Local properties, bialgebras and representations for one-dimensional tilings

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Abstract

For any one-dimensional tiling, we discuss finite dimensional standard modules for the associated tiling bialgebra. We will notice that such modules are completely reducible, and we will parametrize finite dimensional irreducible ones, using the set of all patches. Furthermore, we will discuss the associated completed groups and Iwasawa-type decompositions. We also characterize for one-dimensional tilings to be locally nondistinguishable by the associated tiling bialgebra structures.

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0. Introduction. In this paper, we will give an algebraic approach to tilings, and establish some basic properties. First, we will construct tiling monoids (Section 2) and tiling bialgebras (Section 3). Then we will classify finite dimensional irreducible standard modules (Section 4). Also we will reach a certain decomposition rule for tensor products of such modules (Section 6). Since our bialgebra is coming from a monoid structure, the action on a tensor product can be said to be diagonal roughly. Hence, the decomposition rule looks very simple (Section 5). Furthermore, we will construct some completed groups using formal exponential maps. Then, we will see that there are Gauss decompositions as well as Iwasawa decompositions in such groups (Section 10). We shall review for the readers Gauss decompositions (Section 8) and Iwasawa decompositions (Section 9), both of which are well-known. Finally we will discuss some local property for two tilings to be locally nondistinguishable. We will find that this condition is equivalent to the fact that the corresponding tiling bialgebras are isomorphic as bialgebras with triangular decompositions (Section 11).

1. Tilings. Let $V = \mathbb{R}$ be a real line. A tile in V is a connected closed bounded subset V, namely a closed interval $[a, b]$ whose interior is nonempty. A tiling $\mathcal{T}$ of V

is an infinite set of tiles which cover V overlapping, at most, at their boundaries. A finite subset of a tiling is called a patch if the subset of V covered by its elements is connected. Let $P = P(\mathcal{T})$ be the set of all patches in $\mathcal{T}$. We define two patches τ and τ' to be equivalent if there is a translation f of V such that $f(\tau) = \tau'$. Each equivalence class of patches is called a patch class. The patch class containing a patch τ is denoted by $[\tau]$, and we put $[P] = \{ \ [\tau] \ \mid \ \tau \in P \ \}$. Sometimes, we use the same notation $[P]$ for a set of complete representatives of $[P]$ in P, and the same notation $[\tau]$ for a representative of the patch class $[\tau]$. If a patch consists of one tile, then the corresponding equivalence class is called a tile class. Let $\Omega = \Omega(\mathcal{T}) = \{ \ [\tau] \ \mid \ \tau \in P, \ Card(\tau) = 1 \ \}$ be the set of all tile classes of $\mathcal{T}$. We identify Ω with a set of complete representatives of tile classes. An element of Ω may simply be written as $[\omega]$ or ω instead of $[\{\omega\}]$.

2. Tiling monoids. A doubly pointed patch is a triplet (p, τ, q) with a patch τ and two distinguished tiles p and q appeared in τ. We also define (p, τ, q) and (p', τ', q') to be equivalent if there is a translation f of V such that $f(\tau) = \tau', \ f(p) = p', \ f(q) = q'$. Each equivalence class is called a doubly pointed patch class. The doubly pointed patch class containing (p, τ, q) is denoted by $[p, \tau, q]$. Sometimes we identify $[\omega]$ with $[\omega, \{\omega\}, \omega]$ for each tile $\omega \in \mathcal{T}$. Hence, we can view Ω as the set of $[\omega, \{\omega\}, \omega]$ for all tiles ω in our tiling $\mathcal{T}$. Let $M = M(\mathcal{T})$ be the set of all doubly pointed patch classes of a tiling $\mathcal{T}$ together with two special symbols $\mathbf{z}$ and $\mathbf{e}$, that is, $M = \{ \ \mathbf{z}, \ \mathbf{e}, \ [p, \tau, q] \ \mid \ p, q \in \tau, \ \tau \in P \ \}$. We will introduce a binary operation on M. Let $[p, \tau, q]$ and $[p', \tau', q']$ be two doubly pointed patch classes. If there are translations f and f' of V such that both $f(\tau)$ and $f'(\tau')$ are patches with $f(q) = f'(p')$, then we define $[p, \tau, q] \ [p', \tau', q'] = [p'', \tau'', q'']$, where $p'' = f(p), \ q'' = f'(q')$, and $\tau'' = f(\tau) \cup f'(\tau')$. If there is no such a pair of translations, we define $[p, \tau, q] \ [p', \tau', q'] = \mathbf{z}$. Also we define $m \ \mathbf{z} = \mathbf{z} \ m = \mathbf{z}$ as well as $m \ \mathbf{e} = \mathbf{e} \ m = m$ for all $m \in M$. Then, the set M becomes a monoid with the above operation. We call M the tiling monoid of a given tiling $\mathcal{T}$. In another sense, M can also be regarded as an inverse monoid with zero (cf. [6]).

3. Tiling bialgebras. Let $A = A(\mathcal{T}) = \mathbb{C}[M] = \oplus_{m \in M} \mathbb{C}m$ be the monoid algebra of M over $\mathbb{C}$. Then, we can introduce a coalgebra map $\Delta \ : \ A \longrightarrow A \otimes A$ with $\Delta(m) = m \otimes m \ (m \in M)$, and a counit map $\varepsilon \ : \ A \longrightarrow \mathbb{C}$ with $\varepsilon(m) = 1 \ (m \in M)$. In this way, A has a bialgebra structure (cf. [1]). Then $\mathbb{C}\mathbf{z}$ is a two sided ideal of A and a coideal of A, since $\Delta(\mathbf{z}) = \mathbf{z} \otimes \mathbf{z} \in \mathbb{C}\mathbf{z} \otimes A + A \otimes \mathbb{C}\mathbf{z}$. We set $B = B(\mathcal{T}) = A/\mathbb{C}\mathbf{z}$, which is called the tiling bialgebra of $\mathcal{T}$. We use the same symbol $\mathbf{e}$ and $[p, \tau, q]$ for their images modulo $\mathbb{C}\mathbf{z}$. If we consider $G(B) = \{ \ x \in B \ \mid \ \Delta(x) = x \otimes x, \ \varepsilon(x) = 1 \ \}$, the set of all group-like elements in B, then we easily see $G(B) = \{ \ \mathbf{e}, \ [p, \tau, q] \ \mid \ p, q \in \tau, \ \tau \in P \ \}$. Let $\mathcal{L} = \mathcal{L}(B)$ be the family of principal two sided ideals, $I_h = BhB$, of B generated by h with $h \in G(B)$. We consider this $\mathcal{L}$ as a lattice (with respect to inclusion). We usually use $1 = 1_B$ instead of $\mathbf{e}$ for an identity element of the bialgebra B. We set $H = H(B) = \{ \ [p, \tau, q] \ \mid \ p, q \in \tau, \ \tau \in P, \ |\tau| \ = 2, \ p \neq q \ \}$. Then, we see that B is generated by 1, Ω and H.

For $\omega, \omega' \in \mathcal{T}$, we say $\omega < \omega'$ if $x < x'$ with $x \in \omega^\circ$ and $x' \in \omega'^\circ$, where ω° and ω'° are the interior of ω and ω' respectively. Then, we put

$$B_+ = \sum_{[p,\tau,q]\in M,\ p<q} \mathbb{C}[p,\tau,q], \qquad B_- = \sum_{[p,\tau,q]\in M,\ p>q} \mathbb{C}[p,\tau,q],$$

and $B_0 = \mathbb{C}1 \oplus \sum_{[p,\tau,p]\in M} \mathbb{C}[p,\tau,p]$. Then we obtain triangular decompositions (cf. [8]) : $B = B_- \oplus B_0 \oplus B_+$.

4. Standard modules. A B-module U is called standard if for each $u \in U$, there are only finitely many $x \in G(B)$ such that $xu \neq 0$. Here we put $T = [P] \cup \{1\}$. For $t \in T$, we define $|t|$ by $|1| = 1$ and $|t| = Card(\tau)$ if $t = [\tau] \in [P]$.

Proposition 1. Let U be a finite dimensional standard B-module. Then:
(1) U is completely reducible.
(2) If U is irreducible, then U is isomorphic to U_t for some $t \in T$. (Finite dimensional irreducible standard B-modules are parametrized by T naturally.)
(3) In general, $U \simeq U_{t_1} \oplus U_{t_2} \oplus \cdots \oplus U_{t_s}$ for some $t_1, t_2, \cdots, t_s \in T$.

For $t = 1$, we put $U_1 = \mathbb{C}$, being viewed as a trivial B-module, that is, $1u = u$ and $[p,\tau,q]u = 0$ for all $u \in U_1$ and for all $[p,\tau,q] \in B$. We want to construct the representation corresponding to $t = [\tau] \in [P]$. If $|t| = 1$, then $U_t = \mathbb{C}$ and $[p,\sigma,q]$ with $[\sigma] \neq [\tau]$ acts on U_t as the zero operator, while $[\tau]$ acts as the identity operator. Next suppose that $\tau = \{\omega_1, \ldots, \omega_\ell\}$ with $\ell = |\tau| > 1$ and $\omega_i < \omega_{i+1}$ for $1 \leq i \leq \ell - 1$. Put $U_t = \mathbb{C}^\ell$. We take $x_k = [\alpha_k, \tau_k, \beta_k]$ and $y_k = [\beta_k, \tau_k, \alpha_k]$ in H for each $\tau_k = \{\alpha_k, \beta_k\} \in P$ with $\alpha_k < \beta_k$. Let $[\omega] \in \Omega$. We fix a standard basis of U_t, and then identify $\mathrm{End}(U_t)$ with the matrix algebra $\mathrm{M}_\ell(\mathbb{C})$. The symbol E_{ij} denotes an $\ell \times \ell$ matrix unit, 1 in the (i,j)-position, 0 elsewhere. Then the action of B on U_t is described as follows:

$$\xi_t \ : \ \begin{cases} x_k & \mapsto \ \sum_{1\leq i<\ell} \delta_{[\alpha_k],[\omega_i]} \cdot \delta_{[\beta_k],[\omega_{i+1}]} \cdot E_{i,i+1} \ ; \\ y_k & \mapsto \ \sum_{1\leq i<\ell} \delta_{[\alpha_k],[\omega_i]} \cdot \delta_{[\beta_k],[\omega_{i+1}]} \cdot E_{i+1,i} \ ; \\ [\omega] & \mapsto \ \sum_{1\leq i\leq\ell} \delta_{[\omega],[\omega_i]} \cdot E_{ii} \ . \end{cases}$$

Proof of Proposition 1. Because of the condition for a standard module, we see that the action of B on U can be considered as passing through the quotient algebra $B/J(\nu)$ of B by $J(\nu)$ for a suitable integer $\nu \geq 0$, where $J(\nu)$ is the two-sided ideal of B defined by

$$J(\nu) = \oplus_{Card(\tau)>\nu,\ p,q\in\tau} \mathbb{C}[p,\tau,q],$$

and ν should be large enough. Then, we find that $B/J(\nu)$ is finite dimensional and semisimple, since

$$\xi_\nu \ : \ B \longrightarrow \oplus_{t\in T,\ |t|\le\nu}\ \mathrm{M}_{|t|}(\mathbb{C})$$

with $x \mapsto (\xi_t(x))_{t\in T,\ |t|\le\nu}$ induces $B/J(\nu) \simeq \oplus_{t\in T,\ |t|\le\nu}\ \mathrm{M}_{|t|}(\mathbb{C})$. Therefore, U is completely reducible, and each irreducible component must be isomorphic to one of the U_t above. Q.E.D.

5. Diagonal actions. Let $S = \{\ s_1, \ldots, s_\ell\ \}$ and $S' = \{\ s'_1, \ldots, s'_{\ell'}\ \}$. We consider an operation

$$f_k \ : \ S \cup S' \longrightarrow S \cup S' \cup \{0\}$$

with $f_k(S) \subset S \cup \{0\}$ and $f_k(S') \subset S' \cup \{0\}$. (We assume for convenience that $s_i \ne 0$, $s'_i \ne 0$ and $s_i \ne s'_j$.) Then we define a new operation

$$\tilde{f}_k \ : \ S \times S' \longrightarrow (S \times S') \cup \{0\}$$

by

$$\tilde{f}_k(s, s') = \begin{cases} (f_k(s), f_k(s')) & \text{if} \qquad f_k(s) \ne 0 \text{ and } f_k(s') \ne 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $(s, s') \in S \times S'$. We choose and fix a system of such operations, called $\{\ f_k \ \mid\ k\ \}$. Then we say $(r, r') \sim (s, s')$ for $(r, r'),\ (s, s') \in S \times S'$ if $f_k(r, r') = (s, s')$ or $f_k(s, s') = (r, r')$ for some k. This creates, as its transitive closure, an equivalence relation on $S \times S'$, which is also called $\sim$. If we take the tensor product of two modules with bases S and S' respectively, and if we want to consider some diagonal actions, then the above process is helpful to study a certain decomposition into submodules. Actually we will use this in Section 6. (One may find that a similar idea appeared as crystal graphs in the theory of quantum groups.)

6. Tensor products. Let $\tau, \tau' \in P$. We say that τ'' is a subpatch of τ, and we write $\tau'' \dashv \tau$ if τ'' is a subset of τ such that $\tau'' \in P$. Put $t = [\tau],\ t' = [\tau']$. We set $D^*(t, t') = \{\ (\tau_1, \tau'_1) \ \mid\ \tau_1 \dashv \tau,\ \tau'_1 \dashv \tau',\ [\tau_1] = [\tau'_1]\ \}$. Then, we denote by $D(t, t')$ the subset of $D^*(t, t')$ consisting of the maximal elements in $D^*(t, t')$ relative to the natural double inclusions, called $(\tau_1, \tau'_1) \subset (\tau_2, \tau'_2)$, which is defined by $\tau_1 \subset \tau_2$ and $\tau'_1 \subset \tau'_2$. Then, for each $t'' = [\tau''] \in [P]$, we put $c_{t''}(t, t') = Card(\{\ (\tau_1, \tau'_1) \in D(t, t') \ \mid\ t'' = [\tau_1]\ \})$, $E(t, t') = \{\ [\tau_1] \ \mid\ (\tau_1, \tau'_1) \in D(t, t')\ \}$ and $F(t, t') = \{\ (\tau_1, \tau'_1) \ \mid\ \tau_1 \dashv \tau,\ \tau'_1 \dashv \tau',\ Card(\tau_1) = Card(\tau'_1) = 1,\ [\tau_1] \ne [\tau'_1]\ \}$, and we define $c_1(t, t') = Card(F(t, t'))$. Then, we obtain the following decomposition method for tensor products. We note that $\otimes$ always means $\otimes_{\mathbb{C}}$.

We take standard bases of U_t and $U_{t'}$ as S and S' in Section 5, respectively, and we choose $\{\ x_k,\ y_k \ \mid\ k\ \}$ as a system of operations. Then, the corresponding equivalence relation $\sim$ discussed in Section 5 gives all information about the decomposition of $U_t \otimes U_{t'}$ into irreducible components. On the other hand, transitive closures in Section

5 are corresponding to maximal elements in $D^*(t,t')$ here. Hence, we get the following.

Proposition 2. Let $U_t, U_{t'}$ be finite dimensional irreducible standard B-modules with $t=[\tau], t'=[\tau'] \in T$. Then,

$$U_t \otimes U_{t'} \simeq \left(\bigoplus_{t'' \in E(t,t')} U_{t''}^{\oplus c_{t''}(t,t')} \right) \oplus \left(U_1^{\oplus c_1(t,t')} \right).$$

Using the situation of these combinatrics, we can symbolically explain the following computation as an example:

$$ababa \otimes abaab = 2aba + 2ab + 3a + 12,$$

which implies the decomposition of the corresponding tensor product of modules:

$$U_{[ababa]} \otimes U_{[abaab]} \simeq U_{[aba]}^{\oplus 2} \oplus U_{[ab]}^{\oplus 2} \oplus U_{[a]}^{\oplus 3} \oplus U_1^{\oplus 12}.$$

In this demonstration, we consider *ababa* and *abaab* as subsequences of a tiling of $\mathbb{R}$ with two symbols a, b.

There is a visual simple way to understand this rule. Using the above example, we can express the following table:

$$\begin{array}{c|ccccccccc}
 & a & & b & & a & & a & & b \\
\hline
a & a & & \star & & a & & a & & \star \\
 & & \searrow & & & & & & \searrow & \\
b & \star & & b & & \star & & \star & & b \\
 & & & & \searrow & & & & & \\
a & a & & \star & & a & & a & & \star \\
 & & \searrow & & & & & & \searrow & \\
b & \star & & b & & \star & & \star & & b \\
 & & & & \searrow & & & & & \\
a & a & & \star & & a & & a & & \star
\end{array}$$

In this table, one can easily read the information, along the diagonal direction, that *aba* appears twice and *ab* appears twice, but a appears three times, which shows the multiplicities for $U_{[aba]}$, $U_{[ab]}$, $U_{[a]}$ respectively. Also the number of $\star$ gives the multiplicity for U_1.

Proof of Proposition 2. Let S and S' be standard bases of U_t and $U_{t'}$ respectively. We choose $\{\ x_k,\ y_k\ \}$ as a system of operations as in the previous section. Then we obtain the equivalence relation $\sim$ on $S \times S'$. Let Q be an equivalence class, and put

$U_Q = \oplus_{(s,s')\in Q}\ \mathbf{C}(s\otimes s')$. Then U_Q is a submodule of $U_t \otimes U_{t'}$. Furthermore, since the action of B creates the full $\mathbf{C}$-endomorphism ring of U_Q, we see that U_Q is irreducible. We also find that U_Q is isomorphic to $U_{t''}$ for some $t'' \in T$ corresponding to Q. On the other hand, we obtain $U_t \otimes U_{t'} = \oplus_Q\ U_Q$, where Q runs over all equivalence classes. If U_Q is trivial, then Q is corresponding to an element of $F(t,t')$. If U_Q is nontrivial, then Q is corresponding to an element of $E(t,t')$. Hence, we have the desired result. Q.E.D.

7. Trivial tilings. Let $\mathcal{T}$ be a trivial tiling of $\mathbb{R}$, which means the case when $\Omega = \{\ \omega\ \}$, where we simply denote $\omega = [\omega]$, and also we write

$$\omega = \boxed{}$$

and put

$$x = \boxed{p \mid q}\ , \qquad y = \boxed{q \mid p}\ .$$

Then,

$$B = \mathbb{C}\langle\ x, y\ \rangle = \oplus_{0\le i,k\le j}\ \mathbb{C}x^i y^j x^k.$$

Then, we can describe all finite dimensional standard representations. In fact, the maps of $\{\ x, y\ \}$ into $\mathrm{M}_\ell(\mathbb{C})$ with

$$x \quad \mapsto \quad \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad y \quad \mapsto \quad \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

for $\ell \ge 1$ give all finite dimensional irreducible standard representations of B.

8. Gauss decompositions. Let $GL_n(\mathbb{C})$ be the general linear group over $\mathbb{C}$ of degree n. We take three kinds of standard subgroups, namely $U^+(\mathbb{C})$ as the standard maximal upper-triangular unipotent subgroup, $T(\mathbb{C})$ as the standard maximal diagonal subgroup, and $U^-(\mathbb{C})$ as the standard maximal lower-triangular unipotent subgroup. Then we obtain the following decomposition:

$$GL_n(\mathbb{C}) = U^{\pm}(\mathbb{C})\ U^{\mp}(\mathbb{C})\ T(\mathbb{C})\ U^{\pm}(\mathbb{C}),$$

which is called a Gauss decomposition. The name comes from "Gauss eliminations". Such a decomposition exists in a finite dimensional semisimple algebraic group as well as in a Kac-Moody group (cf. [9]).

9. Iwasawa decompositions. Let $U_n(\mathbb{C})$ be the standard unitary subgroup of $GL_n(\mathbb{C})$, namely

$$U_n(\mathbb{C}) = \{\ X \in GL_n(\mathbb{C})\ \mid\ {}^t\bar{X}X = I_n\ \},$$

where I_n is the identity matrix of degree n. Then, we obtain the following decomposition:

$$GL_n(\mathbb{C}) = U_n(\mathbb{C})\ T(\mathbb{C})\ U^{\pm}(\mathbb{C}),$$

which is called an Iwasawa decomposition and is extremely well-known. There is a strong version. We take the following positive diagonal subgroup:

$$T(\mathbb{R}_{>0}) = \{\ diag(a_1, \ldots, a_n) \in GL_n(\mathbb{R})\ \mid\ a_i > 0\ (1 \le i \le n)\ \}.$$

Then we also have

$$GL_n(\mathbb{C}) = U_n(\mathbb{C})\ T(\mathbb{R}_{>0})\ U^{\pm}(\mathbb{C}).$$

At the right hand side, the expression is unique, since $U_n(\mathbb{C}) \cap (T(\mathbb{R}_{>0})\ U^{\pm}(\mathbb{C})) = \{\ I_n\ \}$.

10. Completed groups. Let B be the tiling bialgebra of a tiling $\mathcal{T}$. As a formal sum, we consider the following infinite sum:

$$\sum_{g \in G(B)}^{\infty} a_g\ g$$

with $a_g \in \mathbb{C}$. Then naturally we can define the sum of such formal sums by

$$\left(\sum^{\infty} a_g\ g \right) + \left(\sum^{\infty} b_g\ g \right) = \sum^{\infty}\ (a_g + b_g)\ g.$$

Also we can define the multiplication of such formal sums by

$$\left(\sum^{\infty} a_g\ g \right) \cdot \left(\sum^{\infty} b_g\ g \right) = \sum^{\infty} c_g\ g$$

with

$$c_g = \sum_{g', g'' \in G(B),\ g'g'' = g} a_{g'} b_{g''},$$

where the number of pairs (g', g'') satisfying $g'g'' = g$ is actually finite. Hence, the number c_g can be determined. We denote by $\hat{B}$ the set of all such formal infinite sums $\sum_{g \in G(B)}^{\infty} a_g\ g$ with $a_g \in \mathbb{C}$. Then $\hat{B}$ becomes an associative algebra containing B. This $\hat{B}$ is called the formal completion of B. In the same way, we can construct, named $\hat{B}_\varepsilon$, the formal completions of B_ε with $\varepsilon = 0, \pm$. Clearly we have $\hat{B} = \hat{B}_- \oplus \hat{B}_0 \oplus \hat{B}_+$. Also we need $\hat{B}^\times$, the set of units in $\hat{B}$, to obtain the associated group. We note that $\hat{B} \simeq \prod_{t \in T} M_{|t|}(\mathbb{C})$ and $\hat{B}^\times \simeq \prod_{t \in T} GL_{|t|}(\mathbb{C})$. Using $\hat{B}^\times$, we define the following exponential map:

$$\mathrm{Exp}\ :\ \hat{B} \longrightarrow \hat{B}^\times$$

by $\mathrm{Exp}(x) = \sum_{i=0}^{\infty} x^i/i!$. Put $\hat{\Gamma} = \langle\ \mathrm{Exp}(x) \ \mid\ x \in \hat{B}\ \rangle = \mathrm{Exp}(\hat{B}) = \hat{B}^{\times}$, and $\hat{\Gamma}_{\varepsilon} = \hat{\Gamma} \cap \hat{B}_{\varepsilon}$ for $\varepsilon = 0,\ \pm$. Then, using Gauss decompositions in Section 8, we obtain the following Gauss-type decomposition:

$$\hat{\Gamma} = \hat{\Gamma}_{\pm}\hat{\Gamma}_{\mp}\hat{\Gamma}_{0}\hat{\Gamma}_{\pm}.$$

Next, we define the anti-automorphism of $\hat{B}$, named $*$, induced by $(\alpha[p,\tau,q])^* = \bar{\alpha}[q,\tau,p]$ and $(\alpha 1)^* = \bar{\alpha}1$ with $\alpha \in \mathbb{C}$. Then, the unitary form in the sense of Lie theory, called $\hat{B}_u$, is the $\mathbb{R}$-span of

$$\sqrt{-1}([p,\tau,q] + [q,\tau,p]), \quad [p,\tau,q] - [q,\tau,p], \quad \sqrt{-1}[p,\tau,p], \quad \sqrt{-1}\cdot 1_B$$

with $p \neq q$. Using this form, we can construct $\hat{\Gamma}_u$ by $\hat{\Gamma}_u = \mathrm{Exp}(\hat{B}_u)$, which may be called a compact-like subgroup of $\hat{\Gamma}$. Then, using Iwasawa decompositions in Section 9, we obtain the following Iwasawa-type decomposition:

$$\hat{\Gamma} = \hat{\Gamma}_u\hat{\Gamma}_0\hat{\Gamma}_{\pm}.$$

If we set $\hat{\Gamma}_a = \mathrm{Exp}(\hat{B}_0^{\mathbb{R}})$, where $\hat{B}_0^{\mathbb{R}} = \{\ \sum_{g\in G(B_0)}^{\infty} a_g g \ \mid\ a_g \in \mathbb{R}\ \} \subset \hat{B}_0 \subset \hat{B}$, then we can rewrite the decomposition $\hat{\Gamma} = \hat{\Gamma}_u\hat{\Gamma}_0\hat{\Gamma}_{\pm}$ by

$$\hat{\Gamma} = \hat{\Gamma}_u\hat{\Gamma}_a\hat{\Gamma}_{\pm}$$

with unique expression in the sense that for every element $\hat{g} \in \hat{\Gamma}$ there uniquely exist $\hat{g}_u \in \hat{\Gamma}_u$, $\hat{g}_a \in \hat{\Gamma}_a$, $\hat{g}_{\pm} \in \hat{\Gamma}_{\pm}$ satisfying $\hat{g} = \hat{g}_u\hat{g}_a\hat{g}_{\pm}$ (cf. Section 9).

Proposition 3. Notation is as above. Then, we have the following Iwasawa-type decompositions:

$$\hat{\Gamma} = \hat{\Gamma}_u\hat{\Gamma}_0\hat{\Gamma}_{\pm}, \qquad \hat{\Gamma} = \hat{\Gamma}_u\hat{\Gamma}_a\hat{\Gamma}_{\pm}.$$

11. Local properties. In this section, we will characterize some local property of one-dimensional tilings by the associated bialgebras. Let $\mathcal{T}$ and $\mathcal{T}'$ be one-dimensional tilings with the set of tiles $\Omega = \Omega(\mathcal{T})$ and $\Omega' = \Omega(\mathcal{T}')$ respectively. We call $\mathcal{T}$ and $\mathcal{T}'$ are locally nondistinguishable if the following conditions are satisfied:

(LND1) there is a bijection ψ of Ω onto Ω', which gives a transformation between the patterns in terms of Ω and the patterns in terms of Ω',
(LND2) if $[\tau] \in [P(\mathcal{T})]$, then the pattern induced from $[\tau]$ by ψ appears in $[P(\mathcal{T}')]$,
(LND3) if $[\tau'] \in [P(\mathcal{T}')]$, then the pattern induced from $[\tau']$ by ψ^{-1} appears in $[P(\mathcal{T})]$.

If the condition (LND1) is satisfied, then without loss of generality we can assume $\Omega = \Omega'$. Sometimes we may also assume $\psi(\omega) = \omega$ for $\omega \in \Omega$ (modulo permutations).

Before showing our main theorem, we will discuss a lattice of certain two sided ideals. For each $h \in G(B)$, we put $I_h = BhB$, the principal two sided ideal of B generated by h. Let $\mathcal{L}$ be the family of principal two sided ideals $I_h = BhB$ for all $h \in G(B)$. There is the top element $I_{\mathbf{e}} = B\mathbf{e}B = B$ in $\mathcal{L}$, which equals the whole algebra. (B is called a level-zero element.) We note that $I_{[p',\tau',q']} = B[p',\tau',q']B$ is spanned by all $[p'',\tau'',q'']$ satisfying $[\tau'] \dashv [\tau'']$. What is a maximal element (= a level-one element) in $\mathcal{L}$? It is of the form $\mathfrak{m} = I_\omega = B\omega B$ for some $\omega \in \Omega$, which is spanned by all $[p'',\tau'',q'']$ satisfying $[\omega] \dashv [\tau'']$. We choose and fix $\tau = \{\alpha,\beta\} \in P$ with $\omega \in \tau$ and $Card(\tau) = 2$. Let $\mathfrak{a} = I_{[p,\tau,q]} = B[p,\tau,q]B$ with $p, q \in \tau$, which is the $\mathbf{C}$-subspace of $\mathfrak{m}$ spanned by all $[p'',\tau'',q'']$ satisfying $[\tau] \dashv [\tau'']$. Then $\mathfrak{a}$ is a level-two element in $\mathcal{L}$ with $\mathfrak{a} \subset \mathfrak{m} \subset B$. Let $\mathfrak{b}$ be the $\mathbf{C}$-subspace of $\mathfrak{a}$ spanned by all $[p'',\tau'',q'']$ satisfying $[\tau] \dashv [\tau'']$ and $[\tau] \neq [\tau'']$. Then $\mathfrak{b}$ is a proper ideal in $\mathfrak{a}$ satisfying $\mathfrak{a}/\mathfrak{b} \simeq M_2(\mathbf{C})$. This $\mathfrak{b}$ is uniquely maximal in $\mathfrak{a}$ in the sense of being generated by group-like elements. More precisely, we have $\mathfrak{a} = (\mathbf{C}[\alpha,\tau,\alpha] \oplus \mathbf{C}[\alpha,\tau,\beta] \oplus \mathbf{C}[\beta,\tau,\alpha] \oplus \mathbf{C}[\beta,\tau,\beta]) \oplus \mathfrak{b}$.

Proposition 4. Let $\mathcal{T}$ and $\mathcal{T}'$ be two tilings of $\mathbb{R}$ with the associated tiling bialgebras B and B' respectively. Then, $\mathcal{T}$ and $\mathcal{T}'$ are locally nondistinguishable if and only if there is a bialgebra isomorphism $\phi : B \longrightarrow B'$ satisfying $\phi(B_\varepsilon) = B'_\varepsilon$ with $\varepsilon = 0, \pm$.

Proof of Proposition 4. We only need to show "if part". Using ϕ, we obtain a bijection, again denoted ϕ, between $G(B)$ and $G(B')$ as well as an isomorphism, also denoted ϕ, between two lattices $\mathcal{L}(B)$ and $\mathcal{L}(B')$. Hence, modulo our identification of tiles, we can assume that $\Omega = \Omega(\mathcal{T}) = \Omega(\mathcal{T}')$. Then we write $\mathcal{L}(B)$ and $\mathcal{L}(B')$ as

$$\begin{array}{ccccccc}
 & & \mathfrak{m}_1 & \leftarrow & \mathfrak{a}_1 & \leftarrow & \cdots \\
 & \swarrow & \vdots & \swarrow & \vdots & & \\
B & \leftarrow & \mathfrak{m}_i & \leftarrow & \mathfrak{a}_j & \leftarrow & \vdots \\
 & \nwarrow & \vdots & \nwarrow & \vdots & & \\
 & & \vdots & \leftarrow & \vdots & \leftarrow & \cdots
\end{array}
\qquad
\begin{array}{ccccccc}
 & & \mathfrak{m}'_1 & \leftarrow & \mathfrak{a}'_1 & \leftarrow & \cdots \\
 & \swarrow & \vdots & \swarrow & \vdots & & \\
B' & \leftarrow & \mathfrak{m}'_i & \leftarrow & \mathfrak{a}'_j & \leftarrow & \vdots \\
 & \nwarrow & \vdots & \nwarrow & \vdots & & \\
 & & \vdots & \leftarrow & \vdots & \leftarrow & \cdots
\end{array}$$

with $\phi(\mathfrak{m}_i) = \mathfrak{m}'_i$ and $\phi(\mathfrak{a}_j) = \mathfrak{a}'_j$, where $\mathfrak{m}_i$ (resp. $\mathfrak{m}'_i$) is a maximal element of $\mathcal{L}(B)$ (resp. $\mathcal{L}(B')$), and $\mathfrak{a}_j$ (resp. $\mathfrak{a}'_j$) is a level-two element of $\mathcal{L}(B)$ (resp. $\mathcal{L}(B')$), and so on. Here the arrows " $\leftarrow$ ", " $\nwarrow$ ", " $\swarrow$ " in the above diagram mean the canonical inclusion maps. For each $\mathfrak{a}_j$, we select x_j and y_j as follows. The elements $x_j, y_j \in G(B)$ are uniquely determined by the property that x_j is an element of B_+ generating $\mathfrak{a}_j$ and y_j is an element of B_- generating $\mathfrak{a}_j$. More precisely there is a uniquely definable maximal proper ideal $\mathfrak{b}_j$ in $\mathfrak{a}_j$ satisfying $\mathfrak{a}_j/\mathfrak{b}_j \simeq M_2(\mathbb{C})$, and

$$\mathfrak{a}_j = (\mathbb{C}z_{11} \oplus \mathbb{C}z_{12} \oplus \mathbb{C}z_{21} \oplus \mathbb{C}z_{22}) \oplus \mathfrak{b}_j$$

with $z_{11}, z_{22} \in G(B) \cap B_0$, $z_{12} \in G(B) \cap B_+$ and $z_{21} \in G(B) \cap B_-$. Then each of the $z_{11}, z_{12}, z_{21}, z_{22}$ generates $\mathfrak{a}_j$, and we put $x_j = z_{12}$ and $y_j = z_{21}$. Similarly we can choose $x'_j, y'_j \in B'$. Then we see $\phi(x_j) = x'_j$ and $\phi(y_j) = y'_j$ for all j. On the other hand, we know that the x_j and y_j together with Ω generate B as a $\mathbb{C}$-algebra, and that the x'_j and y'_j together with Ω generate B' as a $\mathbb{C}$-algebra, respectively. Put $W = \{ \ x_j, \ y_j \ \mid \ j \ \}$, and $W' = \{ \ x'_j, \ y'_j \ \mid \ j \ \}$. If we write

$$x_j = [p_j, \tau_j, q_j], \ y_j = [q_j, \tau_j, p_j], \ x'_j = [p'_j, \tau'_j, q'_j], \ y'_j = [q'_j, \tau'_j, p'_j],$$

then we obtain, using Ω-actions from the left hand side and from the right hand side, $[\tau_j] = [\tau'_j]$ for all j (modulo our identification). On the other hand, there is a natural bijection between the sets

$$\hat{P} = \{ \ w_1 \cdots w_\ell \ (\neq 0) \ \mid \ \ell \geq 1, \ w_k \in W \ (1 \leq k \leq \ell) \ \}$$

and

$$\hat{P}' = \{ \ w'_1 \cdots w'_\ell \ (\neq 0) \ \mid \ \ell \geq 1, \ w'_k \in W' \ (1 \leq k \leq \ell) \ \}.$$

This implies that both local patches for $\mathcal{T}$ and $\mathcal{T}'$ should completely coincide. Therefore, $\mathcal{T}$ and $\mathcal{T}'$ are locally nondistinguishable. Q.E.D.

12. Remarks. There are many ways to construct interesting tilings and study them, e.g. by the usual cut-and-projection scheme, by a certain symbol dynamical system, and so on (cf. [3], [4], [10]). One may be interested in several algebraic approaches to quasicrystals (cf. [2], [7]), which induce tilings, tiling algebras and representations. Algebraically it is very natural and important to produce a suitable representation theory and classify all irreducible representations (cf. [5]). If two tilings $\mathcal{T}$ and $\mathcal{T}'$ of $V = \mathbb{R}$ are opposite, that is, $\mathcal{T}' = -\mathcal{T}$, then there is a bialgebra isomorphism ψ : $B \longrightarrow B'$ satisfying $\psi(B_\varepsilon) = B'_{-\varepsilon}$ with $\varepsilon = 0, \pm$, however it is not necessary for $\mathcal{T}$ and $\mathcal{T}'$ to be locally nondistinguishable in our sense here. The main part (Proposition 4) of this note, at this moment, could not be generalized to the case of higher dimensional tilings. Some difficulty exists to control the geometrical configurations of given tilings only using the associated tiling bialgebras. Other parts (Propositions 1,2,3) can be established, in the same way as here, for any higher dimensional tiling just by a formal generalization.

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