MICROLINEARITY IN FRÖLICHER SPACES
– BEYOND THE REGNANT PHILOSOPHY OF MANIFOLDS –

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Abstract: Frölicher spaces and smooth mappings form a Cartesian closed category. It was shown in our previous paper [Far East Journal of Mathematical Sciences, 35 (2009), 211-223] that its full subcategory of Weil exponentiable Frölicher spaces is Cartesian closed. By emancipating microlinearity from within a well-adapted model of synthetic differential geometry to Frölicher spaces, we get the notion of microlinearity for Frölicher spaces. It is shown in this paper that its full subcategory of Weil exponentiable and microlinear Frölicher spaces is Cartesian closed. The canonical embedding of Weil exponentiable Frölicher spaces into the Cahiers Topos is shown to preserve microlinearity besides finite products and exponentiation.

1. Introduction

Differential geometry of finite-dimensional smooth manifolds has been generalized by many authors to the infinite-dimensional case by replacing finite-dimensional vector spaces by Hilbert spaces, Banach spaces, Fréchet spaces or, more generally, convenient vector spaces as the local prototype. We know well that the category of smooth manifolds of any kind, whether finite-dimensional

or infinite-dimensional, is not Cartesian closed, while Frölicher spaces, introduced by Frölicher and others (cf. [3], [4] and [5]), do form a Cartesian closed category. We are strongly biased in favor of our central dogma that *the basic objects under study of infinite-dimensional differential geometry should form a Cartesian closed category*. It seems that Frölicher and his followers do not know what a kind of Frölicher space, besides convenient vector spaces, should become the basic object of research for infinite-dimensional differential geometry. The category of Frölicher spaces and smooth mappings should be restricted *adequately* to a Cartesian closed subcategory.

Synthetic differential geometry is differential geometry with a cornucopia of nilpotent infinitesimals. For a standard textbook on synthetic differential geometry the reader is referred to [8], whose Chapter III is devoted to its model theory. Roughly speaking, a space of nilpotent infinitesimals of some kind, which exists only within an imaginary world, corresponds to a Weil algebra, which is an entity of the real world. The central object of study in synthetic differential geometry is *microlinear* spaces. Although the notion of a manifold (= a pasting of copies of a certain linear space) is defined on the local level, the notion of microlinearity is defined absolutely on the genuinely infinitesimal level. What we should do so as to get an adequately restricted Cartesian closed category of Frölicher spaces is to emancipate microlinearity from within a well-adapted model of synthetic differential geometry. In other words, we should externalize the notion of microlinearity for Frölicher spaces, which is the princcipal objective in this paper.

Although nilpotent infinitesimals exist only within a well-adapted model of synthetic differential geometry, the notion of Weil functor was formulated for finite-dimensional manifolds (cf. Section 35 of [9]) and for infinite-dimensional manifolds (cf. Section 31 of [10]). It was generalized to that for Frölicher spaces in our previous paper [15], which paved the way to microlinearity for Frölicher spaces. This is the first step towards microlinearity for Frölicher spaces. Therein all Frölicher spaces which believe in fantasy that all Weil functors are really exponentiations by some adequate infinitesimal objects in imagination were shown to form a Cartesian closed category. This is the second step towards microlinearity for Frölicher spaces. We will introduce the notion of "transversal limit diagram of Frölicher spaces" after the manner of that of "transversal pullback" in Section 3, which is a familiar token in the arena of synthetic differential geometry. This is the third and final step towards microlinearity for Frölicher spaces. Just as microlinearity is closed under arbitrary limits within a well-adapted model of synthetic differential geometry, microlinearity for Frölicher spaces is closed under arbitrary transversal limits. We will introduce the central notion

of microlinearity in Section 4, where it is to be shown that Weil exponentiable and microlinear Frölicher spaces, together with smooth mappings among them, form a Cartesian closed category. In Section 5 we will demonstrate that our canonical embedding of the category of Frölicher spaces and smooth mappings into the Cahiers Topos preserves microlinearity. Therein our hasty discussions in Section 5 of [15] will also be elaborated.

2. Preliminaries

2.1. Weil Prolongation

In our previous paper [15] we have discussed how to assign, to each pair (X, W) of a Frölicher space X and a Weil algebra W, another Frölicher space $X \otimes W$, called the *Weil prolongation of* X *with respect to* W, which naturally extends to a bifunctor $\mathbf{FS} \times \mathbf{W} \to \mathbf{FS}$, where $\mathbf{FS}$ is the category of Frölicher spaces and smooth mappings, and $\mathbf{W}$ is the category of Weil algebras. We have shown in [15] that

Theorem 1. *The functor* $\cdot \otimes W : \mathbf{FS} \to \mathbf{FS}$ *is product-preserving for any Weil algebra* W.

2.2. Weil Exponentiability

A Frölicher space X is called *Weil exponentiable* if
$$(X \otimes (W_1 \otimes_\infty W_2))^Y = (X \otimes W_1)^Y \otimes W_2 \tag{1}$$
holds naturally for any Frölicher space Y and any Weil algebras W_1 and W_2. If $Y = 1$, then (1) degenerates into
$$X \otimes (W_1 \otimes_\infty W_2) = (X \otimes W_1) \otimes W_2. \tag{2}$$
If $W_1 = \mathbb{R}$, then (1) degenerates into
$$(X \otimes W_2)^Y = X^Y \otimes W_2. \tag{3}$$

The following propositions and theorem have been established in our previous paper [15].

Proposition 2. *Convenient vector spaces are Weil exponentiable.*

Corollary 3. C^∞-*manifolds in the sense of [10] (cf. Section 27) are Weil exponentiable.*

Proposition 4. *If X is a Weil exponentiable Frölicher space, then so is $X \otimes W$ for any Weil algebra W.*

Proposition 5. *If X and Y are Weil exponentiable Frölicher spaces, then so is $X \times Y$.*

Proposition 6. *If X is a Weil exponentiable Frölicher space, then so is X^Y for any Frölicher space Y.*

Theorem 7. *Weil exponentiable Frölicher spaces, together with smooth mappings among them, form a Cartesian closed subcategory $\mathbf{FS_{WE}}$ of the category $\mathbf{FS}$.*

3. Transversal Limit Diagrams

Generally speaking, limits in the category $\mathbf{FS}$ are bamboozling. The notion of limit in $\mathbf{FS}$ should be elaborated geometrically.

A finite cone $\mathcal{D}$ in $\mathbf{FS}$ is called a *transversal limit diagram* providing that $\mathcal{D} \otimes W$ is a limit diagram in $\mathbf{FS}$ for any Weil algebra W, where the diagram $\mathcal{D} \otimes W$ is obtained from $\mathcal{D}$ by putting $\otimes W$ to the right of every object and every morphism in $\mathcal{D}$. By taking $W = \mathbb{R}$, we see that a transversal limit diagram is always a limit diagram. The limit of a finite diagram of Frölicher spaces is said to be *transversal* providing that its limit diagram is a transversal limit diagram.

Lemma 8. *If $\mathcal{D}$ is a transversal limit diagram whose objects are all Weil exponentiable, then $\mathcal{D}^X$ is also a transversal limit diagram for any Frölicher space X, where $\mathcal{D}^X$ is obtained from $\mathcal{D}$ by putting X as the exponential over every object and every morphism in $\mathcal{D}$.*

Proof. Since the functor $\cdot^X : \mathbf{FS} \to \mathbf{FS}$ preserves limits, we have
$$\mathcal{D}^X \otimes W = (\mathcal{D} \otimes W)^X$$
for any Weil algebra W, so that we have the desired result. $\square$

Lemma 9. *If $\mathcal{D}$ is a transversal limit diagram whose objects are all Weil exponentiable, then $\mathcal{D} \otimes W$ is also a transversal limit diagram for any Weil algebra W.*

Proof. Since the functor $W \otimes_\infty \cdot : \mathbf{W} \to \mathbf{W}$ preserves finite limits, we have
$$(\mathcal{D} \otimes W) \otimes W' = \mathcal{D} \otimes (W \otimes_\infty W')$$
for any Weil algebra W', so that we have the desired result. $\square$

4. Microlinearity

A Frölicher space X is called *microlinear* providing that any finite limit diagram $\mathcal{D}$ in $\mathbf{W}$ yields a limit diagram $X \otimes \mathcal{D}$ in $\mathbf{FS}$, where $X \otimes \mathcal{D}$ is obtained from $\mathcal{D}$ by putting $X \otimes$ to the left of every object and every morphism in $\mathcal{D}$.

The following result should be obvious.

Proposition 10. *Convenient vector spaces are microlinear.*

Corollary 11. *C^∞-manifolds in the sense of [10] (cf. Section 27) are microlinear.*

Proposition 12. *If X is a Weil exponentiable and microlinear Frölicher space, then so is $X \otimes W$ for any Weil algebra W.*

Proof. This follows simply from Proposition 4 and Lemma 9. $\qquad \square$

Proposition 13. *If X and Y are microlinear Frölicher spaces, then so is $X \times Y$.*

Proof. This follows simply from Theorem 1 and the familiar fact that the functor $\cdot \times \cdot : \mathbf{FS} \times \mathbf{FS} \to \mathbf{FS}$ preserves limits. $\qquad \square$

Proposition 14. *If X is a Weil exponentiable and microlinear Frölicher space, then so is X^Y for any Frölicher space Y.*

Proof. This follows simply from (3), Proposition 6 and Lemma 8. $\qquad \square$

We recapitulate:

Theorem 15. *Weil exponentiable and microlinear Frölicher spaces, together with smooth mappings among them, form a Cartesian closed subcategory $\mathbf{FS_{WE,ML}}$ of the category $\mathbf{FS}$.*

We note in passing that microlinearity is closed under transversal limits.

Theorem 16. *If the limit of a diagram $\mathcal{F}$ of microlinear Frölicher spaces is transversal, then it is microlinear.*

Proof. Let X be the limit of the diagram $\mathcal{F}$, i.e.,

$$X = \mathrm{Lim}\ \mathcal{F}.$$

Let W be the limit of an arbitrarily given finite diagram $\mathcal{D}$ of Weil algebras, i.e.,

$$W = \mathrm{Lim}\ \mathcal{D}.$$

We denote by $\mathcal{F} \otimes \mathcal{D}$ the diagram obtained from the diagrams $\mathcal{F}$ and $\mathcal{D}$ by the

application of the bifunctor $\otimes : \mathbf{FS} \times \mathbf{W} \to \mathbf{FS}$. By recalling that double limits in a complete category commute (cf. Section 2 of Chapter IX of [12]), we have

$$\mathrm{Lim}\ (X \otimes \mathcal{D}) = \mathrm{Lim}\ ((\mathrm{Lim}\ \mathcal{F}) \otimes \mathcal{D})$$
$$= \mathrm{Lim}_{\mathcal{D}}\ \mathrm{Lim}_{\mathcal{F}}\ (\mathcal{F} \otimes \mathcal{D})$$
$$[\text{since}\ \mathrm{Lim}\ \mathcal{F}\ \text{is the transversal limit}]$$
$$= \mathrm{Lim}_{\mathcal{F}}\ \mathrm{Lim}_{\mathcal{D}}\ (\mathcal{F} \otimes \mathcal{D})$$
$$[\text{since double limits commute}]$$
$$= \mathrm{Lim}\ (\mathcal{F} \otimes (\mathrm{Lim}\ \mathcal{D}))$$
$$[\text{since every object in}\ \mathcal{F}\ \text{is microlinear}]$$
$$= \mathrm{Lim}\ (\mathcal{F} \otimes W)$$
$$= (\mathrm{Lim}\ \mathcal{F}) \otimes W$$
$$[\text{since}\ \mathrm{Lim}\ \mathcal{F}\ \text{is the transversal limit}]$$
$$= X \otimes W.$$

Therefore we have the desired result. $\square$

Proposition 17. *If a Weil exponentiable Frölicher space X is microlinear, then any finite limit diagram $\mathcal{D}$ in $\mathbf{W}$ yields a transversal limit diagram $X \otimes \mathcal{D}$ in $\mathbf{FS}$.*

Proof. By the same token as in the proof of Lemma 9. $\square$

5. The Embedding into the Cahiers Topos

Let $\mathbf{D}$ be the full subcategory of the category of C^{∞}-algebras in form $C^{\infty}(\mathbb{R}^n) \otimes_{\infty} W$ with a natural number n and a Weil algebra W. Now we would like to extend the Weil prolongation $\mathbf{FS_{WE}} \times \mathbf{W} \overset{\otimes}{\to} \mathbf{FS_{WE}}$ to a bifunctor $\mathbf{FS_{WE}} \times \mathbf{D} \overset{\otimes}{\to} \mathbf{FS_{WE}}$. On objects we define

$$X \otimes C = X^{\mathbb{R}^n} \otimes W \tag{4}$$

for any Weil exponentiable Frölicher space X and any $C = C^{\infty}(\mathbb{R}^n) \otimes_{\infty} W$. By Proposition 6 $X^{\mathbb{R}^n}$ is Weil exponentiable, so that $X \otimes C$ is Weil exponentiable by Proposition 4. It is easy to see that the right hand of (4) is functorial in X, but we have not so far succeeded in establishing its functoriality in C. Therefore we pose it as a conjecture.

Conjecture 18. *The right hand of (4) is functorial in C, so that we have a bifunctor $\mathbf{FS_{WE}} \times \mathbf{D} \overset{\otimes}{\to} \mathbf{FS_{WE}}$.*

In the following we will assume that the conjecture is really true. We define the functor $\mathbf{J} : \mathbf{FS_{WE}} \to \mathbf{Sets^D}$ to be the exponential adjoint to the composite

$$\mathbf{FS_{WE}} \times \mathbf{D} \overset{\otimes}{\to} \mathbf{FS_{WE}} \to \mathbf{Sets},$$

where $\mathbf{FS_{WE}} \to \mathbf{Sets}$ is the underlying-set functor. Now we have

Proposition 19. *For any Weil-exponentiable Frölicher space X and any object $\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W$ in $\mathbf{D}$, we have*

$$\mathbf{J}(X)^{\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W, \cdot)} = \mathbf{J}(X \otimes (\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W)).$$

Proof. For any object $\mathcal{C}^\infty(\mathbb{R}^m) \otimes_\infty W'$ in $\mathbf{D}$, we have

$\mathbf{J}(X)^{\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W, \cdot)}(\mathcal{C}^\infty(\mathbb{R}^m) \otimes_\infty W')$

$= \hom_{\mathbf{sets^D}}(\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^m) \otimes_\infty W', \cdot), \mathbf{J}(X)^{\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W, \cdot)})$

[By Yoneda Lemma]

$= \hom_{\mathbf{sets^D}}(\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^m) \otimes_\infty W', \cdot) \times \hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W, \cdot), \mathbf{J}(X))$

$= \hom_{\mathbf{sets^D}}(\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^m) \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W \otimes_\infty W', \cdot), \mathbf{J}(X))$

$= \hom_{\mathbf{sets^D}}(\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R}^n) \otimes_\infty W \otimes_\infty W', \cdot), \mathbf{J}(X))$

$= \mathbf{J}(X)(\mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R}^n) \otimes_\infty W \otimes_\infty W')$

[By Yoneda Lemma]

$= X^{\mathbb{R}^m \times \mathbb{R}^n} \otimes (W \otimes_\infty W')$

$= (X^{\mathbb{R}^n} \otimes W)^{\mathbb{R}^m} \otimes W'$

$= (X \otimes (\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W))^{\mathbb{R}^m} \otimes W'$

$= \mathbf{J}(X \otimes (\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W))(\mathcal{C}^\infty(\mathbb{R}^m) \otimes_\infty W').$

Therefore we have the desired result. $\qquad\square$

Proposition 20. *For any Weil-exponentiable Frölicher spaces X and Y, we have*

$$\mathbf{J}(X \times Y) = \mathbf{J}(X) \times \mathbf{J}(Y).$$

Proof. For any object $\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W$ in $\mathbf{D}$, we have

$\mathbf{J}(X \times Y)(\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W) = (X \times Y)^{\mathbb{R}^n} \otimes W$

$\qquad\qquad = (X^{\mathbb{R}^n} \times Y^{\mathbb{R}^n}) \otimes W$

$\qquad\qquad = (X^{\mathbb{R}^n} \otimes W) \times (Y^{\mathbb{R}^n} \otimes W)$

$\qquad\qquad$ [By Theorem 1]

$\qquad\qquad = \mathbf{J}(X)(\mathcal{C}^\infty(\mathbb{R}^n) \otimes W) \times \mathbf{J}(Y)(\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W)$

$\qquad\qquad = (\mathbf{J}(X) \times \mathbf{J}(Y))(\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W)$

Therefore we have the desired result.   □

Proposition 21.   *For any Weil exponentiable Frölicher spaces X and Y, we have the following isomorphism in* $\mathbf{Sets}^{\mathbf{D}}$:
$$\mathbf{J}(X^Y) = \mathbf{J}(X)^{\mathbf{J}(Y)}.$$

Proof. Let $\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W$ be an object in $\mathbf{D}$. On the one hand, we have
$$\mathbf{J}(X^Y)(\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W) = (X^Y)^{\mathbb{R}^n} \otimes W$$
$$= X^{Y \times \mathbb{R}^n} \otimes W$$
$$= (X^{\mathbb{R}^n} \otimes W)^Y.$$

On the other hand, we have
$$\mathbf{J}(X)^{\mathbf{J}(Y)}(\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W) = \hom_{\mathbf{Sets}^{\mathbf{D}}}(\hom_{\mathbf{D}}(\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W, \cdot), \mathbf{J}(X)^{\mathbf{J}(Y)})$$
$$[\text{By Yoneda Lemma}]$$
$$= \hom_{\mathbf{Sets}^{\mathbf{D}}}(\hom_{\mathbf{D}}(\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W, \cdot) \times \mathbf{J}(Y), \mathbf{J}(X))$$
$$= \hom_{\mathbf{Sets}^{\mathbf{D}}}(\mathbf{J}(Y), \mathbf{J}(X)^{\hom_{\mathbf{D}}(\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W, \cdot)})$$
$$= \hom_{\mathbf{Sets}^{\mathbf{D}}}(\mathbf{J}(Y), \mathbf{J}(X \otimes (\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W)))$$
$$[\text{By Proposition 19}]$$
$$= \hom_{\mathbf{FS}_{\mathbf{WE}}}(Y, X \otimes (\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W))$$
$$[\text{since } \mathbf{J} \text{ is full and faithful}].$$

Therefore we have the desired result.   □

Theorem 22.   *The functor* $\mathbf{J} : \mathbf{FS}_{\mathbf{WE}} \to \mathbf{Sets}^{\mathbf{D}}$ *preserves the Cartesian closed structure. In other words, it preserves finite products and exponentials. It is full and faithful. It sends the Weil prolongation to the exponentiation by the corresponding infinitesimal object.*

Proof. The first statement follows from Propositions 20 and 21. The second statement that it is full and faithful follows by the same token as in [6] and [7]. The final statement follows from Proposition 19.   □

Now we are concerned with microlinearity. First we will establish

Proposition 23.   *The functor* $\mathbf{J} : \mathbf{FS}_{\mathbf{WE}} \to \mathbf{Sets}^{\mathbf{D}}$ *preserves transversal limit diagrams. In other words, the functor* $\mathbf{J}$ *always sends a transversal limit diagram of Frölicher spaces lying in* $\mathbf{FS}_{\mathbf{WE}}$ *to the limit diagram in* $\mathbf{Sets}^{\mathbf{D}}$.

Proof. We should show that a transversal limit diagram $\mathcal{D}$ in $\mathbf{FS}_{\mathbf{WE}}$ always yields a limit diagram $\mathbf{J}(\mathcal{D})$ in $\mathbf{Sets}^{\mathbf{D}}$. To this end, it suffices to show (cf. [13], pp. 22-23) that $\mathbf{J}(\mathcal{D})(C)$ is a limit diagram in $\mathbf{Sets}$ for any object $C =$

$\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W$ in **D**. Since the forgetful functor **FS** → **Sets** preserves limits, we have only to note that $\mathcal{D}^{\mathbb{R}^n} \otimes W$ is a limit diagram in **FS**, which follows readily from Lemma 8. □

Theorem 24. *The functor* **J** : **FS**$_{\mathbf{WE}}$ → **Sets**$^{\mathbf{D}}$ *preserves microlinearity.*

Proof. This follows simply Propositions 17, 19 and 23. □

The site of definition for the Cahiers Topos $\mathcal{C}$ is the dual category **D**$^{\mathrm{op}}$ of the category **D** together with the open-cover topology, so that we have the canonical embedding

$$\mathcal{C} \hookrightarrow \mathbf{Sets}^{\mathbf{D}}.$$

By the same token as in [6] and [7] we can see that the functor **J** : **FS**$_{\mathbf{WE}}$ → **Sets**$^{\mathbf{D}}$ factors in the above embedding. The resulting functor is denoted by **J**$_{\mathcal{C}}$. Since the above embedding creates limits and exponentials, Theorems 22 and 24 yields directly

Theorem 25. *The functor* **J**$_{\mathcal{C}}$: **FS**$_{\mathbf{WE}}$ → $\mathcal{C}$ *preserves the Cartesian closed structure. In other words, it preserves finite products and exponentials. It is full and faithful. It sends the Weil prolongation to the exponentiation by the corresponding infinitesimal object. The functor* **J**$_{\mathcal{C}}$ *preserves transversal limit diagrams and microlinearity.*

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