{Hom}_\mathcal{A}(X, \mathcal{C}) = 0$ (resp., $\mathrm{Hom}_\mathcal{A}(\mathcal{C}, X) = 0$). For an object $Y \in \mathcal{A}$, we use the notation ${}^\perp Y$ (resp., $Y^\perp$) instead of ${}^\perp\mathrm{add}(Y)$ (resp., $\mathrm{add}(Y)^\perp$).

Remark 3.2.3. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for an abelian category $\mathcal{A}$. Then the following hold.

(1) $\mathcal{F} = \mathcal{T}^\perp$ and $\mathcal{T} = {}^\perp\mathcal{F}$.

(2) $\mathcal{T}$ and $\mathcal{F}$ are closed under extensions.

(3) There exists a subfunctor t of the identity functor $\mathbf{1}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$, called the associated torsion radical, such that $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$ for all $X \in \mathcal{A}$.

Proof. (1) By the conditions (1)–(3), $\mathcal{F} \subset \mathcal{T}^\perp$ and $\mathcal{T} \subset {}^\perp\mathcal{F}$. On the other hand, by the condition (4), $\mathcal{T}^\perp \subset \mathcal{F}$ and ${}^\perp\mathcal{F} \subset \mathcal{T}$.

(2) Immediate by (1).

(3) For each $X \in \mathcal{A}$, take an exact sequence

$$0 \to X' \xrightarrow{\iota_X} X \xrightarrow{\pi_X} X'' \to 0$$

with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$. For any $Z \in \mathcal{T}$, since $\mathrm{Hom}_\mathcal{A}(Z, X'') = 0$, $\mathrm{Hom}_\mathcal{A}(Z, \iota_X)$ is an isomorphism. It follows that X' is maximum in the collection of subobjects of X belonging to $\mathcal{T}$. We set $t(X) = X'$. Next, let

$f : X \to Y$ be a morphism. Since $\mathrm{Hom}_{\mathcal{A}}(X', Y'') = 0$, $\pi_Y \circ f \circ \iota_X = 0$ and there exists a unique morphism $f' : X' \to Y'$ such that $f \circ \iota_X = \iota_Y \circ f'$. We set $t(f) = f'$. Then for any $X \in \mathcal{A}$ we have $\mathrm{id}_X \circ \iota_X = \iota_X \circ \mathrm{id}_{t(X)}$ and hence $t(\mathrm{id}_X) = \mathrm{id}_{t(X)}$. Also, for any consecutive morphisms $f : X \to Y$ and $g : Y \to Z$, since $f \circ \iota_X = \iota_Y \circ t(f)$ and $g \circ \iota_Y = \iota_Z \circ t(g)$, we have $g \circ f \circ \iota_X = \iota_Z \circ t(g) \circ t(f)$ and hence $t(g \circ f) = t(g) \circ t(f)$. □

Although the next lemma is well-known, we include a proof because it will play an indispensable role in the next section.

Lemma 3.2.4. *For any $Y \in$ mod-A, by setting $\mathcal{T} = {}^{\perp}Y$ and $\mathcal{F} = \mathcal{T}^{\perp}$, we have a torsion theory $(\mathcal{T}, \mathcal{F})$ for* mod-A.

Proof. It is obvious that the conditions (1)–(3) of Definition 3.2.1 are satisfied. Let $X \in$ mod-A. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be the set of submodules of X belonging to $\mathcal{T}$ and set $X' = \bigcup_{\lambda \in \Lambda} X_\lambda$. Note that $\mathcal{T}$ is closed under extensions and finite direct sums. In particular, Λ is directed, where $\lambda \leq \mu$ if and only if $X_\lambda \subset X_\mu$, and X' is a submodule of X. Thus we have an epimorphism $\bigoplus_{\lambda \in \Lambda} X_\lambda \to X'$ in Mod-A and, since $\mathrm{Hom}_A(\bigoplus_{\lambda \in \Lambda} X_\lambda, Y) \cong \prod_{\lambda \in \Lambda} \mathrm{Hom}_A(X_\lambda, Y) = 0$, it follows that $X' \in \mathcal{T}$. Next, we claim that $X/X' \in \mathcal{F}$. Let $Z \in \mathcal{T}$ and $f \in \mathrm{Hom}_A(Z, X/X')$. Take a pull-back of f along with the canonical epimorphism $X \to X/X'$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \longrightarrow & W & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X/X' & \longrightarrow & 0.
\end{array}$$

Then, since $W \in \mathcal{T}$, $\mathrm{Im}\, g \subset X'$ and $f = 0$. □

Definition 3.2.5. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$ closed under extensions. Then an object $X \in \mathcal{C}$ is said to be Ext-projective (resp., Ext-injective) if $\mathrm{Ext}^1_{\mathcal{A}}(X, \mathcal{C}) = 0$ (resp., $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{C}, X) = 0$).

Lemma 3.2.6. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for* mod-A. *Then a module $X \in \mathcal{T}$ is* Ext-*injective if and only if $X = t(E)$ with E an injective envelope of X.*

Proof. "If" part. Let $E \in$ mod-A be an injective module and take an exact sequence

$$0 \to t(E) \xrightarrow{\mu} Y \xrightarrow{\varepsilon} Z \to 0$$

with $Z \in \mathcal{T}$. We claim that μ is a split monomorphism. Denote by $\iota : t(E) \to E$ the inclusion. By the injectivity of E, $\iota = \phi \circ \mu$ for some $\phi : Y \to E$. Note that by Remark 3.2.3(2) $Y \in \mathcal{T}$. Thus $\phi(Y) \subset t(E)$ and $\phi = \iota \circ \phi'$ for some $\phi' : Y \to t(E)$. Then $\iota = \iota \circ \phi' \circ \mu$ and $\mathrm{id}_{t(E)} = \phi' \circ \mu$. It follows that $t(E)$ is Ext-injective.

"Only if" part. Let $X \in \mathcal{T}$ and E an injective envelope of X. We consider X as a submodule of E. Then $X \subset t(E)$ and we have an exact sequence

$$0 \to X \xrightarrow{\iota} t(E) \to t(E)/X \to 0.$$

Since $t(E)/X \in \mathcal{T}$, and since X is Ext-injective, the inclusion $\iota : X \to t(E)$ has to be a split monomorphism. On the other hand, E and hence $t(E)$ are essential extensions of X. It follows that $X = t(E)$. □

We refer to [9, Chapter V, Sections 1 and 2] for the following Definitions 2.7, 2.8 and Lemmas 2.9, 2.11.

Definition 3.2.7. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$. Let $f : X \to Y$ be a morphism with $X, Y \in \mathcal{C}$. Then f is said to be right (resp., left) almost split in $\mathcal{C}$ if f is not a split epimorphism (resp., monomorphism) and if every morphism $h : Z \to Y$ (resp., $h : X \to Z$) with $Z \in \mathcal{C}$ factors through f unless h is a split epimorphism (resp., monomorphism).

Definition 3.2.8. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$ closed under extensions. Then a nonsplit exact sequence

$$0 \to Z \xrightarrow{g} Y \xrightarrow{f} X \to 0$$

with $X, Z \in \mathcal{C}$ is said to be an almost split sequence in $\mathcal{C}$ if the following conditions are satisfied:

(1) $\mathrm{End}_{\mathcal{A}}(X)$ and $\mathrm{End}_{\mathcal{A}}(Z)$ are local; and

(2) f (resp., g) is right (resp., left) almost split in $\mathcal{C}$.

Lemma 3.2.9. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$ closed under extensions. Let*

$$0 \to Z_1 \to Y_1 \to X_1 \to 0, \qquad 0 \to Z_2 \to Y_2 \to X_2 \to 0$$

be almost split sequences in $\mathcal{C}$. Then $X_1 \cong X_2$ if and only if $Z_1 \cong Z_2$.

Definition 3.2.10. For each indecomposable module $X \in \text{mod-}A$, we take a minimal projective resolution $P_X^\bullet \to X$ and set $\tau X = \mathrm{Z}^{-1}(\nu P_X^\bullet)$.

Lemma 3.2.11. *Let $X \in \text{mod-}A$ be an indecomposable nonprojective module. Then $\mathrm{Ext}_A^1(X, \tau X) \neq 0$ and the following hold.*

(1) As a right module over $\mathrm{End}_A(X)$, $\mathrm{Ext}_A^1(X, \tau X)$ is embedded in $D\mathrm{End}_A(X)$ and hence has a simple socle.

(2) A nonsplit exact sequence

$$0 \to \tau X \to Y \to X \to 0$$

representing a nonzero element of the socle of $\mathrm{Ext}_A^1(X, \tau X)$ is an almost split sequence in mod-A.

Lemma 3.2.12 ([23, Lemma 2]). *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for* mod-A *and $X \in \mathcal{T}$ an indecomposable module. Then the following hold.*

(1) X is Ext*-projective if and only if $\tau X \in \mathcal{F}$.*

(2) Assume X is not Ext*-projective and let $0 \to \tau X \to Y \to X \to 0$ be an almost split sequence in* mod-A*. Then the induced sequence*

$$0 \to t(\tau X) \to t(Y) \to X \to 0$$

is an almost split sequence in $\mathcal{T}$.

Definition 3.2.13. Assume A is selfinjective and let $\{e_1, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A. Then there exists a permutation ρ of the set $I = \{1, \cdots, n\}$, called the Nakayama permutation, such that $\nu(e_i A) \cong e_{\rho(i)}A$ for all $i \in I$.

Proposition 3.2.14. *Assume A is selfinjective and has a cyclic Nakayama permutation. Then for any tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$ selfinjective we have $l(P^\bullet) = 0$.*

Proof. Set $l = l(P^\bullet)$. We may assume $P^i = 0$ unless $0 \le i \le l$. Suppose to the contrary that $l \ge 1$. Set $X = \mathrm{H}^l(P^\bullet)$ and $Y = \mathrm{H}^0(P^\bullet)$. Since by Lemma 3.1.7 $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, we have $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu^k P^\bullet)$ for all $k \ge 0$. Thus for any $k \ge 0$, since $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[-l]) = 0$, and since $\nu^k P^\bullet \in \mathrm{add}(P^\bullet)$, we have

$$\begin{aligned}\mathrm{Hom}_A(\nu^k X, Y) &\cong \mathrm{Hom}_A(\mathrm{H}^l(\nu^k P^\bullet), \mathrm{H}^0(P^\bullet)) \\ &\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(\nu^k P^\bullet, P^\bullet[-l]) \\ &= 0.\end{aligned}$$

By Lemma 3.2.4 there exists a torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-A such that $\mathcal{T} = {}^\perp Y$ and $\mathcal{F} = \mathcal{T}^\perp$. Let $\{e_1, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A and set $S_i = e_i A / e_i J$ for $1 \le i \le n$, where J is the Jacobson radical of A. Note that $\nu S_i \cong S_{\rho(i)}$ for all $1 \le i \le n$. Let $S \in$ mod-A be a simple module which is a factor module of X. For any $k \ge 0$, since $\nu^k X \in \mathcal{T}$, and since $\nu^k S$ is a factor module of $\nu^k X$, we have $\nu^k S \in \mathcal{T}$. Note that $S \cong S_i$ for some $1 \le i \le n$. Then $\nu^k S \cong S_{\rho^k(i)}$ for all $k \ge 0$. Since ρ is cyclic, it follows that $S_i \in \mathcal{T}$ for all $1 \le i \le n$. Thus $\mathcal{F}$ does not contain any simple module and $\mathcal{F} = \{0\}$. On the other hand, by the construction we have $0 \ne Y \in \mathcal{F}$, a contradiction. □

3.3 Main results

To begin with, we modify [14, Lemma of 2.1] as follows.

Lemma 3.3.1. *Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a complex with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \ne 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Assume there exists a tilting complex $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, T^\bullet[i]) = 0$ unless $-1 \le i \le 0$. Form a distinguished triangle in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$*

$$Q^\bullet \to P^{\bullet(n)} \xrightarrow{f} T^\bullet \to$$

such that $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, f)$ is epic. Then $Q^\bullet \oplus P^\bullet$ is a tilting complex.

Proof. Note first that such a homomorphism f exists. To see this, set $X^\bullet = \mathrm{Hom}^\bullet_A(P^\bullet, T^\bullet) \in \mathcal{K}^{\mathrm{b}}(\text{mod-}R)$. Then $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, T^\bullet) \cong \mathrm{H}^0(X^\bullet) \in \text{mod-}R$, i.e., $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, T^\bullet)$ is finitely generated over R. It then follows that $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, T^\bullet)$ is finitely generated over $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. Take a set of generators $f_1, \cdots, f_n \in \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, T^\bullet)$ over $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$ and set

$$f = (f_1, \cdots, f_n) : P^{\bullet(n)} \to T^\bullet.$$

It then follows by the construction that $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, f)$ is epic.

Obviously, $\mathrm{add}(Q^\bullet \oplus P^\bullet)$ generates $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as a triangulated category. Note also that by Lemma 3.1.5 $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, P^\bullet[i]) = 0$ unless $0 \le i \le 1$.

Claim. The following hold.

(1) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, Q^\bullet[i]) = 0$ for $i \neq 0$.

(2) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$.

(3) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, Q^\bullet[i]) = 0$ for $i > 1$.

(4) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, T^\bullet[i]) = 0$ for $i < -1$.

Proof. (1), (3) and (4) follow by the construction and (2) follows by (1) and Lemma 3.1.5. □

Now, by (1), (3) of Claim $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, Q^\bullet[i]) = 0$ for $i > 0$ and by (2), (4) of Claim $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(Q^\bullet, Q^\bullet[i]) = 0$ for $i < 0$. This finishes the proof of Lemma 3.3.1 □

Corollary 3.3.2. *Assume A is selfinjective. Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a complex with $P^i = 0$ unless $0 \le i \le 1$. Assume $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Then there exists some $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $Q^\bullet \oplus P^\bullet$ is a tilting complex. In particular, if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of the Grothendieck group $K_0(A)$, then $P^\bullet$ is a tilting complex.*

Proof. Applying Lemma 3.3.1 to $T^\bullet = A$, the first assertion follows. The last assertion follows by [39, Proposition 9.3]. □

Recall that A is said to be representation-finite if there exist only a finite number of nonisomorphic indecomposable modules in mod-A.

Remark 3.3.3 *([30] and [40]).* Let A, B be derived equivalent selfinjective Artin R-algebras. Then A is representation-finite if and only if so is B.

Proof. This follows by the fact that A, B are stably equivalent (see [30, Theorem 3.8] and [40, Corollary 2.2]). □

Lemma 3.3.4. *Assume A is selfinjective and representation-finite. Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a complex of length ≥ 1 with $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Then there exists a tilting complex $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ of length 1 such that*

(1) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, P^\bullet[i]) = 0$ for $i \geq l(P^\bullet)$,

(2) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet[i], T^\bullet) = 0$ for $i < 0$, and

(3) $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet)$ is a selfinjective Artin R-algebra whose Nakayama permutation coincides with that of A.

Proof. Set $l = l(P^\bullet)$. We may assume $P^i = 0$ unless $0 \leq i \leq l$. Note that $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$ implies $\mathrm{add}(\mathrm{H}^0(P^\bullet)) = \mathrm{add}(\mathrm{H}^0(\nu P^\bullet))$. Also, by Lemma 3.2.4 there exists a torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-A such that $\mathcal{T} = {}^\perp\mathrm{H}^0(P^\bullet) = {}^\perp\mathrm{H}^0(\nu P^\bullet)$ and $\mathcal{F} = \mathcal{T}^\perp$. We denote by t the associated torsion radical.

Claim 1. $\mathrm{H}^l(P^\bullet) \in \mathcal{T}$ and $\mathrm{H}^0(P^\bullet), \mathrm{H}^0(\nu P^\bullet) \in \mathcal{F}$.

Proof. By the construction $\mathrm{H}^0(P^\bullet)$, $\mathrm{H}^0(\nu P^\bullet) \in \mathcal{F}$. Also, by Lemma 3.1.5

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{H}^l(P^\bullet), \mathrm{H}^0(\nu P^\bullet)) &\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet[-l]) \\ &\cong D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[l]) \\ &= 0\end{aligned}$$

and $\mathrm{H}^l(P^\bullet) \in \mathcal{T}$. □

Claim 2. ν : mod-$A \xrightarrow{\sim}$ mod-A induces $\mathcal{T} \xrightarrow{\sim} \mathcal{T}$ and $\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. In particular, $\nu(t(X)) = t(\nu X)$ for all $X \in$ mod-A.

Proof. We have $\nu\mathcal{T} = {}^\perp(\nu\mathrm{H}^0(P^\bullet)) = {}^\perp\mathrm{H}^0(\nu P^\bullet) = \mathcal{T}$ and then $\nu\mathcal{F} = (\nu\mathcal{T})^\perp = \mathcal{T}^\perp = \mathcal{F}$. □

Let $\{e_1, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A. Set $I = \{1, \cdots, n\}$, $I_1 = \{i \in I \mid e_iA \in \mathcal{T}\}$, $I_2 = \{i \in I \mid e_iA \in \mathcal{F}\}$ and $I_3 = I \setminus I_1 \cup I_2$. For each $i \in I$, we define a complex $T_i^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as follows. Set $T_i^\bullet = e_iA[-1]$ if $i \in I_1$, and set $T_i^\bullet = e_iA$ if $i \in I_2$. Assume $i \in I_3$. Since e_iA is indecomposable injective, $t(e_iA)$ is indecomposable. Also, by Lemma 2.6 $t(e_iA)$ is Ext-injective. To this module $t(e_iA)$, we associate an indecomposable Ext-projective module $X_i \in \mathcal{T}$ as follows. Set $Y_1 = t(e_iA)$ and for $k \geq 1$ set $Y_{k+1} = t(\tau Y_k)$ unless Y_k is Ext-projective. Then, according to Lemma 3.2.9, Y_m has to be Ext-projective for some $m \geq 1$ because $\mathcal{T}$ contains only a finite number of nonisomorphic indecomposable modules. We set $X_i = Y_m$ and define $T_i^\bullet$ as the (-1)-shift of a minimal projective presentation of X_i. Now, we set $T^\bullet = \bigoplus_{i \in I} T_i^\bullet$ (cf. [27, Theorem 5.8]). Also, we denote by ρ the Nakayama permutation of A.

Claim 3. $\nu T_i^\bullet \cong T_{\rho(i)}^\bullet$ for all $i \in I$. In particular, $\nu T^\bullet \cong T^\bullet$ and $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet)$ is a selfinjective Artin R-algebra with ρ the Nakayama permutation.

Proof. By Claim 2 the sets I_i are ρ-stable. Thus $\nu T_i^\bullet \cong T_{\rho(i)}^\bullet$ for $i \in I_1 \cup I_2$. Let $i \in I_3$. Then by Claim 2 $\nu(t(e_iA)) \cong t(\nu(e_iA)) \cong t(e_{\rho(i)}A)$ and hence $\nu X_i \cong X_{\rho(i)}$. Thus $\nu T_i^\bullet \cong T_{\rho(i)}^\bullet$. Now, for any $i \in I$, by Lemma 3.1.5

$$\begin{aligned} D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_i^\bullet, T^\bullet) &\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, \nu T_i^\bullet) \\ &\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, T_{\rho(i)}^\bullet). \end{aligned}$$

□

Claim 4. $\mathrm{H}^1(T^\bullet) \in \mathcal{T}$ and $\mathrm{H}^0(T^\bullet), \mathrm{H}^0(\nu T^\bullet) \in \mathcal{F}$.

Proof. By the construction $H^1(T^\bullet) \in \mathcal{T}$. Also, by Lemma 3.2.12(1) $\mathrm{H}^0(\nu T_i^\bullet) \cong \tau X_i \in \mathcal{F}$ for all $i \in I_3$ and hence $\mathrm{H}^0(\nu T^\bullet) \in \mathcal{F}$. It then follows by Claim 3 that $\mathrm{H}^0(T^\bullet) \in \mathcal{F}$. □

Claim 5. $T^\bullet$ is a tilting complex.

Proof. By Claim 4 $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, T^\bullet[-1]) \cong \mathrm{Hom}_A(\mathrm{H}^1(T^\bullet), \mathrm{H}^0(T^\bullet)) = 0$. Then by Lemma 3.1.5 and Claim 3

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, T^\bullet[1]) &\cong D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, \nu T^\bullet[-1]) \\ &\cong D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, T^\bullet[-1]) \\ &= 0. \end{aligned}$$

Thus by Claim 3 we can apply the last part of Corollary 3.3.2. □

Claim 6. $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, P^\bullet[i]) = 0$ for $i \geq l$ and $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet[i], T^\bullet) = 0$ for $i < 0$.

Proof. For any $i > l$ we have $a(P^\bullet[i]) < b(T^\bullet)$ and $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, P^\bullet[i]) = 0$. Similarly, for any $i < -1$ we have $a(T^\bullet) < b(P^\bullet[i])$ and $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet[i], T^\bullet) = 0$. Also, by Lemma 3.1.5 and Claims 1, 4

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, P^\bullet[l]) &\cong D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet, \nu T^\bullet[-l]) \\ &\cong D\mathrm{Hom}_A(\mathrm{H}^l(P^\bullet), \mathrm{H}^0(\nu T^\bullet)) \\ &= 0, \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet[-1], T^\bullet) &\cong D\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, \nu P^\bullet[-1]) \\ &\cong D\mathrm{Hom}_A(\mathrm{H}^1(T^\bullet), \mathrm{H}^0(\nu P^\bullet)) \\ &= 0. \end{aligned}$$

□

This finishes the proof of Lemma 3.3.4. □

Remark 3.3.5. Consider the case where $l(P^\bullet) = 1$ in the above lemma. Then $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet \oplus P^\bullet, (T^\bullet \oplus P^\bullet)[i]) = 0$ for $i \neq 0$ and by [26, Lemma 1.8] we have $P^\bullet \in \mathrm{add}(T^\bullet)$.

Theorem 3.3.6. *Assume A is selfinjective and representation-finite. Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a complex with $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Then there exists some $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $Q^\bullet \oplus P^\bullet$ is a tilting complex. In particular, if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of the Grothendieck group $K_0(A)$, then $P^\bullet$ is a tilting complex.*

Proof. Set $l = l(P^\bullet)$. We may assume $P^i = 0$ unless $0 \leq i \leq l$. In case $l \leq 1$, this is a special case of Corollary 3.3.2. Assume $l \geq 2$. Let $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex constructed in Lemma 3.3.4 and set $B = \mathrm{End}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(T^\bullet)$. There exists an equivalence of triangulated categories $F : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ which sends $T^\bullet$ to B. Denote by $G : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ a quasi-inverse of F. Set $\bar{P}^\bullet = F(P^\bullet)$. Then $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}B)}(\bar{P}^\bullet, \bar{P}^\bullet[i]) \cong \mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$. Also, by Lemma 3.1.8 $\nu\bar{P}^\bullet \cong F(\nu P^\bullet)$ and hence $\mathrm{add}(\bar{P}^\bullet) = \mathrm{add}(\nu\bar{P}^\bullet)$. Furthermore,

$$\begin{aligned}\mathrm{H}^i(\bar{P}^\bullet) &\cong \mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}B)}(B, \bar{P}^\bullet[i]) \\ &\cong \mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, P^\bullet[i]) \\ &= 0\end{aligned}$$

for $i \geq l$ and $\mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}B)}(\bar{P}^\bullet[i], B) \cong \mathrm{Hom}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet[i], T^\bullet) = 0$ for $i < 0$, so that $l(\bar{P}^\bullet) \leq l-1$. Thus by induction hypothesis there exists some $\bar{Q}^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ such that $\bar{Q}^\bullet \oplus \bar{P}^\bullet$ is a tilting complex. Then, by setting $Q^\bullet = G(\bar{Q}^\bullet)$, $Q^\bullet \oplus P^\bullet$ is a tilting complex. □

Theorem 3.3.7. *Assume A is selfinjective and representation-finite. Then for any selfinjective Artin R-algebra B derived equivalent to A the following hold.*

(1) There exists a sequence of selfinjective Artin R-algebras $A = B_0, B_1, \cdots, B_m = B$ such that for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1.

(2) The Nakayama permutation of B coincides with that of A.

Proof. (1) Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex with $B \cong \mathrm{End}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$. Set $l = l(P^\bullet)$. In case $l \leq 1$, we have nothing to prove. Assume $l \geq 2$. Let $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex constructed in Lemma 3.3.4. Set $B_1 = \mathrm{End}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(T^\bullet)$ and let $F : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \to \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B_1})$ be an equivalence of triangulated categories which sends $T^\bullet$ to B_1. Note that B_1 is selfinjective and representation-finite, and that $P_1^\bullet = F(P^\bullet)$ is a tilting complex with $B \cong \mathrm{End}_{\mathcal{K}(\mathrm{Mod}\text{-}B_1)}(P_1^\bullet)$. Also, as in the proof of Theorem 3.3.6, we have $l(P_1^\bullet) \leq l - 1$. The assertion now follows by induction.

(2) By (1) and Lemma 3.3.4. □

Chapter 4

Frobenius extensions and tilting complexes

Let A be a ring and $e \in A$ an idempotent. Assume A contains a subring R such that $xe = ex$ for all $x \in R$, Ae_R is finitely generated and eA_A is embedded in $\mathrm{Hom}_R(Ae, R_R)_A$ as a submodule. Then A/AeA is finitely presented as a right A-module and $\mathrm{Hom}_A(A/AeA, eA) = 0$. Thus by [25, Proposition 1.2] there exists a tilting complex (see [39]) of the form

$$T^\bullet : \cdots \to 0 \to T^{-1} \to T^0 \to 0 \to \cdots$$

such that $T^0 \in \mathrm{add}((1-e)A_A)$, $T^{-1} \in \mathrm{add}(eA_A)$ and $eA[1] \in \mathrm{add}(T^\bullet)$. This type of tilting complex plays an important role in the theory of derived equivalences. For instance, Rickard [40] showed that the Brauer tree algebras over a field with the same numerical invariants are derived equivalent to each other and then Okuyama pointed out that such derived equivalences are given as iterations of derived equivalences induced by tilting complexes of the above type. Our aim is to provide a way to construct extensions A of a given ring R containing such an idempotent. To do so, we need the notion of Frobenius extensions of rings due to Nakayama-Tsuzuku [36, 37] (cf. also Kasch [28, 29]) which we modify as follows. Let A be a ring containing a ring R as a subring. Then A is said to be a Frobenius extension of R if the following conditions are satisfied: (F1) A_R and ${}_RA$ are finitely generated projective; and (F2) $A_A \cong \mathrm{Hom}_R(A, R_R)_A$ and ${}_AA \cong {}_A\mathrm{Hom}_R(A, {}_RR)$. We will see that Frobenius extensions preserve various homological properties (cf. [28], [29], [35], [36], [37] and so on). For instance, the following hold: inj dim $A_A \le$ inj dim R_R and inj dim ${}_AA \le$ inj dim ${}_RR$; if R is a noetherian ring satisfying the Auslander condition (see [12]) then so is A; and, if R is a quasi-Frobenius ring, i.e., a selfinjective artinian ring then so is A.

For any integer $n \ge 1$, any permutation π of $I = \{1, \cdots, n\}$ and any ring R,

This chapter is based on my joint paper with M. Hoshino [4].

we will provide a way to construct a Frobenius extension A of R which enjoys the following properties: (a) $1 = \sum_{i \in I} e_i$ with the e_i orthogonal idempotents; (b) $e_i x = x e_i$ for all $i \in I$ and $x \in R$; (c) $e_i A e_j \neq 0$ for all $i, j \in I$; (d) $e_i A_A \not\cong e_j A_A$ unless $i = j$; (e) every $e_i A e_i$ is a local ring whenever so is R; (f) $e_i A_A \cong \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)_A$ and ${}_A Ae_{\pi(i)} \cong {}_A\mathrm{Hom}_R(e_i A, {}_R R)$ for all $i \in I$; and (g) there exists a ring automorphism $\eta \in \mathrm{Aut}(A)$ such that $\eta(e_i) = e_{\pi(i)}$ for all $i \in I$. In particular, for any nonempty π-stable subset J of I, we get a desired idempotent $e = \sum_{j \in J} e_j$. In case π is cyclic, we have constructed such a Frobenius extension in [24] (cf. also [31] and [38]). We generalize this construction. Namely, we define an appropriate multiplication on a free right R-module A with a basis $\{e_{ij}\}_{i,j \in I} \cup \{v_i\}_{i \in I_0}$, where $I_0 = \{i \in I \mid \pi(i) = i\}$, and then set $e_i = e_{ii}$ for $i \in I$. To do so, we need a certain pair (t, ω) of an integer $t \geq 1$ and a mapping $\omega : I \times I \to \mathbb{Z}$ and a certain pair (c, σ) of a nonunit $c \in R$ and a ring automorphism $\sigma \in \mathrm{Aut}(R)$. Although the ring structure of A depends on the choice of (t, ω) and (c, σ), the properties (a)–(g) above are always enjoyed. Finally, consider the case where c is regular. Then we will see that if I_0 is empty then A can be embedded as a subring in the $n \times n$ full matrix ring $\mathrm{M}_n(R)$ over R, and that if $i \in I \setminus I_0$ then A is derived equivalent to a generalized triangular matrix ring

$$\begin{pmatrix} e_i A e_i & \mathrm{Ext}^1_A(A/Ae_iA, e_iA) \\ 0 & A/Ae_iA \end{pmatrix}$$

and $\mathrm{Ext}^1_A(A/Ae_iA, e_iA) \cong e_{\pi^{-1}(i)}(A/Ae_iA)$ as right (A/Ae_iA)-modules.

For a ring R, we denote by $\mathrm{Z}(R)$ the center of R, by $R^\times$ the set of units in R and by $\mathrm{Aut}(R)$ the group of ring automorphisms of R. We denote by Mod-R the category of right R-modules and sometimes consider left R-modules as right R^{op}-modules, where R^{op} denotes the opposite ring of R. We use the notation X_R (resp., ${}_R X$) to stress that the module X considered is a right (resp., left) R-module. For a module X, by an injective resolution of X we mean a cochain complex $I^\bullet$ of injective modules such that $I^i = 0$ for $i < 0$, $\mathrm{H}^i(I^\bullet) = 0$ for $i > 0$ and $\mathrm{H}^0(I^\bullet) \cong X$, where $\mathrm{H}^i(-)$ denotes the i^{th} cohomology. We refer to [15] for standard homological algebra in module categories.

4.1 Definition and basic properties

In this chapter, a ring A is said to be an extension of a ring R if A contains R as a subring. We start by modifying the notion of Frobenius extensions of rings due to Nakayama-Tsuzuku [36, 37] (cf. also Kasch [28, 29]) as follows.

Definition 4.1.1. Let A be an extension of a ring R. Then A is said to be a Frobenius extension of R if the following conditions are satisfied:
(F1) A_R and ${}_R A$ are finitely generated projective; and
(F2) $A_A \cong \mathrm{Hom}_R(A, R_R)_A$ and ${}_A A \cong {}_A\mathrm{Hom}_R(A, {}_R R)$.

Remark 4.1.2. Let A be an extension of a ring R and assume there exists an isomorphism of right A-modules $\phi : A_A \xrightarrow{\sim} \mathrm{Hom}_R(A, R_R)_A$. Then the following hold.

(1) There exists a ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. In particular, ϕ is an isomorphism of R-A-bimodules if and only if $\theta(x) = x$ for all $x \in R$.

(2) Assume A_R is finitely generated projective. Then ${}_R\mathrm{Hom}_R(A, R_R)$ is finitely generated projective and we have an isomorphism of A-R-bimodules

$$ {}_A A_R \xrightarrow{\sim} {}_A\mathrm{Hom}_R(\mathrm{Hom}_R(A, R_R), {}_R R)_R, a \mapsto (h \mapsto h(a)). $$

Thus, if ϕ is an isomorphism of R-A-bimodules, then ${}_R A$ is finitely generated projective and we have an isomorphism of A-R-bimodules $\psi : {}_A A_R \xrightarrow{\sim} {}_A\mathrm{Hom}_R(A, {}_R R)_R$ such that $\psi(a)(b) = \phi(b)(a)$ for all $a, b \in A$.

Throughout the rest of this section, A is a Frobenius extension of R. We fix an isomorphism of right A-modules $\phi : A_A \xrightarrow{\sim} \mathrm{Hom}_R(A, R_R)_A$. Then, as remarked above, there exists a ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. For a right (resp., left) A-module M_A (resp., ${}_A L$) we denote by $M_{\theta(R)}$ (resp., ${}_{\theta(R)}L$) the right (resp., left) R-module on which R operates via $\theta : R \to A$. Then ϕ yields an isomorphism of R-A-bimodules $\phi : {}_{\theta(R)}A_A \xrightarrow{\sim} {}_R\mathrm{Hom}_R(A, R_R)_A$. Similarly, we fix an isomorphism of left A-modules $\psi : {}_A A \xrightarrow{\sim} {}_A\mathrm{Hom}_R(A, {}_R R)$. Then there exists a ring homomorphism $\eta : R \to A$ such that $\psi(1)x = \eta(x)\psi(1)$ for all $x \in R$. For a right (resp., left) A-module M_A (resp., ${}_A L$) we denote by $M_{\eta(R)}$ (resp., ${}_{\eta(R)}L$) the right (resp., left) R-module on which R operates via $\eta : R \to A$. Then ψ yields an isomorphism of A-R-bimodules $\psi : {}_A A_{\eta(R)} \xrightarrow{\sim} {}_A\mathrm{Hom}_R(A, {}_R R)_R$. Note that ${}_{\theta(R)}A$ and $A_{\eta(R)}$ are finitely generated projective.

Recall that in [36, 37] A is said to be a Frobenius extension of second kind if θ induces a ring automorphism of R and to be a Frobenius extension of first kind if $\theta(x) = x$ for all $x \in R$. However, we will see in Section 3 that $\theta(R) \neq R$ in general. In the following, we collect several basic properties of Frobenius extensions (cf. [28], [29], [35], [36], [37] and so on). By symmetry, "right" and "left" can be exchanged in the following statements.

Remark 4.1.3. Let $X \in \text{Mod-}R$, $M \in \text{Mod-}A$ and $L \in \text{Mod-}A^{\mathrm{op}}$. Then we have the following bifunctorial isomorphisms:

(1) $\mathrm{Hom}_R(M, X \otimes_R A) \cong \mathrm{Hom}_R(M_{\eta(R)}, X)$;

(2) $\mathrm{Hom}_A(\mathrm{Hom}_R(A, X), M) \cong \mathrm{Hom}_R(X, M_{\theta(R)})$; and

(3) $\mathrm{Hom}_R(A, X) \otimes_A L \cong X \otimes_R {}_{\theta(R)}L$.

Proof. Since ${}_R A$ and A_R are finitely generated projective, we have functorial isomorphisms in Mod-A

$$ X \otimes_R A \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_R(A, {}_R R), X), x \otimes a \mapsto (h \mapsto xh(a)), $$

$$X \otimes_R \mathrm{Hom}_R(A, R_R) \xrightarrow{\sim} \mathrm{Hom}_R(A, X), x \otimes h \mapsto (a \mapsto xh(a))$$

which are special cases of Watt's theorem (cf. [42]). Since ${}_A\mathrm{Hom}_R(A, {}_RR)_R \cong {}_AA_{\eta(R)}$, we have bifunctorial isomorphisms

$$\begin{aligned}\mathrm{Hom}_A(M, X \otimes_R A) &\cong \mathrm{Hom}_A(M, \mathrm{Hom}_R(A_{\eta(R)}, X)) \\ &\cong \mathrm{Hom}_R(M \otimes_A A_{\eta(R)}, X) \\ &\cong \mathrm{Hom}_R(M_{\eta(R)}, X).\end{aligned}$$

Similarly, since ${}_R\mathrm{Hom}_R(A, R_R)_A \cong {}_{\theta(R)}A_A$, we have bifunctorial isomorphisms

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{Hom}_R(A, X), M) &\cong \mathrm{Hom}_A(X \otimes_R {}_{\theta(R)}A, M) \\ &\cong \mathrm{Hom}_R(X, \mathrm{Hom}_A({}_{\theta(R)}A, M)) \\ &\cong \mathrm{Hom}_R(X, M_{\theta(R)}),\end{aligned}$$

$$\mathrm{Hom}_R(A, X) \otimes_A L \cong X \otimes_R {}_{\theta(R)}A \otimes_A L \cong X \otimes_R {}_{\theta(R)}L.$$

□

The first two isomorphisms of the following preliminary lemma are known as Eckmann-Shapiro lemma.

Lemma 4.1.4. *Let $X \in$ Mod-R, $M \in$ Mod-A and $L \in$ Mod-A^{op}. Then for any $i \geq 0$ we have the following bifunctorial isomorphisms:*

(1) $\mathrm{Ext}^i_A(M, \mathrm{Hom}_R(A, X)) \cong \mathrm{Ext}^i_R(M, X)$;

(2) $\mathrm{Ext}^i_A(X \otimes_R A, M) \cong \mathrm{Ext}^i_R(X, M)$;

(3) $\mathrm{Tor}^A_i(X \otimes_R A, L) \cong \mathrm{Tor}^R_i(X, L)$;

(4) $\mathrm{Ext}^i_A(M, X \otimes_R A) \cong \mathrm{Ext}^i_R(M_{\eta(R)}, X)$;

(5) $\mathrm{Ext}^i_A(\mathrm{Hom}_R(A, X), M) \cong \mathrm{Ext}^i_R(X, M_{\theta(R)})$; *and*

(6) $\mathrm{Tor}^A_i(\mathrm{Hom}_R(A, X), L) \cong \mathrm{Tor}^R_i(X, {}_{\theta(R)}L)$.

Proof. See [15, Chapter VI, Section 4] for the first three isomorphisms; (1) follows by the projectivity of A_R and (2), (3) follow by the flatness of ${}_RA$.

Similarly, according to Remark 4.1.3, the last three isomorphisms follow by the exactness of $- \otimes_R A$ and $\mathrm{Hom}_R(A, -)$. □

Proposition 4.1.5. *The following hold.*

(1) If R is right noetherian (resp., artinian), so is A.

(2) $\mathrm{Ext}^i_A(M, A) \cong \mathrm{Ext}^i_R(M, R)$ *for all $M \in$ Mod-A and $i \geq 0$. In particular,* inj dim $A_A \leq$ inj dim R_R.

(3) If $I^\bullet$ is an injective resolution of R_R, then $\mathrm{Hom}_R(A, I^\bullet)$ *is an injective resolution of A_A with* flat dim $\mathrm{Hom}_R(A, I^i)_A \leq$ flat dim I^i_R *for all $i \geq 0$.*

Proof. (1) follows by the fact that A_R is finitely generated. Also, since $A_A \cong \mathrm{Hom}_R(A, R_R)_A$, (2) follows by Lemma 4.1.4(1). Finally, since $\mathrm{Hom}_R(A, -)$ is exact, and since $A_A \cong \mathrm{Hom}_R(A, R_R)_A$, (3) follows by (1), (6) of Lemma 4.1.4. □

Lemma 4.1.6. *Assume the inclusion $R \to A$ is a split monomorphism of R-R-bimodules. Then for any $X \in$ Mod-R the following hold.*

(1) inj dim $\mathrm{Hom}_R(A, X)_A =$ inj dim $X \otimes_R A_A =$ inj dim X_R.

(2) proj dim $\mathrm{Hom}_R(A, X)_A =$ proj dim $X \otimes_R A_A =$ proj dim X_R.

(3) flat dim $\mathrm{Hom}_R(A, X)_A =$ flat dim $X \otimes_R A_A =$ flat dim X_R.

Proof. Note that every $X \in$ Mod-R is a direct summand of both $\mathrm{Hom}_R(A, X)_R$ and $X \otimes_R A_R$.

(1) By Lemma 4.1.4(1) inj dim $\mathrm{Hom}_R(A, X)_A \leq$ inj dim X_R. Conversely, assume inj dim $\mathrm{Hom}_R(A, X)_A = d < \infty$. Then for any $Y \in$ Mod-R and $i > d$ by Lemma 4.1.4(1) $\mathrm{Ext}^i_R(\mathrm{Hom}_R(A, Y), X) = 0$ and hence $\mathrm{Ext}^i_R(Y, X) = 0$. Thus inj dim $X_R \leq d$.

Similarly, by Lemma 4.1.4(4) inj dim $X \otimes_R A_A \leq$ inj dim X_R. Conversely, assume inj dim $X \otimes_R A_A = d < \infty$. Then for any $Y \in$ Mod-R and $i > d$ by Lemma 4.1.4(2) $\mathrm{Ext}^i_R(Y, X \otimes_R A) = 0$ and hence $\mathrm{Ext}^i_R(Y, X) = 0$. Thus inj dim $X_R \leq d$.

(2) and (3) follow by the same arguments as in (1). □

Proposition 4.1.7. *Assume the inclusion $R \to A$ is a split monomorphism of R-R-bimodules. Then the following hold.*

(1) If A is right noetherian (resp., artinian), so is R.

(2) inj dim $A_A =$ inj dim R_R.

(3) If $I^\bullet$ is an injective resolution of R_R, then $\mathrm{Hom}_R(A, I^\bullet)$ *is an injective resolution of A_A with* flat dim $\mathrm{Hom}_R(A, I^i)_A =$ flat dim I^i_R *for all $i \geq 0$.*

Proof. (1) Take a homomorphism of R-R-bimodules $\gamma : A \to R$ such that $\gamma(x) = x$ for all $x \in R$. Then $\gamma(\mathfrak{a}A) = \mathfrak{a}$ for all right ideals $\mathfrak{a}$ of R and the assertion follows.

(2) Since $A_A \cong \mathrm{Hom}_R(A, R_R)_A$, this follows by Lemma 4.1.6(1).

(3) follows by Proposition 4.1.5(3) and Lemma 4.1.6(3). □

Definition 4.1.8. A Frobenius extension A of R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

4.2 Notation

To construct a desired Frobenius extension, we fix the following notation which will be kept throughout this and the next sections.

Let $n \geq 1$ be an integer, π a permutation of $I = \{1, \cdots, n\}$ and $I_0 = \{i \in I \mid \pi(i) = i\}$. Let $t \geq 1$ be an integer, let $\omega : I \times I \to \mathbb{Z}$ be a mapping and define a mapping $\chi : I \to \mathbb{Z}$ as follows:

$$\chi(i) = \begin{cases} t & \text{if } i \in I_0, \\ \omega(i, \pi(i)) & \text{if } i \in I \setminus I_0. \end{cases}$$

We assume the following conditions are satisfied:
(W1) $\omega(i,i) = 0$ for all $i \in I$;
(W2) $\omega(i,j) + \omega(j,k) \geq \omega(i,k)$ for all $i,j,k \in I$;
(W3) $\omega(i,j) + \omega(j,i) \geq 1$ unless $i = j$; and
(W4) $\omega(i,j) + \omega(j,\pi(i)) = \chi(i)$ unless $i = j \in I_0$.

Example 4.2.1. Let $t = 2$ and define $\omega : I \times I \to \mathbb{Z}$ as follows: $\omega(i,j) = 0$ if $i = j$, $\omega(i,j) = 2$ if $j = \pi(i) \neq i$ and $\omega(i,j) = 1$ otherwise. Then the conditions (W1)–(W4) are satisfied.

Lemma 4.2.2. *We have $\omega(\pi(i), \pi(j)) = \omega(i,j) - \chi(i) + \chi(j)$ for all $i, j \in I$.*

Proof. We may assume $i \neq j$. In case $j \neq \pi(i)$, by (W4) $\{\omega(i,j) - \chi(i)\} + \chi(j) = -\omega(j, \pi(i)) + \{\omega(j, \pi(i)) + \omega(\pi(i), \pi(j))\} = \omega(\pi(i), \pi(j))$. Assume $j = \pi(i)$. Then $i \in I \setminus I_0$ and $\omega(i,j) - \chi(i) + \chi(j) = \omega(i, \pi(i)) - \chi(i) + \chi(\pi(i)) = \chi(\pi(i))$. Note that by (W1) $\omega(\pi(j), \pi(\pi(i))) = 0$. Thus, since $\pi(i) \neq \pi(j)$, by (W4) $\chi(\pi(i)) = \omega(\pi(i), \pi(j)) + \omega(\pi(j), \pi(\pi(i))) = \omega(\pi(i), \pi(j))$. □

For the sake of convenience, we define a mapping $\lambda : I \times I \times I \to \mathbb{Z}$ as follows:

$$\lambda(i,j,k) = \omega(i,j) + \omega(j,k) - \omega(i,k)$$

for all $i, j, k \in I$. It is easy to see that the following hold:
(L1) $\lambda(i,j,k) \geq 0$ for all $i,j,k \in I$;
(L2) $\lambda(i,j,k) = 0$ if either $i = j$ or $j = k$;
(L3) $\lambda(i,j,i) = \lambda(j,i,j) \geq 1$ unless $i = j$;
(L4) $\lambda(i,j,\pi(i)) = 0$ for all $i \in I \setminus I_0$ and $j \in I$; and
(L5) $\lambda(i,j,i) = \chi(i)$ for all $i \in I_0$ and $j \in I \setminus \{i\}$.

Lemma 4.2.3. *The following hold.*

(1) $\omega(i,j) + \omega(j,k) = \lambda(i,j,k) + \omega(i,k)$ *for all* $i,j,k \in I$.

(2) $\lambda(i,j,k) + \lambda(i,k,l) = \lambda(i,j,l) + \lambda(j,k,l)$ *for all* $i,j,k,l \in I$.

(3) $\lambda(\pi(i), \pi(j), \pi(k)) = \lambda(i,j,k)$ *for all* $i,j,k \in I$.

(4) $\lambda(i,j,k) = \lambda(j,k,i)$ *for all* $i \in I_0$ *and* $j,k \in I \setminus \{i\}$.

Proof. (1) and (2) follow by the definition and (3) follows by Lemma 4.2.2.

(4) By (2) and (L5) $\lambda(i,j,k) - \lambda(j,k,i) = \lambda(i,j,i) - \lambda(i,k,i) = \chi(i) - \chi(i) = 0$. $\square$

Also, we fix a ring R together with a pair of a nonunit $c \in R \setminus R^\times$ and a ring automorphism $\sigma \in \mathrm{Aut}(R)$ satisfying the following condition:

$$(*) \quad \sigma(c) = c \text{ and } xc = c\sigma(x) \text{ for all } x \in R.$$

This is obviously satisfied if either $c = 0$ and σ is arbitrary, or $c \in \mathrm{Z}(R)$ and $\sigma = \mathrm{id}_R$. We provide a non-trivial example.

Example 4.2.4. Let $k[X]$ be a polynomial ring in one variable X over a commutative ring k and $\mathfrak{a} = (X^m)$ an ideal of $k[X]$ generated by X^m with $m \geq 3$. Set $R = k[X]/\mathfrak{a}$, $x = X + \mathfrak{a}$ and $c = x^r$ with $m > r \geq (m+1)/2$. Then there exists $\sigma \in \mathrm{Aut}(R)$ such that $\sigma(f(x)) = f(x + c)$ for all $f(X) \in k[X]$. It is easy to see that the condition $(*)$ is satisfied.

Here, we deal with the case of $n = 1$. Let S be a free right R-module with a basis $\{e, v\}$ and define the multiplication on S subject to the following axioms:
(S1) $e^2 = e$, $v^2 = -vc^t$ and $ev = v = ve$; and
(S2) $xe = ex$ and $xv = v\sigma^t(x)$ for all $x \in R$.

Lemma 4.2.5. *The following hold.*

(1) *S is an associative ring with* $1 = e$.

(2) *S is a split Frobenius extension of R, where R is considered as a subring of S via the injective ring homomorphism* $R \to S, x \mapsto ex$.

(3) *If R is local, so is S.*

Proof. (1) and (2) will be proved in the next section (see Theorem 4.3.1).

(3) Let $\mathfrak{m} = R \setminus R^\times$ and $\mathfrak{M} = e\mathfrak{m} + vR$. It is easy to see that $\mathfrak{M}$ is an ideal of S. We claim that $\mathfrak{M} = S \setminus S^\times$. Take a basis $\{\alpha, \rho\}$ for ${}_R\mathrm{Hom}_R(S, R_R)$ such that $a = e\alpha(a) + v\rho(a)$ for all $a \in S$. Then for any $a, b \in A$ we have $\alpha(ab) = \alpha(a)\alpha(b)$ and $\rho(ab) = \sigma^t(\alpha(a))\rho(b) + \rho(a)\alpha(b) - c^t\sigma^t(\rho(a))\rho(b)$. For any $a \in S^\times$ we have $\alpha(a) \in R^\times$ and $a \in S \setminus \mathfrak{M}$. Let $a \in S \setminus \mathfrak{M}$. Then $\alpha(a) \in R^\times$ and, since $c^t \in \mathfrak{m}$, $\alpha(a) - c^t\rho(a) \in R^\times$. Thus, by setting $x = \sigma^t(\alpha(a))^{-1}\rho(a)(c^t\rho(a) - \alpha(a))^{-1}$, we have $(e\alpha(a)^{-1} + vx)a = e$. Similarly, a has a right inverse too. Thus $a \in S^\times$. $\square$

4.3 Construction

Let A be a free right R-module with a basis $\{e_{ij}\}_{i,j \in I} \cup \{v_i\}_{i \in I_0}$ and define the multiplication on A subject to the following axioms:
(A1) $e_{ij}e_{kl} = 0$ unless $j = k$;
(A2) $e_{ij}e_{jk} = e_{ik}c^{\lambda(i,j,k)}$ unless $i = k \in I_0$ and $j \in I \setminus \{i\}$;

(A3) $e_{ij}e_{ji} = v_i + e_{ii}c^{\chi(i)}$ for all $i \in I_0$ and $j \in I \setminus \{i\}$;
(A4) $v_iv_j = 0$ unless $i = j$ and $v_i^2 = -v_ic^{\chi(i)}$ for all $i \in I_0$;
(A5) $v_ie_{jk} = 0 = e_{ij}v_k$ unless $i = j = k$ and $v_ie_{ii} = v_i = e_{ii}v_i$ for all $i \in I_0$;
(A6) $xe_{ij} = e_{ij}\sigma^{\omega(i,j)}(x)$ for all $i, j \in I$ and $x \in R$; and
(A7) $xv_i = v_i\sigma^{\chi(i)}(x)$ for all $i \in I_0$ and $x \in R$.

As usual, we require $c^0 = 1$ even if $c = 0$. We fix a basis $\{\alpha_{ij}\}_{i,j\in I} \cup \{\rho_i\}_{i\in I_0}$ for ${}_R\mathrm{Hom}_R(A, R_R)$ such that

$$a = \sum_{i,j\in I} e_{ij}\alpha_{ij}(a) + \sum_{i\in I_0} v_i\rho_i(a)$$

for all $a \in A$. Recall that $\chi(i) = t$ for all $i \in I_0$. For any $a, b \in A$ we have

$$\begin{aligned} ab = &\sum_{i,j,k\in I} e_{ij}e_{jk}\sigma^{\omega(j,k)}(\alpha_{ij}(a))\alpha_{jk}(b) \\ &+ \sum_{i\in I_0} v_i\{\sigma^t(\alpha_{ii}(a))\rho_i(b) + \rho_i(a)\alpha_{ii}(b) - c^t\sigma^t(\rho_i(a))\rho_i(b)\} \\ = &\sum_{i,j,k\in I} e_{ik}c^{\lambda(i,j,k)}\sigma^{\omega(j,k)}(\alpha_{ij}(a))\alpha_{jk}(b) + \sum_{i\in I_0, j\in I\setminus\{i\}} v_i\sigma^{\omega(j,i)}(\alpha_{ij}(a))\alpha_{ji}(b) \\ &+ \sum_{i\in I_0} v_i\{\sigma^t(\alpha_{ii}(a))\rho_i(b) + \rho_i(a)\alpha_{ii}(b) - c^t\sigma^t(\rho_i(a))\rho_i(b)\} \end{aligned}$$

and hence the following hold:
(M1) $\alpha_{ik}(ab) = \sum_{j\in I} c^{\lambda(i,j,k)}\sigma^{\omega(j,k)}(\alpha_{ij}(a))\alpha_{jk}(b)$ for all $i, k \in I$; and
(M2) $\rho_i(ab) = \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a))\alpha_{ji}(b) + \sigma^t(\alpha_{ii}(a))\rho_i(b) + \rho_i(a)\alpha_{ii}(b) - c^t\sigma^t(\rho_i(a))\rho_i(b)$ for all $i \in I_0$.

In the following, we set $e_i = e_{ii}$ and $\alpha_i = \alpha_{ii}$ for $i \in I$. Note that by (W1), (A6) $xe_i = e_ix$ for all $i \in I$ and $x \in R$, and that by (L2), (A1), (A2) and (A5) $1 = \sum_{i\in I} e_i$ with the e_i orthogonal idempotents.

Theorem 4.3.1. *The following hold.*

(1) *A is an associative ring with $1 = \sum_{i\in I} e_i$, where the e_i are orthogonal idempotents.*

(2) *$e_iAe_i = e_iR + v_iR$ for all $i \in I_0$ and $e_iAe_j = e_{ij}R$ unless $i = j \in I_0$. In particular, $e_iAe_i \cong S$ as rings for all $i \in I_0$ and $e_iAe_i \cong R$ as rings for all $i \in I \setminus I_0$.*

(3) *$e_iA_A \not\cong e_jA_A$ unless $i = j$.*

(4) *$e_iA_A \cong \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)_A$ and ${}_AAe_{\pi(i)} \cong {}_A\mathrm{Hom}_R(e_iA, {}_RR)$ for all $i \in I$, so that for any nonempty π-stable subset J of I, by setting $e = \sum_{i\in J} e_i$, we have $eA_A \cong \mathrm{Hom}_R(Ae, R_R)_A$ and ${}_AAe \cong {}_A\mathrm{Hom}_R(eA, {}_RR)$.*

(5) A is a split Frobenius extension of R, where R is considered as a subring of A via the injective ring homomorphism $R \to A, x \mapsto \sum_{i\in I} e_i x$.

Proof. (1) Let $a_1, a_2, a_3 \in A$. For any $i, l \in I$ by (M1) we have

$$\begin{aligned}
&\alpha_{il}(a_1(a_2a_3)) \\
&= \sum_{j\in I} c^{\lambda(i,j,l)}\sigma^{\omega(j,l)}(\alpha_{ij}(a_1))\alpha_{jl}(a_2a_3) \\
&= \sum_{j\in I} c^{\lambda(i,j,l)}\sigma^{\omega(j,l)}(\alpha_{ij}(a_1))\{\sum_{k\in I} c^{\lambda(j,k,l)}\sigma^{\omega(k,l)}(\alpha_{jk}(a_2))\alpha_{kl}(a_3)\} \\
&= \sum_{j,k\in I} c^{\lambda(i,j,l)+\lambda(j,k,l)}\sigma^{\lambda(j,k,l)+\omega(j,l)}(\alpha_{ij}(a_1))\sigma^{\omega(k,l)}(\alpha_{jk}(a_2))\alpha_{kl}(a_3),
\end{aligned}$$

$$\begin{aligned}
&\alpha_{il}((a_1a_2)a_3) \\
&= \sum_{k\in I} c^{\lambda(i,k,l)}\sigma^{\omega(k,l)}(\alpha_{ik}(a_1a_2))\alpha_{kl}(a_3) \\
&= \sum_{k\in I} c^{\lambda(i,k,l)}\sigma^{\omega(k,l)}(\{\sum_{j\in I} c^{\lambda(i,j,k)}\sigma^{\omega(j,k)}(\alpha_{ij}(a_1))\alpha_{jk}(a_2)\})\alpha_{kl}(a_3) \\
&= \sum_{j,k\in I} c^{\lambda(i,k,l)+\lambda(i,j,k)}\sigma^{\omega(k,l)+\omega(j,k)}(\alpha_{ij}(a_1))\sigma^{\omega(k,l)}(\alpha_{jk}(a_2))\alpha_{kl}(a_3)
\end{aligned}$$

and hence by (1), (2) of Lemma 4.2.3 $\alpha_{il}(a_1(a_2a_3)) = \alpha_{il}((a_1a_2)a_3)$. Similarly, for any $i \in I_0$ by (M1), (M2) we have

$$\begin{aligned}
&\rho_i(a_1(a_2a_3)) \\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2a_3) \\
&\quad + \sigma^t(\alpha_i(a_1))\rho_i(a_2a_3) + \rho_i(a_1)\alpha_i(a_2a_3) - c^t\sigma^t(\rho_i(a_1))\rho_i(a_2a_3) \\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\{\sum_{k\in I} c^{\lambda(j,k,i)}\sigma^{\omega(k,i)}(\alpha_{jk}(a_2))\alpha_{ki}(a_3)\} \\
&\quad + \sigma^t(\alpha_i(a_1))\{\sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) + \sigma^t(\alpha_i(a_2))\rho_i(a_3) \\
&\qquad + \rho_i(a_2)\alpha_i(a_3) - c^t\sigma^t(\rho_i(a_2))\rho_i(a_3)\} \\
&\quad + \rho_i(a_1)\{\sum_{j\in I} c^{\lambda(i,j,i)}\sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3)\} \\
&\quad - c^t\sigma^t(\rho_i(a_1))\{\sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) + \sigma^t(\alpha_i(a_2))\rho_i(a_3) \\
&\qquad + \rho_i(a_2)\alpha_i(a_3) - c^t\sigma^t(\rho_i(a_2))\rho_i(a_3)\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,k \in I\setminus\{i\}} c^{\lambda(j,k,i)} \sigma^{\lambda(j,k,i)+\omega(j,i)}(\alpha_{ij}(a_1))\sigma^{\omega(k,i)}(\alpha_{jk}(a_2))\alpha_{ki}(a_3) \\
&\qquad + \sum_{j \in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2)\alpha_i(a_3) \\
&\quad + \sum_{j \in I\setminus\{i\}} \sigma^t(\alpha_i(a_1))\sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) + \sigma^t(\alpha_i(a_1))\sigma^t(\alpha_i(a_2))\rho_i(a_3) \\
&\qquad + \sigma^t(\alpha_i(a_1))\rho_i(a_2)\alpha_i(a_3) - c^t\sigma^{2t}(\alpha_i(a_1))\sigma^t(\rho_i(a_2))\rho_i(a_3) \\
&\quad + \sum_{j \in I\setminus\{i\}} c^t\sigma^t(\rho_i(a_1))\sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) + \rho_i(a_1)\alpha_i(a_2)\alpha_i(a_3) \\
&\quad - \sum_{j \in I\setminus\{i\}} c^t\sigma^t(\rho_i(a_1))\sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) - c^t\sigma^t(\rho_i(a_1))\sigma^t(\alpha_i(a_2))\rho_i(a_3) \\
&\qquad - c^t\sigma^t(\rho_i(a_1))\rho_i(a_2)\alpha_i(a_3) + c^{2t}\sigma^{2t}(\rho_i(a_1))\sigma^t(\rho_i(a_2))\rho_i(a_3),
\end{aligned}
$$

and

$$
\begin{aligned}
&\rho_i((a_1a_2)a_3) \\
&= \sum_{j \in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1a_2))\alpha_{ji}(a_3) \\
&\qquad + \sigma^t(\alpha_i(a_1a_2))\rho_i(a_3) + \rho_i(a_1a_2)\alpha_i(a_3) - c^t\sigma^t(\rho_i(a_1a_2))\rho_i(a_3) \\
&= \sum_{j \in I\setminus\{i\}} \sigma^{\omega(j,i)}(\{\sum_{k\in I} c^{\lambda(i,k,j)}\sigma^{\omega(k,j)}(\alpha_{ik}(a_1))\alpha_{kj}(a_2)\})\alpha_{ji}(a_3) \\
&\quad + \sigma^t(\{\sum_{j\in I} c^{\lambda(i,j,i)}\sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2)\})\rho_i(a_3) \\
&\quad + \{\sum_{j \in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2) + \sigma^t(\alpha_i(a_1))\rho_i(a_2) \\
&\qquad + \rho_i(a_1)\alpha_i(a_2) - c^t\sigma^t(\rho_i(a_1))\rho_i(a_2)\}\alpha_i(a_3) \\
&\quad - c^t\sigma^t(\{\sum_{j \in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2) + \sigma^t(\alpha_i(a_1))\rho_i(a_2) \\
&\qquad\qquad + \rho_i(a_1)\alpha_i(a_2) - c^t\sigma^t(\rho_i(a_1))\rho_i(a_2)\})\rho_i(a_3)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j,k\in I\setminus\{i\}} c^{\lambda(i,k,j)}\sigma^{\omega(j,i)+\omega(k,j)}(\alpha_{ik}(a_1))\sigma^{\omega(j,i)}(\alpha_{kj}(a_2))\alpha_{ji}(a_3) \\
&\qquad + \sum_{j\in I\setminus\{i\}} \sigma^t(\alpha_i(a_1))\sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) \\
&\quad + \sum_{j\in I\setminus\{i\}} c^t\sigma^{t+\omega(j,i)}(\alpha_{ij}(a_1))\sigma^t(\alpha_{ji}(a_2))\rho_i(a_3) \\
&\qquad + \sigma^t(\alpha_i(a_1))\sigma^t(\alpha_i(a_2))\rho_i(a_3) \\
&\quad + \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2)\alpha_i(a_3) + \sigma^t(\alpha_i(a_1))\rho_i(a_2)\alpha_i(a_3) \\
&\qquad + \rho_i(a_1)\alpha_i(a_2)\alpha_i(a_3) - c^t\sigma^t(\rho_i(a_1))\rho_i(a_2)\alpha_i(a_3) \\
&\quad - \sum_{j\in I\setminus\{i\}} c^t\sigma^{t+\omega(j,i)}(\alpha_{ij}(a_1))\sigma^t(\alpha_{ji}(a_2))\rho_i(a_3) \\
&\qquad - c^t\sigma^{2t}(\alpha_i(a_1))\sigma^t(\rho_i(a_2))\rho_i(a_3) - c^t\sigma^t(\rho_i(a_1))\sigma^t(\alpha_i(a_2))\rho_i(a_3) \\
&\qquad + c^{2t}\sigma^{2t}(\rho_i(a_1))\sigma^t(\rho_i(a_2))\rho_i(a_3)
\end{aligned}
$$

and hence by (1), (4) of Lemma 4.2.3 $\rho_i(a_1(a_2a_3)) = \rho_i((a_1a_2)a_3)$.

(2) Immediate by the construction.

(3) Let $i, j \in I$ and assume there exists an isomorphism $h : e_iA_A \xrightarrow{\sim} e_jA_A$. Let $a \in e_iA$ with $e_j = h(a) = h(e_i)a$. Since $h(ae_j) = h(a)e_j = e_j = h(a)$, $a = ae_j \in e_iAe_j$ and $e_j \in e_jAe_iAe_j$. Suppose to the contrary that $i \neq j$. Then by (2) $e_jAe_iAe_j = e_jAe_ie_iAe_j = e_{ji}Re_{ij}R = e_{ji}e_{ij}R$. If $j \in I \setminus I_0$, then $e_{ji}e_{ij}R = e_jc^{\lambda(j,i,j)}R$. Also, if $j \in I_0$, then $e_{ji}e_{ij}R = (v_j + e_jc^t)R$. In either case, we have $e_j \notin e_{ji}e_{ij}R$, a contradiction.

(4) Consider first the case $i \in I \setminus I_0$. We claim that the homomorphism

$$\phi_i : e_iA_A \to \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)_A, a \mapsto \alpha_{i,\pi(i)}a$$

is an isomorphism. For any $a, b \in A$ by (L4), (M1) we have

$$
\begin{aligned}
(\alpha_{i,\pi(i)}a)(b) &= \alpha_{i,\pi(i)}(ab) \\
&= \sum_{j\in I} c^{\lambda(i,j,\pi(i))}\sigma^{\omega(j,\pi(i))}(\alpha_{ij}(a))\alpha_{j,\pi(i)}(b) \\
&= \sum_{j\in I} \sigma^{\omega(j,\pi(i))}(\alpha_{ij}(a))\alpha_{j,\pi(i)}(b).
\end{aligned}
$$

Thus $\alpha_{i,\pi(i)}a = \sum_{j\in I}\sigma^{\omega(j,\pi(i))}(\alpha_{ij}(a))\alpha_{j,\pi(i)}$ for all $a \in A$. In particular, $\alpha_{i,\pi(i)}e_{ij} = \alpha_{j,\pi(i)}$ for all $j \in I$ and ϕ_i is bijective. Next, let $i \in I_0$. We claim that the homomorphism

$$\phi_i : e_iA_A \to \mathrm{Hom}_R(Ae_i, R_R)_A, a \mapsto \rho_i a$$

is an isomorphism. For any $a, b \in A$ by (M2) we have

$$\begin{aligned}(\rho_i a)b &= \rho_i(ab) \\ &= \sum_{j \in I \setminus \{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a))\alpha_{ji}(b) + \sigma^t(\alpha_i(a))\rho_i(b) \\ &\quad + \rho_i(a)\alpha_i(b) - c^t\sigma^t(\rho_i(a))\rho_i(b).\end{aligned}$$

Thus $\rho_i a = \sum_{j \in I \setminus \{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a))\alpha_{ji} + \rho_i(a)\alpha_i + \sigma^t(\alpha_i(a) - c^t\rho_i(a))\rho_i$ for all $a \in A$. For any $a \in e_i A$ with $\rho_i a = 0$, we have $\alpha_{ij}(a) = 0$ for all $j \in I \setminus \{i\}$, $\rho_i(a) = 0$ and $\alpha_i(a) - c^t\rho_i(a) = 0$, so that $a = 0$. Thus ϕ_i is monic. Also, we have $\rho_i e_{ij} = \alpha_{ji}$ for all $j \in I \setminus \{i\}$, $\rho_i e_i = \rho_i$ and $\rho_i(e_i c^t + v_i) = \alpha_i$, so that ϕ_i is epic.

(5) It follows by (1), (4) that $A_A \cong \mathrm{Hom}_R(A, R_R)_A$. Similarly, we have ${}_AA \cong {}_A\mathrm{Hom}_R(A, {}_RR)$. Finally, let $\varphi : R \to A, x \mapsto \sum_{i \in I} e_i x$. For any $i \in I$, α_i is R-R-bilinear and satisfies $\alpha_i \varphi = \mathrm{id}_R$. □

Recall that a ring R is said to be quasi-Frobenius if it is selfinjective and artinian on both sides. It follows by Propositions 4.1.5, 4.1.7 that A is quasi-Frobenius if and only if so is R.

Corollary 4.3.2. *Assume R is local. Then the following hold.*

(1) $e_i A e_i$ is local for all $i \in I$, so that A is semiperfect.

(2) A is connected, i.e., indecomposable as a ring.

(3) A is basic.

(4) If R is quasi-Frobenius, so is A with $\mathrm{soc}(e_i A_A) \cong e_{\pi(i)}A/e_{\pi(i)}\mathfrak{M}$ for all $i \in I$, where $\mathfrak{M}$ is the Jacobson radical of A.

Proof. (1) By Lemma 4.2.5(3) and Theorem 4.3.1(2).

(2) By Theorem 4.3.1(2).

(3) By Theorem 4.3.1(3).

(4) Let $\mathfrak{m} = R \setminus R^\times$. It is not difficult to see that $\mathfrak{M} = \sum_{i \in I} e_i \mathfrak{m} + \sum_{i \in I, j \in I \setminus \{i\}} e_{ij}R + \sum_{i \in I_0} v_i R$. Let $i \in I$. Note that $e_i A_A$ is indecomposable by (1) and is injective by Proposition 4.1.5(2). Also, by Theorem 4.3.1(4) $e_i A_A \cong \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)_A$. Since $Ae_{\pi(i)}/\mathfrak{M}e_{\pi(i)} \cong R/\mathfrak{m}$ as right R-modules, there exists $0 \neq h \in \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)$ with $h(\mathfrak{M}e_{\pi(i)}) = 0$. Then $h\mathfrak{M} = 0$ and $he_{\pi(i)} \neq 0$. Thus $\mathrm{soc}(\mathrm{Hom}_R(Ae_{\pi(i)}, R_R)_A) = hA \cong e_{\pi(i)}A/e_{\pi(i)}\mathfrak{M}$. □

The permutation π of I may be considered as a permutation of $\{e_i\}_{i \in I}$. We claim that this permutation can be extended to a ring automorphism of A. As an additive group, A has an automorphism η such that for any $a \in A$ the following hold:

(H1) $\alpha_{\pi(i),\pi(j)}(\eta(a)) = \sigma^{\chi(j)}(\alpha_{ij}(a))$ for all $i, j \in I$; and

(H2) $\rho_i(\eta(a)) = \sigma^{\chi(i)}(\rho_i(a))$ for all $i \in I_0$.

Proposition 4.3.3. *The mapping η is a ring automorphism of A satisfying the following conditions:*

(1) $\eta(e_{ij}) = e_{\pi(i),\pi(j)}$ *for all* $i, j \in I$*;*

(2) $\eta(v_i) = v_i$ *for all* $i \in I_0$*; and*

(3) $\eta(x) = \sum_{i\in I} e_{\pi(i)}\sigma^{\chi(i)}(x)$ *for all* $x \in R$*.*

Proof. It is easy to see that the required conditions are satisfied. In particular, we have $\eta(1) = 1$. Let $a, b \in A$. For any $i, k \in I$ by (H1), (M1) we have

$$\begin{aligned}
&\alpha_{\pi(i),\pi(k)}(\eta(ab)) \\
&= \sigma^{\chi(k)}(\alpha_{ik}(ab)) \\
&= \sigma^{\chi(k)}(\{\sum_{j\in I} c^{\lambda(i,j,k)}\sigma^{\omega(j,k)}(\alpha_{ij}(a))\alpha_{jk}(b)\}) \\
&= \sum_{j\in I} c^{\lambda(i,j,k)}\sigma^{\chi(k)+\omega(j,k)}(\alpha_{ij}(a))\sigma^{\chi(k)}(\alpha_{jk}(b)),
\end{aligned}$$

$$\begin{aligned}
&\alpha_{\pi(i),\pi(k)}(\eta(a)\eta(b)) \\
&= \sum_{j\in I} c^{\lambda(\pi(i),\pi(j),\pi(k))}\sigma^{\omega(\pi(j),\pi(k))}(\alpha_{\pi(i),\pi(j)}(\eta(a)))\alpha_{\pi(j),\pi(k)}(\eta(b)) \\
&= \sum_{j\in I} c^{\lambda(\pi(i),\pi(j),\pi(k))}\sigma^{\omega(\pi(j),\pi(k))}(\sigma^{\chi(j)}(\alpha_{ij}(a)))\sigma^{\chi(k)}(\alpha_{jk}(b)) \\
&= \sum_{j\in I} c^{\lambda(\pi(i),\pi(j),\pi(k))}\sigma^{\omega(\pi(j),\pi(k))+\chi(j)}(\alpha_{ij}(a)))\sigma^{\chi(k)}(\alpha_{jk}(b))
\end{aligned}$$

and hence by Lemmas 4.2.2, 4.2.3(3) $\alpha_{\pi(i),\pi(k)}(\eta(ab)) = \alpha_{\pi(i),\pi(k)}(\eta(a)\eta(b))$. Also, for any $i \in I_0$, since $\chi(i) = t$, by (H2), (M2) we have

$$\begin{aligned}
&\rho_i(\eta(ab)) \\
&= \sigma^t(\rho_i(ab)) \\
&= \sigma^t(\{\sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a))\alpha_{ji}(b) + \sigma^t(\alpha_i(a))\rho_i(b) \\
&\qquad + \rho_i(a)\alpha_i(b) - c^t\sigma^t(\rho_i(a))\rho_i(b)\}) \\
&= \sum_{j\in I\setminus\{i\}} \sigma^{t+\omega(j,i)}(\alpha_{ij}(a))\sigma^t(\alpha_{ji}(b)) + \sigma^{2t}(\alpha_i(a))\sigma^t(\rho_i(b)) \\
&\qquad + \sigma^t(\rho_i(a))\sigma^t(\alpha_i(b)) - c^t\sigma^{2t}(\rho_i(a))\sigma^t(\rho_i(b)),
\end{aligned}$$

$$
\begin{aligned}
&\rho_i(\eta(a)\eta(b))\\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(\pi(j),\pi(i))}(\alpha_{\pi(i),\pi(j)}(\eta(a)))\alpha_{\pi(j),\pi(i)}(\eta(b))\\
&\qquad + \sigma^t(\alpha_{\pi(i)}(\eta(a)))\rho_i(\eta(b)) + \rho_i(\eta(a))\alpha_{\pi(i)}(\eta(b))\\
&\qquad - c^t\sigma^t(\rho_i(\eta(b)))\rho_i(\eta(b)))\\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(\pi(j),\pi(i))}(\sigma^{\chi(j)}(\alpha_{ij}(a)))\sigma^t(\alpha_{ji}(b))\\
&\qquad + \sigma^t(\sigma^t(\alpha_i(a)))\sigma^t(\rho_i(b)) + \sigma^t(\rho_i(a))\sigma^t(\alpha_i(b))\\
&\qquad - c^t\sigma^t(\sigma^t(\rho_i(a))\sigma^t(\rho_i(b)))\\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(\pi(j),\pi(i))+\chi(j)}(\alpha_{ij}(a))\sigma^t(\alpha_{ji}(b)) + \sigma^{2t}(\alpha_i(a))\sigma^t(\rho_i(b))\\
&\qquad + \sigma^t(\rho_i(a))\sigma^t(\alpha_i(b)) - c^t\sigma^{2t}(\rho_i(a))\sigma^t(\rho_i(b))
\end{aligned}
$$

and hence by Lemma 4.2.2 $\rho_i(\eta(ab)) = \rho_i(\eta(a)\eta(b))$. □

Remark 4.3.4. We have seen in the proof of Theorem 4.3.1(4) that there exists an isomorphism $\phi : A_A \xrightarrow{\sim} \mathrm{Hom}_R(A, R_R)_A$ such that $\phi(1)(a) = \sum_{i\in I\setminus I_0} \alpha_{i,\pi(i)}(a) + \sum_{i\in I_0} \rho_i(a)$ for all $a \in A$. Set $\theta = \eta^{-1} \in \mathrm{Aut}(A)$. Then $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$ (cf. Remark 4.1.2(1)).

Remark 4.3.5. Set $w_i = v_i + e_i c^{\chi(i)}$ for $i \in I_0$ and $w_i = e_{i,\pi(i)}$ for $i \in I \setminus I_0$. Then the following hold.

(1) $\{e_{ij}\}_{i,j\in I} \cup \{w_i\}_{i\in I_0}$ is a basis for A_R and gives rise to another description of the multiplication of A.

(2) $\phi(w_i) = \alpha_{\pi(i)}$ for all $i \in I$, where ϕ is the same as in Remark 4.3.4.

(3) Set $w = \sum_{i\in I} w_i$. Then $\eta(w) = w$ and $aw = w\eta(a)$ for all $a \in A$, so that (R, c, σ) can be replaced by (A, w, η) in our construction.

In the following, we denote by $\mathrm{M}_n(R)$ the $n \times n$ full matrix ring over R. Recall that c is said to be regular if $cx \neq 0$ and $xc \neq 0$ for any $0 \neq x \in R$.

Proposition 4.3.6. *For any $i \in I$ there exists a ring homomorphism*

$$\xi_i : A \to \mathrm{M}_n(R), a \mapsto (c^{\lambda(i,j,k)}\sigma^{-\omega(i,k)}(\alpha_{jk}(a)))_{j,k\in I}.$$

Assume c is regular. Then $\mathrm{Ker}\ \xi_i = \sum_{j\in I_0} v_j R$. *In particular, if I_0 is empty, then ξ_i is injective.*

Proof. It is easy to see that $\xi_i(1)$ is the unit matrix. Let $a, b \in A$. Obviously, $\xi_i(a + b) = \xi_i(a) + \xi_i(b)$. Also, for any $j, l \in I$ by (M1) and (1), (2) of Lemma

4.2.3 we have

$$
\begin{aligned}
& c^{\lambda(i,j,l)}\sigma^{-\omega(i,l)}(\alpha_{jl}(ab)) \\
&= c^{\lambda(i,j,l)}\sigma^{-\omega(i,l)}(\{\sum_{k\in I} c^{\lambda(j,k,l)}\sigma^{\omega(k,l)}(\alpha_{jk}(a))\alpha_{kl}(b)\}) \\
&= \sum_{k\in I} c^{\lambda(i,j,l)+\lambda(j,k,l)}\sigma^{\omega(k,l)-\omega(i,l)}(\alpha_{jk}(a))\sigma^{-\omega(i,l)}(\alpha_{kl}(b)) \\
&= \sum_{k\in I} c^{\lambda(i,j,k)+\lambda(i,k,l)}\sigma^{\lambda(i,k,l)-\omega(i,k)}(\alpha_{jk}(a))\sigma^{-\omega(i,l)}(\alpha_{kl}(b)) \\
&= \sum_{k\in I} \{c^{\lambda(i,j,k)}\sigma^{-\omega(i,k)}(\alpha_{jk}(a))\}\{c^{\lambda(i,k,l)}\sigma^{-\omega(i,l)}(\alpha_{kl}(b))\}
\end{aligned}
$$

and hence $\xi_i(ab) = \xi_i(a)\xi_i(b)$. The last assertion is obvious. □

4.4 Tilting complexes

In this section, we provide a construction of two-term tilting complexes associated with a certain type of idempotent (cf. [25]).

For a ring A we denote by $\mathcal{K}(\text{Mod-}A)$ (resp., $\mathcal{D}(\text{Mod-}A)$) the homotopy (resp., derived) category of cochain complexes over Mod-A and consider modules as complexes concentrated in degree zero. We use the notation $(-)[m]$ to denote the m-shift of complexes. Also, we denote by $\mathcal{P}_A$ the full subcategory of Mod-A consisting of finitely generated projective modules and by $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}(\text{Mod-}A)$ consisting of bounded complexes over $\mathcal{P}_A$. Finally, for an object X in an additive category $\mathfrak{A}$ we denote by $\mathrm{add}(X)$ the full additive subcategory of $\mathfrak{A}$ whose objects are direct summands of finite direct sums of copies of X and by $X^{(m)}$ the direct sum of m copies of X. We refer to [39] for tilting complexes and derived equivalences and to [22], [43] for derived categories.

Let A be an extension of a ring R and $e \in A$ an idempotent. Assume $xe = ex$ for all $x \in R$, Ae_R is finitely generated projective and eA_A is embedded in $\mathrm{Hom}_R(Ae, R_R)_A$ as a submodule. Note first that we have a ring homomorphism $\varphi : R \to eAe, x \mapsto ex$. Let

$$\mu : Ae \otimes_R eA_A \to A_A, a \otimes b \mapsto ab$$

be the multiplication map and $S^\bullet$ its mapping cone. Set $T_1^\bullet = eA[1]$, $T_2^\bullet = (1-e)A \otimes_A S^\bullet$ and $T^\bullet = T_1^\bullet \oplus T_2^\bullet$. Note that $T_2^\bullet$ is the mapping cone of the multiplication map

$$(1-e)A \otimes_A \mu : (1-e)Ae \otimes_R eA_A \to (1-e)A_A.$$

Note also that $Ae \otimes_R eA_A \in \mathrm{add}(eA_A)$. Since the multiplication map

$$eA \otimes_A \mu : eAe \otimes_R eA_A \to eA_A$$

is a split epimorphism and its kernel belongs to $\mathrm{add}(eA_A)$, we have $eA \otimes_A S^\bullet \in \mathrm{add}(T_1^\bullet)$ and hence $S^\bullet \in \mathrm{add}(T^\bullet)$.

Proposition 4.4.1. *The following hold.*

(1) $T^\bullet$ *is a tilting complex.*

(2) *Assume* μ *is monic. Then* $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet)$ *is isomorphic to the following generalized triangular matrix ring*

$$\begin{pmatrix} eAe & \mathrm{Ext}^1_A(A/AeA, eA) \\ 0 & A/AeA \end{pmatrix}.$$

Assume further that φ *is an isomorphism and there exists an idempotent* $f \in A$ *such that* $\mathrm{Hom}_R(Ae, R_R)_A \cong fA_A$. *Then* $\mathrm{Ext}^1_A(A/AeA, eA) \cong f(A/AeA)$ *as right* (A/AeA)*-modules.*

Proof. (1) Obviously, $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, T^\bullet[m]) = 0$ unless $-1 \le m \le 1$. Since $e(A/AeA) = 0$, $A/AeA = (1-e)(A/AeA)$ and $\mathrm{H}^0(T^\bullet) \cong A/AeA$. Thus, since $\mathrm{Hom}_A(eA, A/AeA) \cong (A/AeA)e = 0$, it follows that $\mathrm{Hom}_A(T^{-1}, (1-e)A \otimes_A \mu)$ is epic and $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, T^\bullet[1]) = 0$. Also,

$$\mathrm{Hom}_A(A/AeA, \mathrm{Hom}_R(Ae, R_R)) \cong \mathrm{Hom}_R((A/AeA)e, R_R) = 0$$

and hence $\mathrm{Hom}_A(A/AeA, eA) = 0$. Thus $\mathrm{Hom}_A(\mathrm{H}^0(T^\bullet), T^{-1}) = 0$ and hence $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, T^\bullet[-1]) = 0$. Next, we have a distinguished triangle in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ of the form

$$A \to S^\bullet \to (Ae \otimes_R eA)[1] \to .$$

Since $S^\bullet \in \mathrm{add}(T^\bullet)$, and since $(Ae \otimes_R eA)[1] \in \mathrm{add}(T_1^\bullet)$, it follows that $\mathrm{add}(T^\bullet)$ generates $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as a triangulated category.

(2) We have $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T_1^\bullet) \cong \mathrm{End}_A(eA_A) \cong eAe$. Also, since $(1-e)A \otimes_A \mu$ is monic, we have $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_1^\bullet, T_2^\bullet) = 0$. Furthermore,

$$\begin{aligned}\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T_2^\bullet) &\cong \mathrm{End}_{\mathcal{D}(\text{Mod-}A)}(T_2^\bullet) \\ &\cong \mathrm{End}_{\mathcal{D}(\text{Mod-}A)}(A/AeA) \\ &\cong \mathrm{End}_A(A/AeA) \\ &\cong A/AeA,\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_2^\bullet, T_1^\bullet) &\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(T_2^\bullet, T_1^\bullet) \\ &\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(A/AeA, eA[1]) \\ &\cong \mathrm{Ext}^1_A(A/AeA, eA).\end{aligned}$$

Consequently, $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet)$ is isomorphic to the desired generalized triangular matrix ring.

Next, assume φ is an isomorphism and there exists an idempotent $f \in A$ such that $\mathrm{Hom}_R(Ae, R_R)_A \cong fA_A$. For any $M \in \text{Mod-}A$ we have functorial isomorphisms

$$\mathrm{Hom}_A(M, fA) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_R(Ae, R_R)) \cong \mathrm{Hom}_R(M \otimes_A Ae, R_R).$$

Thus, since $\mu \otimes_A Ae$ is an isomorphism, so is $\mathrm{Hom}_A(\mu, fA)$. Then by applying $\mathrm{Hom}_A(-, fA)$ to the exact sequence

$$0 \to Ae \otimes_R eA \xrightarrow{\mu} A \to A/AeA \to 0,$$

we have $\mathrm{Hom}_A(A/AeA, fA) = 0$ and $\mathrm{Ext}^1_A(A/AeA, fA) = 0$. Note that $fAe \cong R$ as right R-modules. Thus, by applying $fA \otimes_A -$ to the above exact sequence, we get an exact sequence of the form

$$0 \to eA_A \to fA_A \to f(A/AeA)_A \to 0$$

to which we apply $\mathrm{Hom}_A(A/AeA, -)$ to conclude that

$$\begin{aligned} f(A/AeA) &\cong \mathrm{Hom}_A(A/AeA, f(A/AeA)) \\ &\cong \mathrm{Ext}^1_A(A/AeA, eA) \end{aligned}$$

as right (A/AeA)-modules. □

Remark 4.4.2. Let $K = \mathrm{Ker}(eA \otimes_A \mu)$ and assume $\mathrm{add}(K_A) = \mathrm{add}(eA_A)$. Then $\mathrm{add}(S^\bullet) = \mathrm{add}(T^\bullet)$ and $S^\bullet$ is a tilting complex.

Proof. Note that $eA \otimes_A S^\bullet \cong K[1]$ in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$. Since $eA_A \in \mathrm{add}(K_A)$, we have $T_1^\bullet \in \mathrm{add}(eA \otimes_A S^\bullet)$. Thus $T^\bullet \in \mathrm{add}(S^\bullet)$ and hence $\mathrm{add}(S^\bullet) = \mathrm{add}(T^\bullet)$. Then, since $T^\bullet$ is a tilting complex, so is $S^\bullet$. □

In the following examples, A is the Frobenius extension of R constructed in the preceding section. We use the same notation as in the preceding section.

Example 4.4.3. Let J be a nonempty π-stable subset of I and set $e = \sum_{j \in J} e_j$. Then $xe = ex$ for all $x \in R$, Ae_R is finitely generated projective and $eA_A \cong \mathrm{Hom}_R(Ae, R_R)_A$. In this case, the mapping cone of the multiplication map

$$\bigoplus_{j \in J} Ae_j \otimes_R e_jA_A \to A_A$$

is a tilting complex.

Proof. We have seen in the preceding section that all the conditions are satisfied. Let $J_0 = J \cap I_0$ and d the number of elements of J. Set $d_j = d$ for $j \in J_0$ and $d_j = d - 1$ for $j \in J \setminus J_0$. Note that $d_j \geq 1$ for all $j \in J$. Since we have a split exact sequence in Mod-A of the form

$$0 \to \bigoplus_{j \in J} e_jA^{(d_j)} \to \bigoplus_{j \in J} eAe_j \otimes_R e_jA \to eA \to 0,$$

the last assertion follows by the same argument as in Remark 4.4.2. □

Example 4.4.4. Assume c is regular and $I \setminus I_0$ is not empty. Let $i \in I \setminus I_0$ and set $e = e_i$ and $f = e_{\pi^{-1}(i)}$. Then the following conditions are satisfied:

(1) $xe = ex$ for all $x \in R$, Ae_R is finitely generated projective and eA_A is embedded in $\mathrm{Hom}_R(Ae, R_R)_A$ as a submodule;

(2) the multiplication map $Ae \otimes_R eA \longrightarrow A, a \otimes b \mapsto ab$ is monic;

(3) the ring homomorphism $R \longrightarrow eAe, x \mapsto ex$ is an isomorphism; and

(4) $\mathrm{Hom}_R(Ae, R_R)_A \cong fA_A$.

Proof. We denote by $\mu : Ae_i \otimes_R e_iA \longrightarrow A, a \otimes b \mapsto ab$ the multiplication map. Note that $Ae_i \otimes_R e_iA$ is a free right R-module with a basis $\{e_{ji} \otimes e_{il}\}_{j,l \in I}$, and that $e_{ji}e_{il} = e_{jl}c^{\lambda(j,i,l)}$ unless $j = l \in I_0$ and $e_{ji}e_{ij} = v_j + e_jc^{\chi(j)}$ for all $j \in I_0$. Thus, since c is regular, it is easy to see that μ is monic. Also, for any $a \in e_iA$ we have $(\alpha_i a)(e_{ji}) = \alpha_i(ae_{ji}) = c^{\lambda(i,j,i)}\sigma^{\omega(j,i)}(\alpha_{ij}(a))$ for all $j \in I$ and hence $\alpha_i a = \sum_{j \in I} c^{\lambda(i,j,i)}\sigma^{\omega(j,i)}(\alpha_{ij}(a))\alpha_{ji}$, so that by the regularity of c the homomorphism

$$e_iA_A \longrightarrow \mathrm{Hom}_R(Ae_i, R_R)_A, a \mapsto \alpha_i a$$

is monic. We have seen in the preceding section that the remaining conditions are satisfied. □

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