TSUKUBA J. MATH.
Vol. 28 No. 2 (2004), 437-453

RESTRICTED ENERGY INEQUALITIES AND NUMERICAL APPROXIMATIONS

By

Reiko SAKAMOTO

Introduction

Let $\{A, B_j\}$ be linear partial differential operators. Let $\Omega(\subset \boldsymbol{R}^n)$ be a bounded domain with smooth boundary Γ. Our boundary value problem is to find $u \in L^2(\Omega)$ satisfying

$$(P) \qquad \begin{cases} Au = f & \text{in } \Omega, \\ B_j u = f_j & \text{on } \Gamma \ (j \in J) \end{cases}$$

for given data $\{f, f_j\}$. We are particularly interested in a method of numerical approximation of solutions of (P).

The problem (P) is closely connected with its adjoint problem (P^*). The adjoint problem is to find $v \in L^2(\Omega)$ satisfying

$$(P^*) \qquad \begin{cases} A^* v = g & \text{in } \Omega \\ \mathscr{B}_j^* v = g_j & \text{on } \Gamma \ (j \in J^*) \end{cases}$$

for given data $\{g, g_j\}$.

Recently, it has become clear that a solution $u \in L^2(\Omega)$ of (P) can be constructed numerically, assuming an energy inequality

$$(E^*) \qquad \|v\| \leqq C\left(\|A^* v\| + \sum_{j \in J^*} \langle \mathscr{B}_j^* v|_\Gamma \rangle_{\mu_j} \right) \quad (v \in H^q(\Omega))$$

([1]).

Here we have two questions:

(1) In case when L^2-solutions of (P) are not unique, how can we characterize the solution, obtained in [1]?

(2) In case when L^2-solutions of (P^*) are not unique, (E^*) can not be satisfied. Is there any numerical method to approach to one of solutions of (P)?

Received May 6, 2003.
Revised October 20, 2003.

In this paper, in stead of (E^*), we assume a restricted energy inequality:

$$(\mathscr{E}_0^*) \qquad \|v\| \leqq C\|A^*v\| \quad (v \in M^*, \mathscr{B}_j^* v|_\Gamma = 0 \ (j \in J^*)),$$

where M^* is a subspace in $H^q(\Omega)$ defined in § 1. Then we will see that the method in [1] is applicable. Moreover, we will see that the solution obtained by our method is unique in a subspace τ in $L^2(\Omega)$.

§ 1. Restricted Energy Inequalities

Let

$$A = \sum_{|\nu| \leqq m} a_\nu(x) \partial_x^\nu$$

be a differential operator with smooth coefficients defined in a neighborhood of $\overline{\Omega}$. Let

$$B_j = \sum_{|\nu| \leqq j} b_{j\nu}(x) \partial_x^\nu \quad (j \in J, J \subset \{0, 1, \dots, m-1\})$$

be differential operators with smooth coefficients defined in a neighborhood of Γ. We assume that Γ is non-characteristic for $\{A, B_j \ (j \in J)\}$. Namely,

$$\sum_{|\nu| = m} a_\nu(x) \boldsymbol{n}(x)^\nu \neq 0 \quad \text{on } \Gamma,$$

$$\sum_{|\nu| = j} b_{j\nu}(x) \boldsymbol{n}(x)^\nu \neq 0 \quad \text{on } \Gamma,$$

where $\boldsymbol{n}(x)$ is a unit inner normal at $x (\in \Gamma)$.

Set

$$J^c \cup J = \{0, 1, \dots, m-1\}, \quad J^c \cap J = \phi, \quad J^* = \{j \,|\, m-1-j \in J^c\},$$

$$B_j = (d/d\boldsymbol{n})^j \quad (j \in J^c).$$

Then we can define

$$\mathscr{B}_j = \sum_{|\nu| \leqq j} \beta_{j\nu}(x) \partial_x^\nu \quad (j \in \{0, 1, \dots, m-1\}),$$

for which Γ is non-characteristic, such that the following Green's Theorem holds.

LEMMA 1.1 (Green's Theorem). *Suppose that $u, Au \in L^2(\Omega)$, then it holds that*

$$\langle (d/d\boldsymbol{n})^k u|_\Gamma \rangle_{-k-m+1/2} \leqq C(\|u\| + \|Au\|) \quad (k = 0, 1, \ldots, m-1)$$

and

$$(Au, v) - (u, A^*v) = -\sum_{j \in J} \langle B_j u|_\Gamma, \mathscr{B}^*_{m-1-j} v|_\Gamma \rangle$$

$$- \sum_{j \in J^*} \langle B_{m-1-j} u|_\Gamma, \mathscr{B}^*_j v|_\Gamma \rangle \quad (v \in H^{2m-1}(\Omega)),$$

where

$$A^* = \sum_{|\nu| \leqq m} (-\partial_x)^\nu \overline{a_\nu(x)}, \quad \mathscr{B}^*_j = \sum_{|\nu| \leqq j} (-\partial_x)^\nu \overline{\beta_{j\nu}(x)}.$$

NOTATIONS.
(1) $(u, v) = (u, v)_{L^2(\Omega)}$, $\|u\| = \|u\|_{L^2(\Omega)}$,
(2) $\Lambda = (1 - \triangle)^{1/2}$, where $\triangle$ is the Laplace-Beltrami operator on Γ,
(3) $\langle u, v \rangle_\sigma = (u, v)_{H^\sigma(\Gamma)} = (\Lambda^\sigma u, \Lambda^\sigma v)_{L^2(\Gamma)}$, $\langle u \rangle_\sigma = \|u\|_{H^\sigma(\Gamma)}$ for $u, v \in H^\sigma(\Gamma)$ (σ: real),
(4) $u \in H^{-\sigma}(\Gamma)$: $H^\sigma(\Gamma) \ni v \mapsto \langle u, v \rangle \in \mathbf{C}$ $(\sigma > 0)$.

Lemma 1.1 is well known for $u \in H^m(\Omega)$. See Appendix of [3] in case when $u, Au \in L^2(\Omega)$.

REMARK. Set

$$Pu = \{Au, B_j u|_\Gamma \ (j \in J)\}, \quad Qu = \{u, -B_{m-1-j} u|_\Gamma \ (j \in J^*)\},$$

$$P^*v = \{A^*v, \mathscr{B}^*_j v|_\Gamma \ (j \in J^*)\}, \quad Q^*v = \{v, \mathscr{B}^*_{m-1-j} v|_\Gamma \ (j \in J)\},$$

then the problem (P) denotes $Pu = \{f, f_j \ (j \in J)\}$ and the problem (P^*) denotes $P^*v = \{g, g_j \ (j \in J^*)\}$, and Green's Theorem is stated as follows.

GREEN'S THEOREM. *Suppose that $u, Au \in L^2(\Omega)$, then it holds that*

$$\langle (d/d\boldsymbol{n})^k u|_\Gamma \rangle_{-k-m+1/2} \leqq C(\|u\| + \|Au\|) \quad (k = 0, 1, \ldots, m-1)$$

and

$$[Pu, Q^*v] = [Qu, P^*v]_* \quad (v \in H^{2m-1}(\Omega)),$$

where

$$[F, G] = (f, g) + \sum_{j \in J} \langle f_j, g_j \rangle \quad \textit{for } F = \{f, f_j \ (j \in J)\} \textit{ and } G = \{g, g_j \ (j \in J)\},$$

$$[F, G]_* = (f, g) + \sum_{j \in J^*} \langle f_j, g_j \rangle \quad \textit{for } F = \{f, f_j \ (j \in J^*)\} \textit{ and } G = \{g, g_j \ (j \in J^*)\}.$$

NULL SPACES. Set

$$K = \{\phi \in L^2(\Omega) \mid P\phi = 0\}, \quad K^* = \{\phi \in L^2(\Omega) \mid P^*\phi = 0\}.$$

Owing to Green's Theorem, we have

$$K = \{\phi \in L^2(\Omega) \mid [Q\phi, P^*v]_* = 0 \ (\forall v \in H^{2m-1}(\Omega))\},$$
$$K^* = \{\phi \in L^2(\Omega) \mid [Pu, Q^*\phi] = 0 \ (\forall u \in H^{2m-1}(\Omega))\}.$$

Therefore, K and K^* are closed subspaces in $L^2(\Omega)$. Set

$$K^{\perp} = \{f \in L^2(\Omega) \mid (f, \phi) = 0 \ (\forall \phi \in K)\},$$
$$K^{*\perp} = \{f \in L^2(\Omega) \mid (f, \phi) = 0 \ (\forall \phi \in K^*)\}.$$

We assume

(A-I) there exists an integer $p(\geqq 2m-1)$ such that

$$K, K^* \subset H^p(\Omega)$$

throughout this paper.

We define

$$M^* = K^{*\perp} \cap H^q(\Omega) = \{f \in H^q(\Omega) \mid (f, \phi) = 0 \ (\forall \phi \in K^*)\}$$

for an integer q $(m \leqq q \leqq p)$. Then M^* is a closed subspace in $H^q(\Omega)$. Let $u \in H^q(\Omega)$, then there exist $\phi \in K^*$ and $\xi \in M^*$ such that

$$u = \phi + \xi \quad \text{and} \quad \|u\|^2 = \|\phi\|^2 + \|\xi\|^2.$$

We say that *restricted energy inequality* $(\mathscr{E}^*)$ holds, if it holds

$$(\mathscr{E}^*) \qquad \|v\| \leqq C\left(\|A^*v\| + \sum_{j \in J^*} \langle \mathscr{B}_j^* v|_\Gamma \rangle_{\mu_j}\right) \quad (v \in M^*),$$

where $\mu_j = q - 1/2 - j$.

We say that *restricted energy inequality* ($\mathscr{E}_0^*$) holds, if it holds

$$(\mathscr{E}_0^*) \qquad \|v\| \leqq C\|A^*v\| \quad (v \in M^*, \mathscr{B}_j^* v|_\Gamma = 0 \ (j \in J^*)).$$

Since $\{B_j \ (j = 0, 1, \ldots, m-1)\}$ and $\{\mathscr{B}_j^* \ (j = 0, 1, \ldots, m-1)\}$ are Dirichlet sets, we have ([4])

LEMMA 1.2. *Let $s \geqq m$.*

i) *Let $f_j \in H^{s-1/2-j}(\Gamma)$ $(j = 0, 1, \ldots, m-1)$, then there exists $U \in H^s(\Omega)$ such that*

$$B_j U|_\Gamma = f_j \ (j = 0, 1, \ldots, m-1), \quad \|U\|_s \leqq C \sum_{j \in \{0, 1, \ldots, m-1\}} \langle f_j \rangle_{s-1/2-j}.$$

ii) *Let $g_j \in H^{s-1/2-j}(\Gamma)$ $(j = 0, 1, \ldots, m-1)$, then there exists $V \in H^s(\Omega)$ such that*

$$\mathscr{B}_j^* V|_\Gamma = g_j \ (j = 0, 1, \ldots, m-1), \quad \|V\|_s \leqq C \sum_{j \in \{0, 1, \ldots, m-1\}} \langle g_j \rangle_{s-1/2-j}.$$

LEMMA 1.3. *($\mathscr{E}^*$) holds iff ($\mathscr{E}_0^*$) holds.*

PROOF. Suppose that ($\mathscr{E}_0^*$) holds. Let $v \in M^* (\subset H^q(\Omega))$.

(1) Set

$$g = A^*v \in H^{q-m}(\Omega), \quad g_j = \mathscr{B}_j^* v|_\Gamma \in H^{q-1/2-j}(\Gamma) \ (j \in J^*).$$

Then there exists $V \in H^q(\Omega)$ such that

$$\mathscr{B}_j^* V|_\Gamma = g_j \ (j \in J^*), \quad \|V\|_q \leqq C \sum_{j \in J^*} \langle g_j \rangle_{q-1/2-j}$$

from (ii) of Lemma 1.2 $(s = q)$.

(2) Set $w = v - V$, then $w \in H^q(\Omega)$ satisfies

$$\begin{cases} A^*w = g - A^*V, \\ \mathscr{B}_j^* w|_\Gamma = 0 \quad (j = J^*). \end{cases}$$

Since $w \in H^q(\Omega)$, there exist $\phi \in K^*$ and $\xi \in M^*$ such that

$$w = \phi + \xi, \quad \|w\|^2 = \|\phi\|^2 + \|\xi\|^2.$$

Therefore $\xi \in M^*$ satisfies

$$\begin{cases} A^*\xi = g - A^*V, \\ \mathscr{B}_j^*\xi|_\Gamma = 0 \quad (j \in J^*). \end{cases}$$

Since $(\mathscr{E}_0^*)$ holds, we have

$$\|\xi\| \leqq C\|A^*\xi\| = C\|g - A^*V\| \leqq C'\left(\|g\| + \sum_{j \in J^*} \langle g_j \rangle_{q-1/2-j}\right).$$

(3) In the same way, since $V \in H^q(\Omega)$, there exist $\psi \in K^*$ and $\eta \in M^*$ such that

$$V = \psi + \eta, \quad \|V\|^2 = \|\psi\|^2 + \|\eta\|^2.$$

Hence we have

$$v = w + V = (\phi + \psi) + (\xi + \eta), \quad \phi + \psi \in K^*, \quad \xi + \eta \in M^*.$$

Since $v \in M^*$, we have

$$v = \xi + \eta.$$

Hence we have

$$\|v\| \leqq \|\xi\| + \|\eta\| \leqq \|\xi\| + \|V\| \leqq C\left(\|g\| + \sum_{j \in J^*} \langle g_j \rangle_{q-1/2-j}\right). \qquad \square$$

We assume

(A-II) $(\mathscr{E}_0^*)$

throughout this paper. Then we can define a Hilbert space $\mathscr{H}$ as the closure of M^* by the norm $[\ ]$:

$$[v]^2 = \|A^*v\|^2 + \sum_{j \in J^*} \langle \mathscr{B}_j^* v|_\Gamma \rangle_{\mu_j}^2.$$

Inner product of $\mathscr{H}$ is defined by

$$[w, v] = (A^*w, A^*v) + \sum_{j \in J^*} \langle \mathscr{B}_j^* w|_\Gamma, \mathscr{B}_j^* v|_\Gamma \rangle_{\mu_j}.$$

For a fixed $f \in L^2(\Omega)$, define

$$f : \mathscr{H} \ni v \mapsto (v, f) \in \mathbf{C}$$

then f is a continuous linear functional on $\mathscr{H}$. In fact, it holds

$$|(v, f)| \leqq \|v\| \|f\| \leqq C[v]\|f\| \quad (v \in \mathscr{H})$$

from Lemma 1.3. Therefore, owing to Riesz' Theorem in $\mathscr{H}$, there exists $w \in \mathscr{H}$ such that

$$(f, v) = [w, v] \quad (v \in \mathscr{H}),$$

where we say that $w \in \mathscr{H}$ is a *Riesz function* of f $(\in L^2(\Omega))$.

§ 2. Existence and Uniqueness

THEOREM 2.1. *Assume* (A-I) *and* (A-II). *Suppose that* $f \in K^{*\perp}$. *Let* $w \in \mathscr{H}$ *be a Riesz function of* f. *Set* $u = A^*w \in L^2(\Omega)$, *then* u *satisfies*

$$(P_0) \qquad \begin{cases} Au = f & \text{in } \Omega, \\ B_j u|_\Gamma = 0 & (j \in J), \end{cases}$$

and

$$B_{m-1-j}u|_\Gamma = -\Lambda^{2\mu_j}\mathscr{B}_j^* w|_\Gamma \quad (j \in J^*).$$

PROOF. (1) Since $w \in \mathscr{H}$ satisfy

$$(f, v) = (A^*w, A^*v) + \sum_{j \in J^*} \langle \mathscr{B}_j^* w|_\Gamma, \mathscr{B}_j^* v|_\Gamma \rangle_{\mu_j} \quad (v \in \mathscr{H}),$$

$u = A^*w$ satisfies

$$(f, v) - (u, A^*v) = \sum_{j \in J^*} \langle \mathscr{B}_j^* w|_\Gamma, \mathscr{B}_j^* v|_\Gamma \rangle_{\mu_j} \quad (v \in \mathscr{H}) \ldots\ldots ①.$$

(2) Moreover, we have

$$(f, v) - (u, A^*v) = \sum_{j \in J^*} \langle \mathscr{B}_j^* w|_\Gamma, \mathscr{B}_j^* v|_\Gamma \rangle_{\mu_j} \quad (v \in H^q(\Omega)) \ldots\ldots ①'.$$

In fact, let $v \in H^q(\Omega)$. Then there exist $\phi \in K^*$ and $\xi \in M^*$ such that $v = \phi + \xi$. Since $\xi \in M^* \subset \mathscr{H}$, we have from ①

$$(f, \xi) - (u, A^*\xi) = \sum_{j \in J^*} \langle \mathscr{B}_j^* w|_\Gamma, \mathscr{B}_j^* \xi|_\Gamma \rangle_{\mu_j}.$$

We remark that ϕ satisfies

$$A^*\phi = 0, \quad \mathscr{B}_j^* \phi|_\Gamma = 0 \quad (j \in J^*),$$

and $(f, \phi) = 0$. Hence we have

$$(f, v) - (u, A^*v) = \sum_{j \in J^*} \langle \mathscr{B}_j^* w|_\Gamma, \mathscr{B}_j^* v|_\Gamma \rangle_{\mu_j}.$$

(3) From ①$'$, we have

$$(f, v) - (u, A^*v) = 0 \quad (v \in \mathscr{D}'(\Omega)).$$

which means

$$Au = f \quad \text{in } \mathscr{D}'(\Omega).$$

Therefore we have

$$(Au, v) - (u, A^*v) = \sum_{j \in J^*} \langle \mathscr{B}_j^* w|_\Gamma, \mathscr{B}_j^* v|_\Gamma \rangle_{\mu_j} \quad (v \in H^q(\Omega)) \ldots\ldots ①''.$$

(4) Owing to Green's Theorem, we have

$$(Au, v) - (u, A^*v) = -\sum_{j \in J} \langle B_j u|_\Gamma, \mathscr{B}_{m-1-j}^* v|_\Gamma \rangle$$

$$-\sum_{j \in J^*} \langle B_{m-1-j} u|_\Gamma, \mathscr{B}_j^*|_\Gamma v \rangle \quad (v \in H^{2m-1}(\Omega)) \ldots\ldots ②.$$

Hence, from ①$''$ and ②, we have

$$\sum_{j \in J^*} \langle \mathscr{B}_j^* w|_\Gamma, \mathscr{B}_j^* v|_\Gamma \rangle_{\mu_j} = -\sum_{j \in J} \langle B_j u|_\Gamma, \mathscr{B}_{m-1-j}^* v|_\Gamma \rangle$$

$$-\sum_{j \in J^*} \langle B_{m-1-j} u|_\Gamma, \mathscr{B}_j^* v|_\Gamma \rangle \quad (v \in H^{q'}(\Omega),\ q' = \max(q, 2m-1)),$$

which means

$$B_j u|_\Gamma = 0 \quad (j \in J), \quad B_{m-1-j} u|_\gamma = -\Lambda^{2\mu_j} \mathscr{B}_j^* w|_\Gamma \quad (j \in J^*). \qquad \square$$

COROLLARY 2.1. *Assume* (A-I) *and* (A-II). *Suppose that* $\{f \in L^2(\Omega),$ $f_j \in H^{m-1/2-j}(\Gamma)\ (j \in J)\}$ *satisfy*

$$(\mathscr{R}) \qquad (f, \phi) + \sum_{j \in J} \langle f_j, \mathscr{B}_{m-1-j}^* \phi|_\Gamma \rangle = 0 \quad (\phi \in K^*),$$

that is,

$$(\mathscr{R}) \qquad [\{f, f_j\}, Q^*\phi] = 0 \quad (\phi \in K^*).$$

Let $U \in H^m(\Omega)$ *satisfy* $\{B_j U|_\Gamma = f_j \ (j \in J)\}$. *Set* $u = A^* w + U$, *where* w *is a Riesz function of* $f - AU$. *Then* $u \in L^2(\Omega)$ *satisfies* (P).

PROOF. Since $K^* \subset H^{2m-1}(\Omega)$, we have

$$[PU, Q^*\phi] = [QU, P^*\phi]_* = 0 \quad (\phi \in K^*),$$

owing to Green's Theorem. Namely, we have

$$[\{AU, B_j U|_\Gamma \ (j \in J)\}, \{\phi, \mathscr{B}^*_{m-1-j}\phi|_\Gamma \ (j \in J)\}] = 0 \quad (\phi \in K^*),$$

which means

$$(AU, \phi) + \sum_{j \in J} \langle f_j, \mathscr{B}^*_{m-1-j}\phi|_\Gamma \rangle = 0 \quad (\phi \in K^*).$$

Set $F = f - AU$, then we have

$$(F, \phi) = (f - AU, \phi) = (f, \phi) + \sum_{j \in J} \langle f_j, \mathscr{B}^*_{m-1-j}\phi|_\Gamma \rangle \quad (\phi \in K^*).$$

Therefore, we have $F \in K^{*\perp}$, iff $\{f, f_j\}$ satisfies $(\mathscr{R})$.

Now we apply Theorem 2.1, then there exists $v = A^* w \in L^2(\Omega)$ satisfying

$$Av = F, \quad B_j v|_\Gamma = 0 \quad (j \in J),$$

where w is a Riesz function of F. Hence $u = v + U \in L^2(\Omega)$ is a solution of (P). □

Now we define a *subspace* τ in $L^2(\Omega)$:

$$\tau = \left\{ u \in L^2(\Omega) \mid Au \in L^2(\Omega), (u, \phi) + \sum_{j \in J^*} \langle B_{m-1-j} u|_\Gamma, B_{m-1-j}\phi|_\Gamma \rangle_{-\mu_j} = 0 \ (\forall \phi \in K) \right\}.$$

We remark that $K \cap \tau = \{0\}$, because $u \in K \cap \tau$ satisfies

$$(u, u) + \sum_{j \in J^*} \langle B_{m-1-j} u|_\Gamma, B_{m-1-j} u|_\Gamma \rangle_{-\mu_j} = 0.$$

THEOREM 2.2. *Assume* (A-I) *and* (A-II). *Suppose that* $f \in K^{*\perp}$. *Let* w *be a Riesz function of* f. *Then* $u = A^* w \in \tau$ *and* u *is a solution of* (P_0). *Moreover, a solution* u *of* (P_0) *is unique in* τ.

PROOF. (1) From Theorem 2.1, we have

$$\begin{aligned} Qu &= \{u, -B_{m-1-j}u|_\Gamma \ (j \in J^*)\} \\ &= \{A^*w, \Lambda^{2\mu_j}\mathscr{B}_j^* w|_\Gamma \ (j \in J^*)\}, \end{aligned}$$

that is,

$$\begin{aligned} P^*w &= \{A^*w, \mathscr{B}_j^* w|_\Gamma \ (j \in J^*)\} \\ &= \{u, -\Lambda^{-2\mu_j}B_{m-1-j}u|_\Gamma \ (j \in J^*)\}. \end{aligned}$$

(2) Since w, $A^*w \in L^2(\Omega)$, we have, owing to Green's Theorem,

$$[P\phi, Q^*w] = [Q\phi, P^*w]_* \quad (\phi \in H^{2m-1}(\Omega)).$$

Since $K \subset H^{2m-1}(\Omega)$, we have

$$[Q\phi, P^*w]_* = 0 \quad (\phi \in K),$$

that is,

$$(\phi, u) + \sum_{j \in J^*} \langle B_{m-1-j}\phi|_\Gamma, \Lambda^{-2\mu_j}B_{m-1-j}u|_\Gamma \rangle = 0 \quad (\phi \in K),$$

which means $u \in \tau$.

(3) (Uniqueness) Let u_1 and u_2 be solutions of (P_0), belonging to τ. Then $u = u_1 - u_2 \in K \cap \tau$. Since $K \cap \tau = \{0\}$, we have $u = 0$. □

Finally, we consider a method to construct a function belonging to $K - \{0\}$.

LEMMA 2.1. *Let $U \in H^m(\Omega)$ satisfy $\{B_j U|_\Gamma = 0 \ (j \in J)\}$ and $U \notin \tau$. Set $\phi = A^*w + U$, where w is a Riesz function of $-AU$. Then $\phi \in K - \{0\}$.*

PROOF. From Theorem 2.1, we have $\phi = A^*w + U \in K$. On the other hand, from Theorem 2.2, we have $A^*w \in \tau$. Then we have $\phi \notin \tau$. In fact, if we suppose $\phi \in \tau$, then $U = \phi - A^*w \in \tau$, which contradicts to $U \notin \tau$. The fact $\phi \neq 0$ follows from $\phi \notin \tau$. □

LEMMA 2.2. *Assume that there exists $\phi_0 \in K - \{0\}$ such that $\phi_0(x) > 0$ $(x \in \Omega)$. Let U be a non-negative function satisfying*

$$U \in H^m(\Omega) - \{0\} \quad \textit{and} \quad \mathrm{supp}[U] \Subset \Omega,$$

then $U \notin \tau$.

PROOF. We have $U \notin \tau$, because

$$(U, \phi_0) + \sum_{j \in J^*} \langle B_{m-1-j} U|_\Gamma, \Lambda^{-2\mu j} B_{m-1-j} \phi_0|_\Gamma \rangle = (U, \phi_0) > 0. \qquad \square$$

§ 3. Numerical Approximation

Let us say that $\{v_k \ (k = 1, 2, \ldots)\}$ is a *basis* of $\mathscr{H}$, if any finite subset of $\{v_k \ (k = 1, 2, \ldots)\}$ is linearly independent and the space spanned by $\{v_k \ (k = 1, 2, \ldots)\}$ is dense in $\mathscr{H}$.

The solution u, obtained in § 2, can be approximated by the method proposed in [1], that is,

THEOREM 3.1. *Assume* (A-I) *and* (A-II). *Let* $u = A^* w$, *where* w *is a Riesz function of given* $f \in K^{*\perp}$. *Let* $\{v_k \ (k = 1, 2, \ldots)\}$ *be a basis of* $\mathscr{H}$. *Set*

$$u_N = ((f, v_1), \ldots, (f, v_N)) \Gamma_N^{-1} \begin{pmatrix} A^* v_1 \\ \vdots \\ A^* v_N \end{pmatrix},$$

where

$$\Gamma_N = \begin{pmatrix} [v_1, v_1] & \cdots & [v_1, v_N] \\ \vdots & & \vdots \\ [v_N, v_1] & \cdots & [v_N, v_N] \end{pmatrix}.$$

Then

$$u_N \to u \quad (N \to \infty) \quad \text{in } L^2(\Omega).$$

PROOF. (1) (Theory of Fourier Series in $\mathscr{H}$) Let $\{v_1^\wedge, v_2^\wedge, \ldots\}$ be Schmidt's orthonormalization of $\{v_1, v_2, \ldots\}$ in $\mathscr{H}$. For $w \in \mathscr{H}$, we have

$$w_N = \sum_{1 \leqq k \leqq N} [w, v_k^\wedge] v_k^\wedge \to w \quad \text{in } \mathscr{H},$$

that is,

$$w_N = ([w, v_1], \ldots, [w, v_N]) \Gamma_N^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \to w \quad \text{in } \mathscr{H},$$

where

$$\Gamma_N = ([v_j, v_k])_{j,k=1,\dots,N}.$$

Moreover, since $w_N \to w$ in $\mathscr{H}$, we have

$$u_N = A^* w_N \to A^* w = u \quad \text{in } L^2(\Omega).$$

(2) Especially, since $w \in \mathscr{H}$ is a Riesz function of f, that is,

$$[w, v] = (f, v) \quad (v \in \mathscr{H}),$$

we have

$$[w, v_k] = (f, v_k) \quad (k = 1, 2, \dots).$$

Hence we have

$$w_N = ((f, v_1), \dots, (f, v_N))\Gamma_N^{-1}\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix},$$

$$u_N = A^* w_N = ((f, v_1), \dots, (f, v_N))\Gamma_N^{-1}\begin{pmatrix} A^* v_1 \\ \vdots \\ A^* v_N \end{pmatrix}. \qquad \square$$

Since the boundary of Ω is smooth, we have

LEMMA 3.1. *Let* $\mathrm{diam}(\Omega) < a\pi$ $(a > 0)$. *Then*

$$\{\exp(ia^{-1}\alpha \cdot x) \mid \alpha \in \boldsymbol{Z}^n\}$$

is a basis of $H^q(\Omega)$.

As is shown easily, we have

LEMMA 3.2. *Let* $\{v_k \ (k = 1, 2, \dots)\}$ *be a basis of* $H^q(\Omega)$. *Set*

$$v_k = \phi_k + \xi_k \quad (\phi_k \in K^*, \xi_k \in M^*),$$

then the space spanned by $\{\xi_k \ (k = 1, 2, \dots)\}$ *is dense in* $\mathscr{H}$. *Therefore, we can obtain a subset* $\{\xi'_j \ (j = 1, 2, \dots)\} = \{\xi_{k_j} \ (j = 1, 2, \dots)\}$ *such that* $\{\xi'_j \ (j = 1, 2, \dots)\}$ *is a basis of* $\mathscr{H}$.

Let $\{v'_j \ (j = 1, 2, \dots)\} = \{v_{k_j} \ (j = 1, 2, \dots)\}$ be a subset of $\{v_k \ (k = 1, 2, \dots)\}$, corresponding to $\{\xi'_j \ (j=1,2,\dots)\} = \{\xi_{k_j} \ (j=1,2,\dots)\}$ in Lemma 3.2. Remark that it holds

$$([\![\xi_k,\xi_s]\!])_{k,s=1,2,\dots,N} = ([\![v_k,v_s]\!])_{k,s=1,2,\dots,N},$$

$$(A^*\xi_k)_{k=1,2,\dots,N} = (A^*v_k)_{k=1,2,\dots,N},$$

and

$$((f,\xi_k))_{k=1,2,\dots,N} = ((f,v_k))_{k=1,2,\dots N} \quad (f \in K^{*\perp}).$$

Hence we have

COROLLARY 3.1. *Assume* (A-I) *and* (A-II). *Let* $u = A^*w$, *where* w *is a Riesz function of* $f \in K^{*\perp}$. *Let* $\{v_k \ (k=1,2,\dots)\}$ *be a basis of* $H^q(\Omega)$. *Let* $\{v'_k \ (k=1,2,\dots)\}$ *be a subset of* $\{v_k \ (k=1,2,\dots)\}$ *chosen in the above way. Set*

$$u_N = ((f,v'_1),\dots,(f,v'_N))\Gamma_N^{-1}\begin{pmatrix} A^*v'_1 \\ \vdots \\ A^*v'_N \end{pmatrix},$$

where

$$\Gamma_N = ([\![v'_k,v'_s]\!])_{k,s=1,2,\dots,N}.$$

Then

$$u_N \to u \quad (N \to \infty) \quad \text{in } L^2(\Omega).$$

Finally, we consider the approximation of $\phi \in K - \{0\}$ in Theorem 2.3.

THEOREM 3.2. *Assume* (A-I) *and* (A-II). *Assume that there exists* $\phi_0 \in K - \{0\}$ *such that* $\phi_0(x) > 0$ $(x \in \Omega)$. *Let* U *be a non-negative function satisfying*

$$U \in H^m(\Omega) - \{0\} \quad \text{and} \quad \operatorname{supp}[U] \Subset \Omega.$$

Let $\{v_k \ (k=1,2,\dots)\}$ *be a basis of* $\mathscr{H}$. *Set*

$$\phi_N = U - ((AU,v_1),\dots,(AU,v_N))\Gamma_N^{-1}\begin{pmatrix} A^*v_1 \\ \vdots \\ A^*v_N \end{pmatrix},$$

where

$$\Gamma_N = ([\![v_k,v_s]\!])_{k,s=1,2,\dots,N}.$$

Then

$$\phi_N \to \phi \quad (N \to \infty) \quad \text{in } L^2(\Omega),$$

and $\phi \in K - \{0\}$.

§4. Examples

EXAMPLE 1. Consider Neumann problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ (d/d\boldsymbol{n})u = f_1 & \text{on } \Gamma, \end{cases}$$

where $\Omega \Subset (-\pi, \pi)^n$. Then $K(= K^*)$ is a space spanned by 1.

LEMMA 4.1. *It holds*

$$(\mathscr{E}_0) \qquad \|u\| \leqq C\|\Delta u\| \quad (u \in M, (d/d\boldsymbol{n})u|_\Gamma = 0),$$

where

$$M = \{u \in H^2(\Omega) \mid (u, 1) = 0\}.$$

PROOF. Let $\{\phi_k \ (k = 0, 1, \dots)\}$ be a complete set of eigen-functions, corresponding to eigen-values $\{\lambda_k \ (k = 0, 1, \dots)\}$ such that

$$-\Delta\phi_k = \lambda_k\phi_k \quad \text{in } \Omega, \quad (d/d\boldsymbol{n})\phi_k = 0 \quad \text{on } \Gamma,$$

and $(\phi_j, \phi_k) = \delta_{jk}$, where $0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$.

Let $u \in M$ satisfy

$$\begin{cases} -\Delta u = f \in L^2(\Omega) & \text{in } \Omega, \\ (d/d\boldsymbol{n})u = 0 & \text{on } \Gamma. \end{cases}$$

Then, owing to Green's Theorem, we have

$$(f, \phi_k) = \lambda_k(u, \phi_k) \quad (k = 0, 1, \dots).$$

Therefore, we have $f \in K^\perp$,

$$u = \sum_{k \neq 0} (1/\lambda_k)(f, \phi_k)\phi_k,$$

and

$$\|u\|^2 = \sum_{k \neq 0} |\lambda_k|^{-2} |(f, \phi_k)|^2 \leqq c^{-2} \|f\|^2,$$

where $c = |\lambda_1|$. □

Let $\mathscr{H}$ be a Hilbert space, defined by the completion of M by the norm $[\ ]$:

$$[v]^2 = \|\Delta v\|^2 + \langle (d/d\boldsymbol{n}) v|_\Gamma \rangle_{1/2}^2.$$

Since $\{e^{ik\cdot x}\ (k \in \boldsymbol{Z}^n)\}$ is a basis of $H^2(\Omega)$,

$$\{e^{ik\cdot x} - |\Omega|^{-1}(e^{ik\cdot x}, 1)\ \ (k \in \boldsymbol{Z}^n - \{0\})\}$$

is a basis of $\mathscr{H}$. From Theorem 2.1, Theorem 2.2 and Corollary 3.1, we have

PROPOSITION 4.1. *Suppose that $f \in L^2(\Omega)$ and $(f, 1) = 0$. Let $w \in \mathscr{H}$ be a Riesz function of f in $\mathscr{H}$. Set $u = -\Delta w$, then $u \in L^2(\Omega)$ satisfies*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ (d/d\boldsymbol{n}) u = 0 & \text{on } \Gamma \end{cases}$$

and $(u, 1) + \langle u|_\Gamma, 1 \rangle_{-1/2} = 0$. Moreover, set

$$f_N = ((f, e^{ik\cdot x}))_{0<|k|\leqq N},$$

$$\begin{aligned}\Gamma_N &= ([e^{ik\cdot x}, e^{is\cdot x}])_{0<|k|\leqq N, 0<|s|\leqq N} \\ &= (|k|^2|s|^2(e^{ik\cdot x}, e^{is\cdot x}) + (k \cdot \boldsymbol{n})(s \cdot \boldsymbol{n})\langle e^{ik\cdot x}|_\Gamma, e^{is\cdot x}|_\Gamma \rangle_{1/2})_{0<|k|\leqq N, 0<|s|\leqq N},\end{aligned}$$

$$V_N = (-\Delta e^{ik\cdot x})_{0<|k|\leqq N} = (|k|^2 e^{ik\cdot x})_{0<|k|\leqq N},$$

and

$$u_N = {}^t f_N \Gamma_N^{-1} V_N,$$

then it holds

$$u_N \to u \quad (N \to \infty) \quad \text{in } L^2(\Omega).$$

EXAMPLE 2. Consider Dirichlet problem:

$$\begin{cases} (-\Delta - \lambda_0) u = f & \text{in } \Omega, \\ u = f_0 & \text{on } \Gamma, \end{cases}$$

where $\Omega \Subset (-\pi, \pi)^n$ and λ_0 is the least eigen-value for the eigen-value problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

Then, $K(= K^*)$ is a space spanned by ϕ_0, where ϕ_0 is an eigen-function corresponding to the eigen-value λ_0.

LEMMA 4.2. *It holds*

$$(\mathscr{E}_0) \qquad \|u\| \leqq C\|(-\Delta - \lambda_0)u\| \quad (u \in M, u|_\Gamma = 0),$$

where $M = \{u \in H^2(\Omega) \mid (u, \phi_0) = 0\}$.

PROOF. Let $\{\phi_k\ (k = 0, 1, \dots)\}$ be a complete set of eigen-functions, corresponding to eigen-values $\{\lambda_k\ (k = 0, 1, \dots)\}$ such that

$$-\Delta\phi_k = \lambda_k \phi_k \quad \text{in } \Omega, \quad \phi_k = 0 \quad \text{on } \Gamma,$$

and $(\phi_j, \phi_k) = \delta_{jk}$, where $0 < \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$.

Let $u \in M$ satisfy

$$\begin{cases} (-\Delta - \lambda_0)u = f \in L^2(\Omega) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

Then, owing to Green's Theorem, we have

$$(f, \phi_k) = (\lambda_k - \lambda_0)(u, \phi_k) \quad (k = 0, 1, \dots).$$

Therefore, we have $f \in K^\perp$,

$$u = \sum_{k \neq 0} (\lambda_k - \lambda_0)^{-1}(f, \phi_k)\phi_k,$$

and

$$\|u\|^2 = \sum_{k \neq 0} (\lambda_k - \lambda_0)^{-2}|(f, \phi_k)|^2 \leqq c^{-2}\|f\|^2,$$

where $c = |\lambda_1 - \lambda_0|$. □

Let $\mathscr{H}$ be a Hilbert space, defined by the completion of M by the norm [ ]:

$$[v]^2 = \|(-\Delta - \lambda_0)v\|^2 + \langle v|_\Gamma \rangle_{1+1/2}^2.$$

Since $\{e^{ik\cdot x}\ (k \in \boldsymbol{Z}^n)\}$ is a basis of $H^2(\Omega)$,

$$\{\xi_k(x) = e^{ik\cdot x} - (e^{ik\cdot x}, \phi_0)\phi_0 \ \ (k \in \boldsymbol{Z}^n)\}$$

is a basis of $\mathscr{H}$, we have from Theorem 2.1, Theorem 2.2 and Corollary 3.1, we have

PROPOSITION 4.2. *Suppose that* $f \in L^2(\Omega)$ *and* $(f, \phi_0) = 0$. *Let* $w \in \mathscr{H}$ *be a Riesz function of* f *in* $\mathscr{H}$. *Set* $u = (-\Delta - \lambda_0)w$, *then* $u \in L^2(\Omega)$ *satisfies*

$$\begin{cases} (-\Delta - \lambda_0)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma \end{cases}$$

and $(u, \phi_0) + \langle (d/d\boldsymbol{n})u|_\Gamma, (d/d\boldsymbol{n})\phi_0|_\Gamma \rangle_{-1-1/2} = 0$. *Moreover, set*

$$f_N = ((f, e^{ik\cdot x}))_{|k| \leqq N},$$

$$\Gamma_N = ([e^{ik\cdot x}, e^{is\cdot x}])_{|k| \leqq N, |s| \leqq N}$$

$$= ((|k|^2 - \lambda_0)(|s|^2 - \lambda_0)(e^{ik\cdot x}, e^{is\cdot x}) + \langle e^{ik\cdot x}, e^{is\cdot x} \rangle_{1+1/2})_{|k| \leqq N, |s| \leqq N},$$

$$V_n = ((-\Delta - \lambda_0)e^{ik\cdot x})_{|k| \leqq N} = ((|k|^2 - \lambda_0)e^{ik\cdot x})_{|k| \leqq N},$$

and

$$u_N = {}^t f_N \Gamma_N^{-1} V_N,$$

then it holds

$$u_N \to u \quad (N \to \infty) \quad \text{in } L^2(\Omega).$$

References

[1] R. Sakamoto, Numerical approximation of weak solution for boundary value problems, Tsukuba J. Math. **26** (2002), 79–94.
[2] R. Sakamoto, Dirichlet-Neumann problem in a domain with piecewise-smooth boundary, Tsukuba J. Math. **26** (2002), 387–406.
[3] R. Sakamoto, Energy method for numerical analysis, to appear.
[4] L. Hörmander, Linear partial differential operators, Springer, Berlin (1963).

28-2-507 Fukakusa-Sekiyashiki-cho
Fushimi-ku, Kyoto, Japan, 612-0037