

## Remarks on Krein-Kotani's correspondence between strings and Herglotz functions

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**Abstract:** In the study of spectral functions of Strum-Liouville operators, S. Kotani found a one-to-one correspondence between the operators and Herglotz functions determining the spectral measures. In the present paper we give another correspondence to a class of Lévy processes obtained as a compensated integrals of Brownian local times. As an application the continuity theorem of Kotani's correspondence is extended.

**Key words:** Strum-Liouville operator; spectral measure; Krein's correspondence; Herglotz function; Lévy process.

**1. Introduction.** In [3] the authors gave a Brownian representation for a class of Lévy processes whose Lévy measures are absolutely continuous with completely monotone densities. However, some theorems were left half done in the sense that we mentioned a little but did not discuss, for instance, the inverse problem because probabilistic approach is not suited for such kind of problems and we need analytical results instead. The aim of the present paper is a supplement based on the results of recent paper of S. Kotani [4] on spectral theory of second-order differential operators. Also our argument, in turn, gives an extension to Kotani's result.

We first explain the notation and review quickly the results of Kotani cited above. By a *string* we mean a function

$$m : (-\infty, +\infty) \rightarrow [0, +\infty]$$

which is nondecreasing, right-continuous and normalized so that  $m(-\infty) = 0$ . We exclude the trivial case where  $m$  vanishes identically.

For a string  $m$  we define

$$\ell = \ell(m) = \sup\{x; m(x) < \infty\} \quad (\leq +\infty),$$

and we are interested in the spectral theory of the

generalized Strum-Liouville operator

$$L = -\frac{d}{dm(x)} \frac{d}{dx}, \quad -\infty < x < \ell.$$

This operator appears not only in the theory of vibration of the strings but also in Feller's theory of diffusion processes.

We say that a string  $m$  has left boundary of *limit circle type* if, for some  $c (< \ell)$ ,

$$(1) \quad \int_{-\infty}^c x^2 dm(x) < \infty.$$

Throughout the paper we denote by  $\mathcal{M}_{circ}$  the totality of strings satisfying the condition (1). For each  $m \in \mathcal{M}_{circ}$ , we can define  $\varphi_\lambda(x)$ , ( $x < \ell$ ), for every  $\lambda \in \mathbf{C}$ , as the unique solution of the following integral equation:

$$\varphi_\lambda(x) = 1 - \lambda \int_{-\infty}^x (x-y)\varphi_\lambda(y) dm(y), \quad x < \ell.$$

Let  $L_0^2((-\infty, \ell), dm)$  denote the space of all square integrable functions  $f$  such that  $\text{Supp}(f) \subset (-\infty, \ell)$  and, for  $f \in L_0^2((-\infty, \ell), dm)$ , define

$$\widehat{f}(\lambda) = \int_{-\infty}^{\ell} f(x) \varphi_\lambda(x) dm(x).$$

Then a nonnegative Radon measure  $\sigma(d\xi)$  on  $[0, \infty)$  is called a *spectral measure* if

$$\|f\|_{L_0^2((-\infty, \ell), dm)} = \|\widehat{f}\|_{L^2([0, \infty), d\sigma)}.$$

According to Kotani's paper we can compute the spectral measure  $\sigma(d\xi)$  by the following procedure: Let

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$$(2) \quad H(\lambda) = a + \int_{-\infty}^a \left( \frac{1}{\varphi_\lambda(x)^2} - 1 \right) dx + \int_a^\ell \frac{dx}{\varphi_\lambda(x)^2},$$

which exists for every  $\lambda < 0$  and does not depend on the choice of  $a (< \ell)$ . Then  $H$  is a Herglotz function with the following representation:

$$(3) \quad H(\lambda) = \alpha + \int_{-0}^\infty \left( \frac{1}{\xi - \lambda} - \frac{\xi}{\xi^2 + 1} \right) \sigma(d\xi),$$

where  $\alpha \in \mathbf{R}$  and  $\sigma(d\xi)$  is a nonnegative Radon measure on  $[0, \infty)$  satisfying

$$\int_0^\infty \frac{\sigma(d\xi)}{\xi^2 + 1} < \infty.$$

Then  $\sigma(d\xi)$  is the spectral measure in question. We call  $H$  the *characteristic Herglotz function* of the string  $m$ . Kotani [4] proved that the correspondence between  $m$  and  $H$  is not only one-to-one but also *onto* (i.e., for any  $\alpha \in \mathbf{R}$  and  $\sigma(d\xi)$  as above,  $H(\lambda)$  defined by (3) is the characteristic Herglotz function of some string  $m \in \mathcal{M}_{\text{circ}}$ ). Note that M. G. Krein's correspondence is the case where  $\int_0^\infty \sigma(d\xi)/(\xi + 1) < \infty$  and, therefore, Kotani's correspondence is its extension.

Krein's correspondence is bicontinuous in some sense (see [5]) and it remains true in Kotani's as well;

**Theorem A** (S. Kotani [4]). *Let  $m_n, m \in \mathcal{M}_{\text{circ}}$  and let  $H_n, H$  be their characteristic Herglotz functions. Then  $H_n(\lambda) \rightarrow H(\lambda)$  as  $n \rightarrow \infty$  for all  $\lambda < 0$  if and only if the following two conditions hold:*

$$(M1) \quad m_n(x) \Rightarrow m(x),$$

$$(M2) \quad \lim_{c \rightarrow -\infty} \sup_{n \geq 1} \int_{-\infty}^c \left( \int_{-\infty}^y m_n(u) du \right) dy = 0.$$

Here, " $\Rightarrow$ " denotes the convergence at all continuity points of the limit function.

The aim of the present paper is to study the following two problems: The first one is to give a probabilistic representation of  $H$  in terms of the string  $m$  and Brownian local times. This will supplement one of the results in [3] and therefore its inverse problem is completely reduced to Kotani's result. The second one is to study the case where the condition (M2) fails. These two problems may look independent but will turn out helpful each other.

**2. Brownian representation of  $H(\lambda)$ .**

The Herglotz function  $H$  is determined by  $m$  via (2). But in this section we shall see another

(probabilistic) relationship between  $H$  and  $m$ .

For a string  $m \in \mathcal{M}_{\text{circ}}$  we define its *dual*  $m^* : (0, \infty) \rightarrow (-\infty, \infty]$  by

$$m^*(x) (= m^{-1}(x)) = \inf\{u; m(u) > x\} \quad (x > 0).$$

Notice that the condition (1) can be rewritten as

$$(4) \quad \int_{(0,a)} m^*(x)^2 dx < \infty$$

for some (small)  $a > 0$ . For such  $m^*$ , we can associate a Lévy process represented by the Brownian local time as follows: Let  $\{B(t); t \geq 0\}$  be a standard Brownian motion ( $B(0) = 0$ ) and let  $\ell(t, x)$  be its local time with respect to  $2dx$ ; i.e.,

$$\int_0^t f(B(u)) du = 2 \int_{-\infty}^\infty f(x) \ell(t, x) dx$$

for all bounded continuous function  $f$ . For functions  $m^*$  satisfying (4) we defined in [3] a Lévy process  $T(m^*; t)$  in the following way; for any  $\varepsilon_1 > \varepsilon_2 > \dots \rightarrow 0$ , the following limits (5) exist almost surely for any  $0 \leq t < \zeta_\pm := \inf\{\ell(t, 0); B(t) = \pm \ell(m^*)\}$  and define mutually independent but equi-distributed Lévy processes  $T^\pm(m^*; t)$  with life times  $\zeta_\pm$ , respectively;

$$(5) \quad T^\pm(m^*; t) = \lim_{n \rightarrow \infty} \int_{x > \varepsilon_n} \ell(\ell^{-1}(t, 0), \pm x) dm^*(x) + m^*(\varepsilon_n)t.$$

Then  $T(m^*; t)$  is a Lévy process with this common distribution (cf. [3]). Of course the interesting case is when  $m^*(\varepsilon_n) \rightarrow -\infty$ . We refer to [3] for details but we note that (5) is a natural extension of the following way to construct an  $\alpha$ -stable Lévy motion ( $0 < \alpha < 1$ ):

$$Z_\alpha(t) = \int_{x > 0} \ell(\ell^{-1}(t, 0), x) x^{(1/\alpha)-2} dx.$$

Our first result is the following formula which was already referred to in [3] without proof. It is an extension of the well-known formula for Krein's strings.

**Theorem 1.** *Let  $m \in \mathcal{M}_{\text{circ}}$  and let  $\zeta$  be the life-time of  $\{T(m^*; t)\}_{t \geq 0}$ . Then,*

$$(6) \quad E[e^{\lambda T(m^*; t)}; \zeta > t] = e^{t\lambda H(\lambda)}, \quad \lambda < 0.$$

*Proof.* (Step 1) If  $m(-0) = 0$ , then  $m$  is a Krein string and, as mentioned above, the assertion is already known. For details see [5] (where  $h(s) = H(-s)$ ).

(Step 2) If  $m(c) = 0$  for some  $c < 0$ , then put  $m_c(x) = m(x - c)$ , which is a Krein string (i.e.,  $m(-0) = 0$ ). By a simple change of the variables, we see that the characteristic Herglotz function of  $m_c$  is  $H_c(\lambda) = H(\lambda) + c$  and also it is easy to see that  $m_c^*(x) = m^*(x) + c$  and hence  $T(m_c^*; t) = T(m^*; t) + ct$ . Thus the problem may be reduced to Step 1.

(Step 3) For  $m \in \mathcal{M}_{\text{circ}}$  such that  $m(+\infty) = \infty$  so that  $\zeta = +\infty$ , consider the truncated strings  $m_n \in \mathcal{M}_{\text{circ}}$  defined by  $dm_n(x) = 1_{(-n, \infty)}(x) dm(x)$  and apply Step 2 to have

$$(7) \quad E[e^{\lambda T(m_n^*; t)}] = e^{t\lambda H_n(\lambda)}, \quad \lambda < 0.$$

Now let  $n \rightarrow \infty$ . The right side converges to  $e^{t\lambda H(\lambda)}$  by Kotani's theorem, while the left-hand side converges to  $E[e^{\lambda T(m^*; t)}]$  since it holds  $T(m_n^*; t) \rightarrow T(m^*; t)$  a.s. by Theorem 2.5 of [3]. Precisely speaking, since  $X_n = T(m_n^*; t)$  ( $n \geq 1$ ) may take negative values in general, the convergence in law does not necessarily imply that of the Laplace transforms without additional conditions. However, in the present case it is not difficult to see the convergence (see Lemma 1 in Appendix).

(Step 4) For  $m \in \mathcal{M}_{\text{circ}}$  such that  $m(+\infty) < \infty$ , we need a little modification because some of the arguments in Step 3 are not trivial because  $\zeta < \infty$ . In such a case consider the truncated strings  $m_n(x) = m(x) + \infty 1_{[n, \infty)}(x)$ , for which we can apply Step 3. Since  $T(m_n^*; t)$  is nondecreasing in  $n$ , it is easy to complete the proof by the continuity of the correspondence.  $\square$

We can use Theorem 1 in two ways. One is to study the law of  $T(m^*; t)$  by using  $H(\lambda)$  and the other is to apply results on  $T(m^*; t)$  to the study of  $H(\lambda)$ . To begin with let us compute Lévy-Khintchine's canonical representation of the Fourier transform of  $T(m^*; t)$ : Let  $m \in \mathcal{M}_{\text{circ}}$  and let  $H$  be its characteristic Herglotz function of the form (3), and define

$$M(x) = - \int_{-0}^{\infty} e^{-x\xi} \sigma(d\xi) \quad (x > 0).$$

Recall that, in general, such a function expressed as a Laplace transform of a Radon measure on  $[0, \infty)$  is said to be *completely monotone* (see Feller [1]). Then by a direct computation we can rewrite (6) as

$$\begin{aligned} \log E[e^{-sT(m^*; 1)}] &= -sH(-s) \\ &= -\alpha' s + \int_0^{\infty} \left( e^{-sx} - 1 + \frac{sx}{1+x^2} \right) dM(x) \end{aligned}$$

for  $s > 0$ , where  $\alpha'$  is a real constant. Thus,  $dM(x)$ , which is a positive Radon measure on  $(0, \infty)$ , is the Lévy measure of the Lévy process  $\{T(m^*; t)\}$ . This fact is an extension of the well-known formula for the case of Krein's string. Thus Theorem 1 implies the following properties of  $T(m^*; t)$ , which were referred to without proofs (except (ii)) in our previous paper [3].

**Corollary 1.** *For every  $t > 0$ , the law of  $T(m^*; t)$  is infinitely divisible such that*

- (i) *the Gaussian part vanishes,*
- (ii) *the Lévy measure vanishes on  $(-\infty, 0)$ , and*
- (iii) *the function  $-M(x) = \int_{(x, \infty)} dM(y)$ ,  $x > 0$ , defined by the Lévy measure, is completely monotone.*

*Conversely, such an infinitely divisible law can be realized by  $T(m^*; 1) + a$  for suitably chosen  $m \in \mathcal{M}_{\text{circ}}$  and  $a \in \mathbf{R}$ .*

Thus, Kotani's correspondence between  $m \in \mathcal{M}_{\text{circ}}$  and  $H(\lambda)$  also corresponds to a Lévy process satisfying (i)–(iii).

**3. An extended continuity theorem.** At the end of the previous section we mentioned an application of Theorem 2 to a class of Lévy process, and in this section we give, in return, an application of probability theory to Theorem A via Theorem 1: We study the case where the condition (M1) in Theorem A holds but (M2) fails. Our main result is

**Theorem 2.** *Let  $\sigma^2 \geq 0$  and  $m_n, m \in \mathcal{M}_{\text{circ}}$  and let  $H_n, H$  be their characteristic Herglotz functions, respectively. Then*

$$(8) \quad H_n(\lambda) \rightarrow H(\lambda) + \sigma^2 \lambda, \quad n \rightarrow \infty$$

*for every  $\lambda < 0$  if and only if the following two conditions hold:*

$$(M1) \quad m_n \Rightarrow m,$$

$$(M2a) \quad \lim_{c \rightarrow -\infty} \limsup_{n \rightarrow \infty} \left| \int_{-\infty}^c x^2 dm_n(x) - \sigma^2 \right| = 0.$$

Before we proceed to the proof, we see that this theorem is compatible with Theorem A. To this end note that

$$\int_{-\infty}^c \left( \int_{-\infty}^y m_n(u) du \right) dy = \frac{1}{2} \int_{-\infty}^c (c-x)^2 dm_n(x)$$

while

$$\begin{aligned} \int_{-\infty}^{2c} (2c-x)^2 dm(x) &\leq \int_{-\infty}^{2c} x^2 dm(x) \\ &\leq 4 \int_{-\infty}^c (c-x)^2 dm(x). \end{aligned}$$

Here the inequalities hold because  $x < 2c < 0$  implies  $|x - 2c| \leq |x| \leq 2|x - c|$ . Therefore, it is easy to see that, when  $\sigma^2 = 0$ , (M2a) is equivalent to (M2).

**Proof of Theorem 2.** Suppose (M1) and (M2a) are satisfied. Then, by Theorem 2.10 of [3], we see that  $T(m_n^*; t)$  converges in law to  $T(m^*; t) + \sqrt{2}\sigma B^*(t)$ , where  $\{B^*(t)\}_{t \geq 0}$  is a standard Brownian motion independent of  $\{T(m^*; t)\}_t$ . This implies that, for  $\lambda < 0$ ,

$$E[e^{\lambda T(m_n^*; t)}] \rightarrow E[e^{\lambda T(m^*; t) + \sqrt{2}\sigma B^*(t)}]$$

by Lemma 1 in Appendix. Therefore, by Theorem 1, this probabilistic result may be translated as (8). Thus we have the “if” part. Conversely, suppose (8) holds. Then as in the proof of Theorem 1 of [4], we can choose a subsequence  $\{m_{n_k}\}_{k \geq 1}$  and a string  $m_\infty$  such that

$$m_{n_k}(x) \implies m_\infty(x).$$

By Fatou's inequality, we see  $m_\infty \in \mathcal{M}_{circ}$  although (M2) may fail in general. However, instead, his proof implies

$$\lim_{c \rightarrow -\infty} \sup_{k \geq 1} \int_{-\infty}^c \left( \int_{-\infty}^y m_{n_k}(u) du \right) dy < \infty$$

which condition is equivalent to

$$\lim_{c \rightarrow -\infty} \limsup_{k \rightarrow \infty} \int_{-\infty}^c x^2 dm_{n_k}(x) < \infty.$$

(We do not go into details because this condition is in fact not mandatory in the sequel if we allow a simple abuse of notation).

Now choosing a subsequence again, if necessary, we may assume that

$$\lim_{c \rightarrow -\infty} \limsup_{j \rightarrow \infty} \left| \int_{-\infty}^c x^2 dm_{n_j}(x) - \sigma_\infty^2 \right| = 0$$

for some  $\sigma_\infty^2$  (consider the measures  $x^2 dm_{n_j}(x)$  on  $[-\infty, \ell)$ ). Then, apply the “if” part to deduce

$$H_{n_j}(\lambda) \rightarrow H_\infty(\lambda) + \sigma_\infty^2 \lambda, \quad \lambda < 0.$$

Since the left side converges to  $H(\lambda) + \sigma^2 \lambda$  by assumption, we see that  $H_\infty(\lambda) + \sigma_\infty^2 \lambda = H(\lambda) + \sigma^2 \lambda$ , which implies that  $H_\infty(\lambda) = H(\lambda)$  and  $\sigma_\infty^2 = \sigma^2$  by the uniqueness of the representation of Herglotz functions. Since the subsequence was arbitrary, it is routine to complete the proof.  $\square$

**Example.** Let  $m \in \mathcal{M}_{circ}$  and Define

$$m_n(x) = m(x) + c_n 1_{[-n, \infty)}(x), \quad (x \in \mathbf{R})$$

for a sequence  $c_1 > c_2 > \dots \rightarrow 0$ . Of course  $m_n \implies m$ . Let  $H$  and  $H_n$  be the characteristics of  $m$  and  $m_n$ . Then, by Theorem 1 we have  $H_n(\lambda) \rightarrow H(\lambda)$  if and only if  $n^2 c_n \rightarrow 0$ . But Theorem 2 insists that, more generally,  $H_n(\lambda) \rightarrow H(\lambda) + \sigma^2 \lambda$  if and only if  $n^2 c_n \rightarrow \sigma^2$ . This fact can be confirmed directly if  $m$  is of Krein's type; i.e.,  $m(a) = 0$  for some  $a$ . In such a case, we have, for  $n > -a$ ,

$$\begin{aligned} H_n(\lambda) &= -n + \frac{1}{-c_n \lambda + \frac{1}{n + H(\lambda)}} \\ &= \frac{H(\lambda) + n c_n \lambda H(\lambda) + n^2 c_n \lambda}{-n c_n \lambda - c_n \lambda H(\lambda) + 1} \end{aligned}$$

(cf. Examples 1.1, 1.3 of [5]). Thus the assertion can easily be verified.

#### 4. Appendix.

**Lemma 1.** Let  $X_1, X_2, \dots$  be random variables with infinitely divisible laws such that

$$\begin{aligned} \log E[e^{i\xi X_n}] &= i c_n \xi - \frac{1}{2} \beta_n^2 \xi^2 \\ &+ \int_{+0}^{\infty} \left( e^{i\xi x} - 1 - \frac{i\xi x}{1 + x^2} \right) dM_n(x). \end{aligned}$$

If  $X_n$  converges in law to a random variable  $X$ , then, for all  $s > 0$ ,

$$(9) \quad E[e^{-s X_n}] \rightarrow E[e^{-s X}], \quad n \rightarrow \infty.$$

*Proof.* Since it is well known that the law of  $X$  also has a similar representation, let  $c_\infty, \beta_\infty, M_\infty(x)$  be their characteristic. Then it holds that

$$c_n \rightarrow c_\infty, \quad M_n(x) \implies M_\infty(x) \quad (x > 0),$$

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \left| \int_{(0, \varepsilon]} x^2 dM_n(x) + \frac{1}{2} (\sigma_n^2 - \sigma_\infty^2) \right| = 0$$

(see p. 88 of [2]). Therefore, it is a routine to prove (9).  $\square$

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