Derived equivalences for triangular matrix rings

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Abstract

We generalize derived equivalences for triangular matrix rings induced by a certain type of classical tilting module introduced by Auslander, Platzeck and Reiten to generalize reflection functors in the representation theory of quivers due to Bernstein, Gelfand and Ponomarev.

Let R be a finite dimensional algebra over a field k and M a finitely generated projective right R-module. Set

$$A = \begin{pmatrix} k & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A.$$

As pointed out by Brenner and Butler (see [**?**, p.111]), we know from [**?**] (cf. also [**?**]) that $\mathrm{Ext}^1_A(A/AeA, A) \oplus Ae \in \text{Mod-}A^{\mathrm{op}}$ is a classical tilting module, i.e., a tilting module of projective dimension at most one (see [**?**]) with

$$\mathrm{End}_{A^{\mathrm{op}}}(\mathrm{Ext}^1_A(A/AeA, A) \oplus Ae)^{\mathrm{op}} \cong \begin{pmatrix} R & \mathrm{Hom}_R(M, R) \\ 0 & k \end{pmatrix}.$$

Our aim is to extend this type of derived equivalence to the case where M_R has finite projective dimension. Let R, S be rings and M an S-R-bimodule such that M admits a projective resolution $P^\bullet \to M$ in Mod-R with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_R)$ and $\mathrm{Ext}^i_R(M, R) = 0$ for $i < d = \text{proj dim } M_R$. Set

$$A = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A.$$

We will construct a tilting complex $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ associated with e such that

$$\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet) \cong \begin{pmatrix} R & \mathrm{Ext}^d_A(M, R) \\ 0 & S \end{pmatrix}$$

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(Theorems **??** and **??**). Assume further that ${}_S M$ is faithful and that if $d > 0$ then $S \xrightarrow{\sim} \mathrm{End}_R(M)$ canonically and $\mathrm{Ext}^i_R(M, M) = 0$ for $1 \le i < d$. Then we will see that

$$\mathrm{Hom}^\bullet_A(T^\bullet, A)[d+1] \cong \mathrm{Ext}^{d+1}_A(A/AeA, A) \oplus Ae$$

in $\mathcal{D}(\mathrm{Mod}\text{-}A^{\mathrm{op}})$ and $\mathrm{Ext}^{d+1}_A(A/AeA, A) \oplus Ae \in \mathrm{Mod}\text{-}A^{\mathrm{op}}$ is a tilting module of projective dimension $d+1$ (see [**?**]) with

$$\mathrm{End}_{A^{\mathrm{op}}}(\mathrm{Ext}^{d+1}_A(A/AeA, A) \oplus Ae)^{\mathrm{op}} \cong \begin{pmatrix} R & \mathrm{Ext}^d_A(M, R) \\ 0 & S \end{pmatrix}$$

(Corollary **??** and Remark **??**).

For a ring A, we denote by Mod-A the category of right A-modules, by mod-A the full subcategory of Mod-A consisting of finitely presented modules and by $\mathcal{P}_A$ the full subcategory of Mod-A consisting of finitely generated projective modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation X_A (resp., ${}_A X$) to stress that the module X considered is a right (resp., left) A-module. We denote by $\mathcal{K}(\mathrm{Mod}\text{-}A)$ (resp., $\mathcal{D}(\mathrm{Mod}\text{-}A)$) the homotopy (resp., derived) category of cochain complexes over Mod-A and by $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}(\mathrm{Mod}\text{-}A)$ consisting of bounded complexes over $\mathcal{P}_A$. We consider modules as complexes concentrated in degree zero. For any integer $n \in \mathbb{Z}$ we denote by $\mathrm{H}^n(-)$ the n-th homology and by $(-)[n]$ the n-shift of complexes. Also, we use the notation $\mathrm{Hom}^\bullet(-,-)$ to denote the single complex associated with the double hom complex. Finally, for an object X in an additive category $\mathfrak{A}$ we denote by $\mathrm{add}(X)$ the full subcategory of $\mathfrak{A}$ consisting of direct summands of finite direct sums of copies of X.

We refer to [**?**] for tilting complexes and derived equivalences and to [**?**], [**?**] for derived categories.

1 General case

Throughout this section, A is a ring and $e \in A$ is an idempotent satisfying the following conditions:

- (E1) Ae admits a projective resolution $\varepsilon : P^\bullet \to Ae$ in Mod-eAe with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{eAe})$, in particular, $d = \mathrm{proj\ dim}\ Ae_{eAe} < \infty$;
- (E2) $\mu : Ae \otimes_{eAe} eA \to A, x \otimes y \mapsto xy$ is monic;
- (E3) $\varphi : eA \to \mathrm{Hom}_{eAe}(Ae, eAe), x \mapsto (y \mapsto xy)$ is monic;
- (E4) if $d > 0$ then φ is an isomorphism and $\mathrm{Ext}^i_{eAe}(Ae, eAe) = 0$ for $1 \le i < d$; and
- (E5) $\mathrm{Tor}^{eAe}_i(Ae, eA) = 0$ for $i \ne 0$.

We define a complex $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as follows: Set $T_1^\bullet = eA[d+1]$, let $T_2^\bullet$ be the mapping cone of the composite

$$\mu \circ (\varepsilon \otimes_{eAe} eA) : P^\bullet \otimes_{eAe} eA \to Ae \otimes_{eAe} eA \to A$$

and set $T^\bullet = T_1^\bullet \oplus T_2^\bullet$.

Theorem 1.1. *The complex $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is a tilting complex with*

$$\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet) \cong \begin{pmatrix} eAe & \mathrm{Ext}_A^{d+1}(A/AeA, eA) \\ 0 & A/AeA \end{pmatrix}.$$

Proof. We may assume $P^i = 0$ unless $-d \le i \le 0$. Note that by (E5) we have a projective resolution $P^\bullet \otimes_{eAe} eA \to Ae \otimes_{eAe} eA$ in Mod-A with $P^\bullet \otimes_{eAe} eA \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and that by (E2) we have an exact sequence in Mod-A

$$0 \to Ae \otimes_{eAe} eA \xrightarrow{\mu} A \to A/AeA \to 0.$$

Thus $T_2^\bullet$ is a projective resolution of A/AeA in Mod-A. In particular, $T_2^\bullet \cong A/AeA$ in $\mathcal{D}(\text{Mod-}A)$.

Claim 1. $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_1^\bullet, T_1^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T_1^\bullet) \cong eAe$.

Proof. The first assertion is obvious and $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T_1^\bullet) \cong \mathrm{End}_A(eA) \cong eAe$. □

Claim 2. $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_1^\bullet, T_2^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$.

Proof. We have

$$\begin{aligned} \mathrm{Hom}_A^\bullet(T_1^\bullet, T_2^\bullet) &\cong \mathrm{Hom}_A^\bullet(eA, T_2^\bullet)[-d-1] \\ &\cong (T_2^\bullet \otimes_A Ae)[-d-1]. \end{aligned}$$

Also, $T_2^\bullet \otimes_A Ae$ is isomorphic to the mapping cone of $\varepsilon : P^\bullet \to Ae$. Thus $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_1^\bullet, T_2^\bullet[i]) \cong \mathrm{H}^i(\mathrm{Hom}_A^\bullet(T_1^\bullet, T_2^\bullet)) \cong \mathrm{H}^{i-d-1}(T_2^\bullet \otimes_A Ae) = 0$ for all $i \in \mathbb{Z}$. □

Claim 3. $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_2^\bullet, T_1^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_2^\bullet, T_1^\bullet) \cong \mathrm{Ext}_A^{d+1}(A/AeA, eA)$.

Proof. Note first that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_2^\bullet, T_1^\bullet[i]) &\cong \mathrm{H}^i(\mathrm{Hom}_A^\bullet(T_2^\bullet, T_1^\bullet)) \\ &\cong \mathrm{H}^i(\mathrm{Hom}_A^\bullet(T_2^\bullet, eA)[d+1]) \\ &\cong \mathrm{H}^{i+d+1}(\mathrm{Hom}_A^\bullet(T_2^\bullet, eA)) \end{aligned}$$

for all $i \in \mathbb{Z}$. Since $\mathrm{Hom}_A^\bullet(T_2^\bullet, eA)$ is isomorphic to the (-1)-shift of the mapping cone of the composite

$$\mathrm{Hom}_A(\varepsilon, eAe) \circ \varphi : eA \to \mathrm{Hom}_{eAe}(Ae, eAe) \to \mathrm{Hom}_{eAe}^\bullet(P^\bullet, eAe),$$

by (E3), (E4) we have $\mathrm{H}^j(\mathrm{Hom}_A^\bullet(T_2^\bullet, eA)) = 0$ for $j \neq d+1$. Also, since $T_2^\bullet$ is a projective resolution of A/AeA, $\mathrm{H}^{d+1}(\mathrm{Hom}_A^\bullet(T_2^\bullet, eA)) \cong \mathrm{Ext}_A^{d+1}(A/AeA, eA)$. □

Claim 4. $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_2^\bullet, T_2^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T_2^\bullet) \cong A/AeA$.

Proof. Since $\mathrm{Hom}_A(eA, A/AeA) = 0$, $\mathrm{Hom}_A^\bullet(T_2^\bullet, A/AeA) \cong A/AeA$ as complexes and

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_2^\bullet, T_2^\bullet[i]) &\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(T_2^\bullet, T_2^\bullet[i]) \\
&\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(T_2^\bullet, A/AeA[i]) \\
&\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T_2^\bullet, A/AeA[i]) \\
&\cong \mathrm{H}^i(\mathrm{Hom}_A^\bullet(T_2^\bullet, A/AeA)) \\
&= 0
\end{aligned}$$

for $i \neq 0$. Also,

$$\begin{aligned}
\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T_2^\bullet) &\cong \mathrm{End}_{\mathcal{D}(\text{Mod-}A)}(T_2^\bullet) \\
&\cong \mathrm{End}_{\mathcal{D}(\text{Mod-}A)}(A/AeA) \\
&\cong \mathrm{End}_A(A/AeA) \\
&\cong A/AeA.
\end{aligned}$$

□

Now, by the Claims above $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet, T^\bullet[i]) = 0$ for $i \neq 0$ and

$$\mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(T^\bullet) \cong \begin{pmatrix} eAe & \mathrm{Ext}_A^{d+1}(A/AeA, eA) \\ 0 & A/AeA \end{pmatrix}.$$

Next, since $P^i \otimes_{eAe} eA \in \mathrm{add}(eA)$ for all $i \in \mathbb{Z}$, $P^\bullet \otimes_{eAe} eA$ belongs to the full triangulated subcategory of $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ generated by $\mathrm{add}(T_1^\bullet)$. Then, since we have a distinguished triangle in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$

$$P^\bullet \otimes_{eAe} eA \to A \to T_2^\bullet \to,$$

it follows that A belongs to the full triangulated subcategory of $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ generated by $\mathrm{add}(T^\bullet)$. Thus $\mathrm{add}(T^\bullet)$ generates $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as a triangulated category and $T^\bullet$ is a tilting complex.

This finishes the proof of Theorem **??**. □

Note that we have $\mathrm{H}^i(\mathrm{Hom}_A^\bullet(T^\bullet, A)) \cong \mathrm{Ext}_A^i(A/AeA, A)$ for $i \neq d+1$ and $\mathrm{H}^{d+1}(\mathrm{Hom}_A^\bullet(T^\bullet, A)) \cong \mathrm{Ext}_A^{d+1}(A/AeA, A) \oplus Ae$. We consider next the case where $\mathrm{Ext}_A^{d+1}(A/AeA, A) \oplus Ae \in \text{Mod-}A^{\mathrm{op}}$ is a tilting module (see [**?**]). Recall that a module is a tilting module if and only if it is isomorphic to a tilting complex in the derived category (see e.g. [**?**, Proposition 3.9]). Since we have an anti-equivalence of triangulated categories

$$\mathrm{Hom}_A^\bullet(-, A) : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\mathrm{op}}}),$$

$\mathrm{Hom}^\bullet_A(T^\bullet, A) \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{A^{\mathrm{op}}})$ is a tilting complex. Thus, if $\mathrm{Ext}^i_A(A/AeA, A) = 0$ for $i \neq d+1$, then

$$\mathrm{Hom}^\bullet_A(T^\bullet, A)[d+1] \cong \mathrm{Ext}^{d+1}_A(A/AeA, A) \oplus Ae$$

in $\mathcal{D}(\text{Mod-}A^{\mathrm{op}})$ and $\mathrm{Ext}^{d+1}_A(A/AeA, A) \oplus Ae \in \text{Mod-}A^{\mathrm{op}}$ is a tilting module. We denote by $\gamma : A \to \mathrm{End}_{eAe}(Ae), a \mapsto (x \mapsto ax)$ the ring homomorphism given by the left multiplication. Then it is not difficult to see that $\mathrm{Hom}^\bullet_A(T^\bullet, A)$ is isomorphic to the (-1)-shift of the mapping cone of the composite

$$\mathrm{Hom}_A(\varepsilon, Ae) \circ \gamma : A \to \mathrm{End}_{eAe}(Ae) \to \mathrm{Hom}^\bullet_{eAe}(P^\bullet, Ae).$$

Consequently, we have the following.

Corollary 1.2. *Assume that γ is injective and that if $d > 0$ then γ is an isomorphism and $\mathrm{Ext}^i_{eAe}(Ae, Ae) = 0$ for $1 \leq i < d$. Then $\mathrm{Ext}^i_A(A/AeA, A) = 0$ for $i \neq d+1$ and $\mathrm{Ext}^{d+1}_A(A/AeA, A) \oplus Ae \in \text{Mod-}A^{\mathrm{op}}$ is a tilting module with*

$$\mathrm{End}_{A^{\mathrm{op}}}(\mathrm{Ext}^{d+1}_A(A/AeA, A) \oplus Ae)^{\mathrm{op}} \cong \begin{pmatrix} eAe & \mathrm{Ext}^{d+1}_A(A/AeA, eA) \\ 0 & A/AeA \end{pmatrix}.$$

Example 1.3. Let R be a commutative ring and $c \in R$ a regular element which is not a unit. Let $n \geq 2$ be an integer and (m_{ij}) an $n \times n$ matrix of non-negative integers such that $m_{ii} = 0$ for all $1 \leq i \leq n$ and $m_{ij} + m_{jk} \geq m_{ik}$ for all $1 \leq i, j, k \leq n$. Let A be the subset of $\mathrm{M}_n(R)$, the $n \times n$ full matrix algebra over R, consisting of matrices $(x_{ij}) \in \mathrm{M}_n(R)$ with $x_{ij} \in c^{m_{ij}}R$ for all $1 \leq i, j \leq n$ and denote by e the matrix $(x_{ij}) \in A$ such that $x_{nn} = 1$ and $x_{ij} = 0$ unless $i = j = n$. Then A is an R-subalgebra of $\mathrm{M}_n(R)$ and $e \in A$ is an idempotent. Also, $eAe \cong R$ as rings and Ae is a free R-module of rank n. It is not difficult to see that $\mu : Ae \otimes_{eAe} eA \to A$ is monic and $\gamma : A \to \mathrm{End}_{eAe}(Ae)$ is an injective ring homomorphism.

2 Triangular matrix rings

Throughout this section, R and S are rings and M is an S-R-bimodule satisfying the following conditions:

(M1) M admits a projective resolution $P^\bullet \to M$ in Mod-R with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_R)$, in particular, $d = \text{proj dim } M_R < \infty$; and

(M2) $\mathrm{Ext}^i_R(M, R) = 0$ for $i < d$.

Theorem 2.1. *The triangular matrix rings*

$$\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R & \mathrm{Ext}^d_R(M, R) \\ 0 & S \end{pmatrix}$$

are derived equivalent to each other.

Proof. Set

$$A = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A.$$

It is not difficult to see that the conditions (E1)–(E5) in the preceding section are satisfied. Note also that $eAe \cong R$ and $A/AeA \cong S$ as rings. Thus we have only to show that $\mathrm{Ext}_A^{d+1}(A/AeA, eA) \cong \mathrm{Ext}_R^d(M, R)$.

Recall that a module $X \in \text{Mod-}A$ is given by a triple (X_1, X_2, ϕ) of $X_1 \in \text{Mod-}S$, $X_2 \in \text{Mod-}R$ and $\phi \in \mathrm{Hom}_R(X_1 \otimes_S M, X_2)$ and for modules $X = (X_1, X_2, \phi)$ and $Y = (Y_1, Y_2, \psi)$ a homomorphism $f \in \mathrm{Hom}_A(X, Y)$ is given by a pair (f_1, f_2) of $f_1 \in \mathrm{Hom}_S(X_1, Y_1)$ and $f_2 \in \mathrm{Hom}_R(X_2, Y_2)$ such that $f_2 \circ \phi = \psi \circ (f_1 \otimes_S M)$ (see e.g. [?] for details). We may assume $P^i = 0$ unless $-d \le i \le 0$. Since $A/AeA = (S, 0, 0)$, we have a projective resolution $Q^\bullet \to A/AeA$ in Mod-A such that $Q^i = (0, P^{i+1}, 0)$ for $i \ne 0$ and $Q^0 = (S, M, \mathrm{id}_M)$, where id_M denotes the canonical isomorphism $S \otimes_S M \xrightarrow{\sim} M$. Also, since $eA = (0, R, 0)$, it follows that $\mathrm{Hom}_A^\bullet(Q^\bullet, eA) \cong \mathrm{Hom}_R^\bullet(P^\bullet, R)[-1]$ and hence

$$\begin{aligned} \mathrm{Ext}_A^{d+1}(A/AeA, eA) &\cong \mathrm{H}^{d+1}(\mathrm{Hom}_A^\bullet(Q^\bullet, eA)) \\ &\cong \mathrm{H}^{d+1}(\mathrm{Hom}_R^\bullet(P^\bullet, R)[-1]) \\ &\cong \mathrm{H}^{d}(\mathrm{Hom}_R^\bullet(P^\bullet, R)) \\ &\cong \mathrm{Ext}_R^d(M, R). \end{aligned}$$

□

Remark 2.2. Denote by $\sigma : S \to \mathrm{End}_R(M), a \mapsto (x \mapsto ax)$ the ring homomorphism given by the left multiplication. Assume that σ is injective and that if $d > 0$ then σ is an isomorphism and $\mathrm{Ext}_R^i(M, M) = 0$ for $1 \le i < d$. Then $\mathrm{Ext}_A^i(A/AeA, A) = 0$ for $i \ne d + 1$.

Proof. Since $Q^\bullet$ is a projective resolution of A/AeA, and since $\mathrm{Hom}_A^\bullet(Q^\bullet, eA) \cong \mathrm{Hom}_R^\bullet(P^\bullet, R)[-1]$, by (M2) we have $\mathrm{Ext}_A^i(A/AeA, eA) \cong \mathrm{Ext}_R^{i-1}(M, R) = 0$ for $i \ne d + 1$. Also, since $(1 - e)A = (S, M, \mathrm{id}_M)$, it is not difficult to see that $\mathrm{Hom}_A^\bullet(Q^\bullet, (1 - e)A)$ is isomorphic to the (-1)-shift of the mapping cone of the composite

$$S \xrightarrow{\sigma} \mathrm{End}_R(M) \to \mathrm{Hom}_R^\bullet(P^\bullet, M).$$

Thus by the assumption we have $\mathrm{Ext}_A^i(A/AeA, (1 - e)A) = 0$ for $i \ne d + 1$. □

Remark 2.3. Consider the case where R is a finite dimensional algebra over a field k and $S = k$. By Theorem **??**

$$\begin{pmatrix} k & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R & \mathrm{Ext}_R^d(M, R) \\ 0 & k \end{pmatrix}$$

are derived equivalent. Also, since $\mathrm{Ext}_R^d(M, R)$ is a finite dimensional k-vector space, it follows again by Theorem **??** that

$$\begin{pmatrix} R & \mathrm{Ext}_R^d(M, R) \\ 0 & k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k & D\mathrm{Ext}_R^d(M, R) \\ 0 & R \end{pmatrix}$$

are derived equivalent, where $D = \mathrm{Hom}_k(-, k)$. Thus

$$\begin{pmatrix} k & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k & D\mathrm{Ext}^d_R(M, R) \\ 0 & R \end{pmatrix}$$

are derived equivalent, which is a consequence of [?, Corollary 5.4] (see also [?]) if inj dim ${}_RR$ = inj dim $R_R < \infty$, since the algebras above are trivial extensions of $\Lambda = k \times R$ by M and $D\mathrm{Ext}^d_R(M, R)$, respectively (see [?]), since $M \otimes^{\mathbf{L}}_{\Lambda} D\Lambda[-d] \cong M \otimes^{\mathbf{L}}_R DR[-d] \cong \mathrm{Tor}^R_d(M, DR) \cong D\mathrm{Ext}^d_R(M, R)$ in $\mathcal{D}(\text{Mod-}\Lambda)$, and since $D\Lambda \in \text{Mod-}\Lambda$ is a tilting module with $\Lambda \cong \mathrm{End}_{\Lambda}(D\Lambda)$ if inj dim ${}_RR$ = inj dim $R_R < \infty$ (see e.g. [?, Proposition 1.6]).

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