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A MASCHKE TYPE THEOREM FOR HOPF π-COMODULES

By

Shuan-hong WANG

Abstract. Let H be a Hopf π-coalgebra and let A be a right group H-comodule algebra with a total integral ϕ. In this article we will find some sufficient conditions under which an epimorphism of (H, A)-Hopf π-comodule splits if it splits A-linearly. As an application, we obtain a characterization for an (H, A)-Hopf π-comodule to be projective as an A-module, generalizing the one of the Maschke type theorem found in [D2].

1. Introduction

Let H be a Hopf algebra over a commutative ring R and A a right H-comodule algebra. Doi ([D2] under review) gave some sufficient conditions under which an epimorphism of (H, A)-Hopf module splits if it splits A-linearly. As an application in the case when R is a field, he got that an (H, A)-Hopf module is finitely generated projective as an A-module if and only if it is a Hopf module direct summand of $M \otimes A$ for some finite dimensional H-comodule M.

In [Tur] Turaev introduced, for a group π, the notion of a Hopf π-coalgebra, which can induce a π-category, i.e., group-category, and showed that such a category gives rise to a three-dimensional homotopy quantum field theory with target space $K(\pi, 1)$. Virelizier [Vir1] studied some algebraic properties of Hopf π-coalgebras; the results are then applied in [Vir2] to construct Hennings-like (see [KR]) and Kuperberg-like (see [Ku]) invariants of principal π-bundles over link complements and over 3-manifolds.

Now, it is natural to ask whether there exists a Maschke type theorem in the generalized context of Hopf π-coalgebras. This question motivates the present research.

In this paper we will give a positive answer to the above question.

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2. Preliminaries and Basic Definitions

Throughout this paper, R denotes a commutative ring. We will work over R. We always let π be a discrete group with a neutral element 1, and let $\otimes$ denote $\otimes_R$. If U and V are R-modules, $T_{U,V} : U \otimes V \to V \otimes U$ will denote the flip map defined by $T_{U,V}(u \otimes v) = v \otimes u$ for any $u \in U$ and $v \in V$.

Similar to [Vir1], A π-coalgebra is a family of R-modules $C = \{C_\alpha\}_{\alpha \in \pi}$ together with a family of R-linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta \in \pi}$ (called a comultiplication) and an R-linear map $\varepsilon : C_1 \to R$ (called a counit), such that Δ is coassociative in the sense that, for any $\alpha, \beta, \gamma \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}, \tag{2.1}$$

and for all $\alpha, \beta \in \pi$,

$$(id_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha})\Delta_{1,\alpha}. \tag{2.2}$$

REMARK. $(C_1, \Delta_{1,1}, \varepsilon)$ is an ordinery coalgebra in the usual sense of the word (cf. [Sw] or [Mon]).

Following the Sweedler's notation for Hopf π-coalgebras introduced in [Vir1], we have that, for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$,

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)} \in C_\alpha \otimes C_\beta.$$

The coassociativity axiom (2.1) gives that, for any $\alpha, \beta, \gamma \in \pi$ and $c \in C_{\alpha\beta\gamma}$,

$$c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)} \otimes c_{(2,\gamma)} = c_{(1,\alpha)} \otimes c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(2,\beta\gamma)(2,\gamma)}, \tag{2.3}$$

which is written as $c_{(1,\alpha)} \otimes c_{(2,\beta)} \otimes c_{(3,\gamma)}$. Inductively, we can define $c_{(1,\alpha_1)} \otimes c_{(2,\alpha_2)} \otimes \cdots \otimes c_{(n,\alpha_n)}$ for any $c \in C_{\alpha_1\alpha_2\cdots\alpha_n}$.

A π-grouplike element of a π-coalgebra C is a family of elements $x = (x_\alpha)_{\alpha \in \pi} \in \prod_{\alpha \in \pi} C_\alpha$ such that $\Delta_{\alpha,\beta}(x_{\alpha\beta}) = x_\alpha \otimes x_\beta$ for all $\alpha, \beta \in \pi$ and $\varepsilon(x_1) = 1$ (or equivalently $x_1 \neq 0$). Note that x_1 is then an ordinary grouplike element of the coalgebra C_1.

Given a π-coalgebra C. A right $\pi - C$-comodule is a family $M = \{M_\alpha\}_{\alpha \in \pi}$ of R-modules endowed with a family $\rho^M = \{\rho^M_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes C_\beta\}_{\alpha,\beta \in \pi}$ of R-linear maps (the structure maps) such that for any $\alpha, \beta, \gamma \in \pi$,

$$(\rho^M_{\alpha,\beta} \otimes id_{C_\gamma})\rho^M_{\alpha\beta,\gamma} = (id_{M_\alpha} \otimes \Delta_{\beta,\gamma})\rho^M_{\alpha,\beta\gamma}, \tag{2.4}$$

and for any $\alpha \in \pi$,

$$(id_{M_\alpha} \otimes \varepsilon)\rho^M_{\alpha,1} = id_{M_\alpha}. \tag{2.5}$$

Similarly, we use the Sweedler's notation for coactions, for any $\alpha, \beta \in \pi$ and $m \in M_{\alpha\beta}$,

$$\rho^M_{\alpha,\beta}(m) = m_{(0,\alpha)} \otimes m_{(1,\beta)} \in M_\alpha \otimes C_\beta.$$

Axiom (2.4) gives that, for any $\alpha, \beta, \gamma \in \pi$ and $m \in M_{\alpha\beta\gamma}$,

$$m_{(0,\alpha\beta)(0,\alpha)} \otimes m_{(0,\alpha\beta)(1,\beta)} \otimes m_{(1,\gamma)} = m_{(0,\alpha)} \otimes m_{(1,\beta\gamma)(1,\beta)} \otimes m_{(1,\beta\gamma)(2,\gamma)}, \tag{2.6}$$

which is written as $m_{(0,\alpha)} \otimes m_{(1,\beta)} \otimes m_{(2,\gamma)}$. By iterating the procedure, we define inductively $m_{(0,\alpha_0)} \otimes m_{(1,\alpha_1)} \otimes \cdots \otimes m_{(n,\alpha_n)}$ for any $m \in M_{\alpha_0\alpha_1\cdots\alpha_n}$.

REMARK. M_1 endowed with the structure map $\rho_{1,1}$ is an ordinery right comodule over the coalgebra C_1 (cf. [Sw] or [Mon]).

A π-comodule map between two right $\pi - C$-comodules M and N is a family $f = \{f_\alpha : M_\alpha \to N_\alpha\}_{\alpha \in \pi}$ of right R-module maps such that $\rho^N_{\alpha,\beta} f_{\alpha\beta} = (f_\alpha \otimes id_{C_\beta})\rho^M_{\alpha,\beta}$ for all $\alpha, \beta \in \pi$.

In a similar way, we can define the notion of a left $\pi - C$-comodule and the concept of a $\pi - (C, C)$-bicomodule.

The category of right $\pi - C$-comodules is denoted by $\mathcal{M}^{\pi-C}$, and their morphisms are π-comodule maps. Similarly, we can introduce the categories ${}^{\pi-C}\mathcal{M}$ of left $\pi - C$-comodules, and the category ${}^{\pi-C}\mathcal{M}^{\pi-C}$ of $\pi - C$-bicomodules. For $M \in {}^{\pi-C}\mathcal{M}$, we will use the Sweedler's notation, for any $\alpha, \beta \in \pi$ and $m \in M_{\alpha\beta}$,

$${}^M\rho_{\alpha,\beta}(m) = m_{(-1,\alpha)} \otimes m_{(0,\beta)} \in C_\alpha \otimes M_\beta.$$

Similar to [Tur] or [Vir1], a Hopf π-coalgebra is a π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)$ together with a family $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha \in \pi}$ of R-linear maps (called an antipode) such that

each H_α is an R-algebra with multiplication m_α and unit element 1_α, (2.7)

$$\text{for all } \alpha,\beta \in \pi,\ \Delta_{\alpha,\beta} \text{ and } \varepsilon : H_1 \to R \text{ are algebra maps,} \tag{2.8}$$

$$\text{for any } \alpha \in \pi,\ m_\alpha(S_{\alpha^{-1}} \otimes id_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon 1_\alpha = m_\alpha(id_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}. \tag{2.9}$$

Let $H = (\{H_\alpha, m_\alpha, 1_\alpha,\}, \Delta, \varepsilon, S)$ be a Hopf π-coalgebra. Then we have
(a) $S_{\alpha^{-1}}(hg) = S_{\alpha^{-1}}(g)S_{\alpha^{-1}}(h)$ for any $\alpha \in \pi$ and $h, g \in H_{\alpha^{-1}}$;
(b) $S_{\alpha^{-1}}(1_{\alpha^{-1}}) = 1_\alpha$ for any $\alpha \in \pi$;
(c) $\Delta_{\beta,\alpha}S_{\alpha^{-1}\beta^{-1}} = T_{H_\alpha,H_\beta}(S_{\alpha^{-1}} \otimes S_{\beta^{-1}})\Delta_{\alpha^{-1},\beta^{-1}}$ for any $\alpha,\beta \in \pi$;
(d) $\varepsilon S_1 = \varepsilon$.

Note that $(H_1, m_1, 1_1, \Delta_{1,1}, \varepsilon, S_1)$ is an ordinery Hopf algebra (cf. [Sw] or [Mon]) and that the notion of a Hopf π-coalgebra is not self-dual.

Let H be a Hopf π-coalgebra and $M = \{M_\alpha\}_{\alpha\in\pi}$ a right $\pi - H$-comodule with structure maps $\rho = \{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$. The coinvariants of H on M are the elements of the R-module

$$M^{coH} = \left\{ m = (m_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} M_\alpha \mid \rho_{\alpha,\beta}(m_{\alpha\beta}) = m_\alpha \otimes 1_\beta \text{ for all } \alpha,\beta \in \pi \right\}.$$

For any $\alpha \in \pi$, let M_α^{coH} be the image of the canonical projection of this set onto M_α. Similar to [Vir1, Example 2.1], $M^{coH} = \{M_\alpha^{coH}\}_{\alpha\in\pi}$ is a right π-subcomodule of M, called a π-subcomodule of coinvariants.

DEFINITION 2.1. Let $H = (\{H_\alpha, m_\alpha, 1_\alpha, \Delta, \varepsilon\})$ be a Hopf π-coalgebra and let $A = \{A_\alpha, m_\alpha, 1_\alpha\}_{\alpha\in\pi}$ be a family of R-algebras. A is called a right $\pi - H$-comodule algebra if there is a family $\rho^A = \{\rho^A_{\alpha,\beta} : A_{\alpha\beta} \to A_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of R-linear maps such that

$$(A, \rho^A) \text{ is a right } \pi - C\text{-comodule,} \tag{2.10}$$

$$\rho^A_{\alpha,\beta}(ab) = a_{(0,\alpha)}b_{(0,\alpha)} \otimes a_{(1,\beta)}b_{(1,\beta)}, \quad \text{for all } \alpha,\beta \in \pi \text{ and } a, b \in A_{\alpha\beta}, \tag{2.11}$$

$$\rho^A_{\alpha\beta}(1_{\alpha\beta}) = 1_\alpha \otimes 1_\beta \quad \text{for any } \alpha,\beta \in \pi. \tag{2.12}$$

In this occasion, we say that (A, ρ^A) is a right $\pi - H$-comodule algebra. Note that A_1 endowed with the $\rho^{A_1}_{1,1}$ is an ordinery right H_1-comodule algebra (cf. [D1] and [Mon]).

Similarly, we can define the notions of a left $\pi - H$-comodule algebra and a $\pi - H$-bicomodule algebra.

In what follows, let H be a Hopf π-coalgebra. Let A be a right $\pi - H$-comodule algebra and let

$$C = \left\{ a = (a_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} A_\alpha \mid \rho_{\alpha,\beta}(a_{\alpha\beta}) = a_\alpha \otimes 1_\beta \text{ for all } \alpha,\beta \in \pi \right\}.$$

We denote by $\mathcal{M}_A^{\pi-H}$ the category of right (H,A)-Hopf π-comodules; its object is a family $M=\{M_\alpha\}_{\alpha\in\pi}$ of right A_α-modules M_α which is also a right $\pi-H$-comodue such that

$$\rho_{\alpha,\beta}(ma)=m_{(0,\alpha)}a_{(0,\alpha)}\otimes m_{(1,\beta)}a_{(1,\beta)},\quad \text{for all } m\in M_{\alpha\beta},\ a\in A_{\alpha\beta}. \tag{2.13}$$

Its morphism is a family of A_α-module maps which is also a $\pi-H$-comodule map.

REMARK. We remark that the category $\mathcal{M}_{A_1}^{H_1}$ is an ordinary relative Hopf module category studied in [D1].

EXAMPLE 2.2. 1) Obviously, A is an object in $\mathcal{M}_A^{\pi-H}$.

2) For every $M\in\mathcal{M}_A^{\pi-H}$, ρ^M induces a family of R-linear maps $\bar{\rho}=\{\bar{\rho}_\alpha: M_\alpha\to M_1\otimes H_\alpha\}_\alpha$. Then we can view $M_1\otimes H=\{(M_1\otimes H)_\alpha=M_1\otimes H_\alpha\}_{\alpha\in\pi}$ as a right (H,A)-Hopf module by $(m\otimes x)\cdot a=ma_{(0,1)}\otimes xa_{(1,\alpha)}$ for any $m\in M_1$, $x\in H_\alpha$, $a\in A_\alpha$ and $\delta_{\alpha,\beta}(m\otimes x)=m\otimes x_{(1,\alpha)}\otimes x_{(2,\beta)}$ for all $m\in M_1$, $x\in H_{\alpha\beta}$, and then $\bar{\rho}: M\to M_1\otimes H$ becomes a morphism of $\mathcal{M}_A^{\pi-H}$.

3. A Maschke Type Theorem for $\mathcal{M}_A^{\pi-H}$

In this section we will prove that there exists a Maschke type theorem in the generalized context of Hopf π-comodules.

Now, we have some definitions as follows:

DEFINITION 3.1. A total integral is a family of R-maps $\phi=\{\phi_\alpha\}_{\alpha\in\pi}: H_\alpha\to A_\alpha$ such that ϕ is a π-comodule map, i.e., $\rho^A_{\alpha,\beta}\phi_{\alpha\beta}=(\phi_\alpha\otimes id_{C_\beta})\Delta_{\alpha,\beta}$ and $\phi_\alpha(1_\alpha)=1_\alpha$ for any $\alpha,\beta\in\pi$.

DEFINITION 3.2. For $M\in\mathcal{M}_A^{\pi-C}$, a trace map associated with ϕ is a family of maps $tr^M=\{tr^M_\alpha: M_1\to M_\alpha\}$ defined by

$$tr^M_\alpha(m)=m_{(0,\alpha)}\phi_\alpha(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})})) \tag{3.1}$$

for any $m\in M_1$.

PROPOSITION 3.3. *For any $m\in M_1$, we have $tr^M(m)\in M^{coH}$. Moreover, the condition $\phi(1_\alpha)=1_\alpha$ for any $\alpha\in\pi$, implies that tr^M is the identity on M_1.*

PROOF. For any $m\in M_1$, we have

$$\begin{aligned}
&\rho_{\alpha,\beta}(tr^M_{\alpha\beta}(m))\\
&\overset{(3.1)}{=} \rho_{\alpha,\beta}(m_{(0,\alpha\beta)}\phi_{\alpha\beta}S_{\beta^{-1}\alpha^{-1}}(m_{(1,\beta^{-1}\alpha^{-1})}))\\
&\overset{(2.11)}{=} m_{(0,\alpha\beta)(0,\alpha)}\phi_\alpha S_{\alpha^{-1}}(m_{(1,\beta^{-1}\alpha^{-1})(2,\alpha^{-1})})) \otimes m_{(0,\alpha\beta)(1,\beta)}S_{\beta^{-1}}(m_{(1,\beta^{-1}\alpha^{-1})(1,\beta^{-1})}))\\
&= m_{(0,\alpha)}\phi_\alpha S_{\alpha^{-1}}(m_{(1,\alpha^{-1})(2,\alpha^{-1})})) \otimes \underline{m_{(1,\alpha^{-1})(1,1)(1,\beta)}S_{\beta^{-1}}(m_{(1,\alpha^{-1})(1,1)(2,\beta^{-1})}))}\\
&= m_{(0,\alpha)}\phi_\alpha S_{\alpha^{-1}}(m_{(1,\alpha^{-1})})) \otimes 1_\beta\\
&= tr^M_\alpha(m) \otimes 1_\beta,
\end{aligned}$$

where we have used the properties of the antipode and the π-comodule structure for the second and third equations, respectively. ■

Now, we define $\lambda = \{\lambda_\alpha\} : M_1 \otimes H_\alpha \to M_\alpha$ by

$$\lambda_\alpha(m \otimes x) = m_{(0,\alpha)}\phi_\alpha(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})}S_\alpha(x)))$$

for any $x \in H_\alpha$.

We will denote $\rho^M_{1,\alpha} : M_\alpha \to M_1 \otimes H_\alpha$ by ρ_α as follows.

LEMMA 3.4. *For any $\alpha \in \pi$, we have $\lambda_\alpha \circ \rho_\alpha = id_{M_\alpha}$.*

PROOF. For any $m \in M_\alpha$ and $\alpha \in \pi$, one has

$$\begin{aligned}
\lambda_\alpha \circ \rho_\alpha(m) &= m_{(0,1)(0,\alpha)}\phi(S_{\alpha^{-1}}(m_{(0,1)(1,\alpha^{-1})}S_\alpha(m_{(1,\alpha)})))\\
&= m_{(0,\alpha)}\phi(S_{\alpha^{-1}}(m_{(1,1)(1,\alpha^{-1})}S_\alpha(m_{(1,1)(2,\alpha)})))\\
&= m_{(0,\alpha)}\phi(S_{\alpha^{-1}}(\varepsilon(m_{(1,1)})1_{\alpha^{-1}})))\\
&= m\phi S_{\alpha^{-1}}(1_{\alpha^{-1}}) = m,
\end{aligned}$$

where one has used the properties of the antipode. ■

Define the center of A as

$$Z(A) = \{Z(A_\alpha) = \{x \in A_\alpha \mid ax = xa \text{ for all } a \in A_\alpha\}\}_{\alpha\in\pi}.$$

We now have the following two lemmas.

LEMMA 3.5. *For any $\alpha \in \pi$, if $\phi_\alpha(H_\alpha) \subset Z(A_\alpha)$, then λ_α is an A_α-module map,* *here $M_1 \otimes A_\alpha$ affords the module structure given by Example* 2.2(2). *Moreover, for any $\alpha \in \pi$, the following diagram is commutative*:

$$\begin{tikzcd}
M_1 \otimes_C A_\alpha \arrow[r, dashed, "\mu_{M_1}"] \arrow[d, "tr^M \otimes_C id_{A_\alpha}"'] & M_1 \otimes H_\alpha \arrow[d, "\lambda_\alpha"] \\
M^{coH} \otimes_C A_\alpha \arrow[r, dashed, "\Psi_M"] & M_\alpha
\end{tikzcd}$$

where we define $\mu_{M_1}(m \otimes a) = ma_{(0,1)} \otimes a_{(1,\alpha)}$ and $\Psi_M((m) \otimes a) = m_\alpha a$ for any $m \in M_1$ and $a \in A_\alpha$.

Proof. For any $\alpha \in \pi$, $a \in A_\alpha$ and $b \in H_\alpha$, we have

$$\begin{aligned}
&\lambda_\alpha(ma_{(0,1)} \otimes ba_{(1,\alpha)}) \\
&\quad = (ma_{(0,1)})_{(0,\alpha)}\phi(S_{\alpha^{-1}}((ma_{(0,1)})_{(1,\alpha^{-1})}S_\alpha(ba_{(1,\alpha)}))) \\
&\quad = m_{(0,\alpha)}a_{(0,1)(0,\alpha)}\phi(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})}a_{(0,1)(1,\alpha^{-1})}S_\alpha(ba_{(1,\alpha)}))) \\
&\quad = m_{(0,\alpha)}a_{(0,\alpha)}\phi(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})}a_{(2,1)(1,\alpha^{-1})}S_\alpha(a_{(2,1)(2,\alpha)})S_\alpha(b)) \\
&\quad = m_{(0,\alpha)}a\phi(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})}S_\alpha(b)) \\
&\quad = m_{(0,\alpha)}\phi(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})}S_\alpha(b))a \\
&\quad = \lambda_\alpha(m \otimes b)a,
\end{aligned}$$

here we used the properties of the antipode in the fourth equation and the condition: $\phi_\alpha(H_\alpha) \subset Z(A_\alpha)$ in the fifth equation. Hence this proves that λ_α is an A_α-module map, i.e., $\lambda_\alpha(ma_{(0,1)} \otimes ba_{(1,\alpha)}) = \lambda_\alpha(m \otimes b)a$.

For the second assertion, we only note that $\Psi_M \circ (tr^M \otimes_C id_{A_\alpha})(m \otimes a) = tr_\alpha^M(m)a$ and $\lambda_\alpha(m \otimes 1_\alpha) = tr_\alpha^M(m)$ for any $m \in M_1$. So the diagram clearly commutes from this and the formula $\lambda_\alpha(ma_{(0,1)} \otimes a_{(1,\alpha)}) = \lambda_\alpha(m \otimes 1)a$. ■

Lemma 3.6. *For any $\alpha \in \pi$, λ is a morphism in $\mathcal{M}_A^{\pi\text{-}H}$ if either of the following two conditions is fulfilled*:

(i) *A is faithful as an R-module and $\phi_\alpha(A_\alpha) \subset R$,*

(ii) *H is involutory (i.e., $S_{\alpha^{-1}}S_\alpha = id$ for all $\alpha \in \pi$), $\phi_\alpha(H_\alpha) \subset Z(A_\alpha)$ for all $\alpha \in \pi$ and*

$$\phi_\alpha(ab) = \phi_\alpha(ba) \quad \text{for all } a, b \in A_\alpha,\ \alpha \in \pi.$$

PROOF. By Lemma 3.5, it suffices to verify that λ is a $\pi - H$-comodule map.

If (i) holds true, then, for all $x \in H_{\alpha\beta}$, $\alpha, \beta \in \pi$ we have $\phi_\alpha(x_{(1,\alpha)})x_{(2,\beta)} = \phi_{\alpha\beta}(x)_{(0,\alpha)}1_{(1,\beta)} := \phi_{\alpha\beta}(x)1_\beta$ since ϕ is a $\pi - H$-comodule map and $\phi_\alpha(A_\alpha) \subset R \subset A^{coH}$. This implies that $\phi_{\beta^{-1}}(S_\beta(x_{(2,\beta)}))S_\alpha(x_{(1,\alpha)}) = \phi_{\beta^{-1}\alpha^{-1}}(S_{\alpha\beta}(x))_{(0,\beta^{-1})}1_{(1,\alpha^{-1})}$. In what follows, we claim:

$$x_{(1,\alpha)}\phi_{\beta^{-1}}S_\beta(x_{(2,\beta)})_{(0,\beta^{-1}\alpha^{-1})} = \phi_{\beta^{-1}\alpha^{-1}}(S_{\alpha\beta}(x))1_\alpha. \tag{3.2}$$

In fact, we have

$$\begin{aligned}
&x_{(1,\alpha)}\phi_{\beta^{-1}}S_\beta(x_{(2,\beta)})_{(0,\beta^{-1}\alpha^{-1})}\\
&= x_{(1,\alpha)}1_{(1,\alpha)}\phi_{\beta^{-1}\alpha^{-1}\alpha}S_{\alpha^{-1}\alpha\beta}(x_{(2,\alpha^{-1}\alpha\beta)})_{(0,\beta^{-1}\alpha^{-1})}\\
&= x_{(1,\alpha)}\phi_{\beta^{-1}\alpha^{-1}}S_{\alpha\beta}(x_{(2,\alpha^{-1}\alpha\beta)(2,\alpha\beta)})S_{\alpha^{-1}}(x_{(2,\alpha^{-1}\alpha\beta)(1,\alpha^{-1})})\\
&= x_{(1,1)(1,\alpha)}S_{\alpha^{-1}}(x_{(1,1)(2,\alpha^{-1})})\phi_{\beta^{-1}\alpha^{-1}}S_{\alpha\beta}(x_{(2,\alpha\beta)})\\
&= \phi_{\beta^{-1}\alpha^{-1}}S_{\alpha\beta}(x)1_\alpha,
\end{aligned}$$

as required.

We will next show that λ is a $\pi - H$-comodule map. For all $m \in M_1$, $x \in H_{\alpha\beta}$, one has

$$\begin{aligned}
(\lambda_\alpha \otimes id)\delta_{\alpha,\beta}(m \otimes x) &= \lambda_\alpha(m \otimes x_{(1,\alpha)}) \otimes x_{(2,\beta)}\\
&= m_{(0,\alpha)}\phi_\alpha S_{\alpha^{-1}}(m_{(1,\alpha^{-1})}S_\alpha(x_{(1,\alpha)})) \otimes x_{(2,\beta)}\\
&= m_{(0,\alpha)} \otimes \phi_\alpha S_{\alpha^{-1}}(m_{(1,\alpha^{-1})}S_\alpha(x_{(1,\alpha)}))x_{(2,\beta)}\\
&= m_{(0,\alpha)} \otimes \phi_{\alpha\beta\beta^{-1}}S_{\beta\beta^{-1}\alpha^{-1}}(m_{(1,\beta\beta^{-1}\alpha^{-1})}S_{\alpha\beta\beta^{-1}}(x_{(1,\alpha\beta\beta^{-1})}))1_\beta x_{(2,\beta)}\\
&\overset{(3.2)}{=} m_{(0,\alpha)} \otimes (m_{(1,\beta\beta^{-1}\alpha^{-1})}S_{\alpha\beta\beta^{-1}}(x_{(1,\alpha\beta\beta^{-1})}))_{(1,\beta)}\\
&\qquad \phi_{\alpha\beta\beta^{-1}}S_{\beta\beta^{-1}\alpha^{-1}}(m_{(1,\beta\beta^{-1}\alpha^{-1})}S_{\alpha\beta\beta^{-1}}(x_{(1,\alpha\beta\beta^{-1})}))_{(2,\beta^{-1}\alpha^{-1})}x_{(2,\beta)}\\
&= m_{(0,\alpha)} \otimes m_{(1,\alpha^{-1})(1,\beta)}S_{\beta^{-1}}(x_{(1,\alpha)(2,\beta^{-1})}))x_{(2,\beta)}\\
&\qquad \phi_{\alpha\beta}S_{\beta^{-1}\alpha^{-1}}(m_{(1,\alpha^{-1})(2,\beta^{-1}\alpha^{-1})}S_{\alpha\beta}(x_{(1,\alpha)(1,\alpha\beta)}))\\
&= m_{(0,\alpha)} \otimes m_{(1,\alpha^{-1})(1,\beta)}S_{\beta^{-1}}(x_{(2,1)(1,\beta^{-1})})x_{(2,1)(2,\beta)}\\
&\qquad \phi_{\alpha\beta}S_{\beta^{-1}\alpha^{-1}}(m_{(1,\alpha^{-1})(2,\beta^{-1}\alpha^{-1})}S_{\alpha\beta}(x_{(1,\alpha\beta)}))\\
&= m_{(0,\alpha)}\phi_{\alpha\beta}S_{\beta^{-1}\alpha^{-1}}(m_{(1,\alpha^{-1})(2,\beta^{-1}\alpha^{-1})}S_{\alpha\beta}(x)) \otimes m_{(1,\alpha^{-1})(1,\beta)}
\end{aligned}$$

$$= m_{(0,\alpha\beta)(0,\alpha)}\phi_\alpha S_{\alpha^{-1}}(m_{(1,\beta^{-1}\alpha^{-1})}S_{\alpha\beta}(x)) \otimes m_{(0,\alpha\beta)(1,\beta)}$$

$$= \rho^M_{\alpha,\beta}\lambda_{\alpha\beta}(m \otimes x),$$

where we used the properties of Δ_H and S_H in the sixth equation; the association of Δ_H in the seventh equation and the association of ρ^M in the ninth equation.

If (ii) holds, then $\lambda_\alpha(m \otimes x) = m_{(0,\alpha)}\phi_\alpha(xS_{\alpha^{-1}}(m_{(1,\alpha^{-1})}))$, and so

$$\begin{aligned}
\rho^M_{\alpha,\beta}\lambda_{\alpha\beta}(m \otimes x) &= m_{(0,\alpha\beta)(0,\alpha)}\phi_{\alpha\beta}(xS_{\beta^{-1}\alpha^{-1}}(m_{(1,\beta^{-1}\alpha^{-1})}))_{(0,\alpha)} \\
&\quad \otimes m_{(0,\alpha\beta)(1,\beta)}\phi_{\alpha\beta}(xS_{\beta^{-1}\alpha^{-1}}(m_{(1,\beta^{-1}\alpha^{-1})}))_{(1,\beta)} \\
&= m_{(0,\alpha\beta)(0,\alpha)}\phi_\alpha(S_{\alpha^{-1}}(m_{(1,\beta^{-1}\alpha^{-1})(2,\alpha^{-1})}x_{(1,\alpha)}) \\
&\quad \otimes m_{(0,\alpha\beta)(1,\beta)}S_{\beta^{-1}}(m_{(1,\beta^{-1}\alpha^{-1})(1,\beta^{-1})})x_{(2,\beta)} \\
&= m_{(0,\alpha)}\phi_\alpha(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})(3,\alpha^{-1})}x_{(1,\alpha)}) \\
&\quad \otimes m_{(1,\alpha^{-1})(1,\beta)}S_{\beta^{-1}}(m_{(1,\alpha^{-1})(2,\beta^{-1})})x_{(2,\beta)} \\
&= m_{(0,\alpha)}\phi_\alpha(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})}x_{(1,\alpha)}) \otimes x_{(2,\beta)} \\
&= (\lambda_\alpha \otimes id)\delta_{\alpha,\beta}(m \otimes x).
\end{aligned}$$

Here we have used the properties of Δ_H, S_H, and the π-comodule map ϕ and the condition: $\phi_\alpha(ab) = \phi_\alpha(ba)$ for all $a, b \in A_\alpha$, $\alpha \in \pi$ in the second equation; the association of ρ^M in the third equation.

This concludes the proof of the lemma 3.6. ■

Let $M = \{M_\alpha\}_{\alpha\in\pi}$ and $N = \{N_\alpha\}_{\alpha\in\pi}$ be a family of $A = \{A_\alpha\}_{\alpha\in\pi}$-module respectively. Let $f = \{f_\alpha : M_\alpha \to N_\alpha\}_{\alpha\in\pi} : M \to N$ be a family of A-module maps. By definition, an epimorphism $f : M \to N$ of A-modules consists of A_α-module epimorphisms $f_\alpha : M_\alpha \to N_\alpha$.

We now show the main result of this paper as follows.

THEOREM 3.7. *Let H be a Hopf π-coalgebra. Let A be a right $\pi - H$-comodule algebra with a total integral ϕ, and suppose that* (i) *or* (ii) *in Lemma* 3.6 *holds. Then an epimorphism $f : M \to N$ of (H, A)-Hopf π-comodules splits, if the A_1-linear epimorphism $f_1 : M_1 \to N_1$ between the neutral components splits.*

PROOF. Let $j = \{j_\alpha : M_\alpha \to N_\alpha\}_{\alpha\in\pi}$ be a morphism of $\mathcal{M}_A^{\pi-H}$ such that there is an A_1-module map $p_1 : N_1 \to M_1$ with $p_1 j_1 = id_{M_1}$. Define an R-linear map

$\tilde{p} = \{\tilde{p}_\alpha : N_\alpha \to M_\alpha\}_{\alpha\in\pi}$ by $\tilde{p}_\alpha = \lambda_\alpha(p_1 \otimes id)\rho^N_{1,\alpha}$. Then for any $\alpha \in \pi$, $n \in N_\alpha$ and $x \in H_\alpha$, we have

$$\begin{aligned}\tilde{p}(n \cdot x) &\overset{(2.13)}{=} \lambda_\alpha(p_1(n_{(0,1)} \cdot x_{(1,1)}) \otimes n_{(1,\alpha)}x_{(2,\alpha)}) \\ &= \lambda_\alpha(p_1(n_{(0,1)}) \cdot x_{(1,1)} \otimes n_{(1,\alpha)}x_{(2,\alpha)}) \\ &= \lambda_\alpha(p_1(n_{(0,1)}) \otimes n_{(1,\alpha)})x \quad \text{(by Lemma 3.5)} \\ &= \tilde{p}(n)x,\end{aligned}$$

and so $\tilde{p}$ is an A_α-module map.

Next we show that $\tilde{p}$ is a $\pi - H$-comodule map. Indeed, for all $\alpha, \beta \in \pi$, $n \in N_{\alpha,\beta}$, we have

$$\begin{aligned}(\tilde{p} \otimes id)\rho^N_{\alpha,\beta}(n) &= \lambda_\alpha(p_1(n_{(0,\alpha)(0,1)}) \otimes n_{(0,\alpha)(1,\alpha)}) \otimes n_{(1,\beta)} \\ &= \lambda_\alpha(p_1(n_{(0,1)}) \otimes n_{(1,\alpha\beta)(1,\alpha)}) \otimes n_{(1,\alpha\beta)(2,\beta)} \\ &= (\lambda_\alpha \otimes id)\delta_{\alpha,\beta}(p_1(n_{(0,1)}) \otimes n_{(1,\alpha\beta)}) \quad \text{(by Example 2.2)} \\ &= \rho^M_{\alpha,\beta}\lambda_{\alpha\beta}(p_1(n_{(0,1)}) \otimes n_{(1,\alpha\beta)}) \quad \text{(by Lemma 3.6)} \\ &= \rho^M_{\alpha,\beta}\lambda_{\alpha\beta}\tilde{p}(n).\end{aligned}$$

In final, one has $\tilde{p}_\alpha j_\alpha = \lambda_\alpha(p_1 \otimes id)\rho^N_{1,\alpha} j_\alpha = \lambda_\alpha(p_1 \otimes id)(j_1 \otimes id_{H_\alpha})\rho^M_{1,\alpha} = \lambda_\alpha \rho_\alpha = id_{M_\alpha}$, completing the proof. ■

Similar to [D2, Theorem 2], as an application of Theorem 3.7, we have:

THEOREM 3.8. *Let H be a Hopf π-coalgebra. Let A be a right $\pi - H$-comodule algebra with a total integral ϕ, and suppose that* (i) *or* (ii) *in Lemma* 3.6 *holds. Then an (H, A)-Hopf π-comodule P is projective as an A-module iff P is an (H, A)-Hopf π-comodule direct summand of $W \otimes A$ for some $\pi - H$-comodule W, where $W \otimes A$ is regarded as an (H, A)-Hopf π-comodule via $(w \otimes a)b = w \otimes ab$ for any $w \in W_\alpha$, $a, b \in A_\alpha$ and $\rho^{W\otimes A} = \{\rho^{W\otimes A}_{\alpha,\beta} : (W \otimes A)_{\alpha,\beta} \to (W \otimes A)_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ is given by $\rho^{W\otimes A}_{\alpha,\beta}(w \otimes a) = w_{(0,\alpha)} \otimes a_{(0,\alpha)} \otimes w_{(1,\beta)}a_{(1,\beta)}$ for any $\alpha, \beta \in \pi$, $w \in W_{\alpha\beta}$ and $a \in A_{\alpha\beta}$.*

PROOF. For an (H, A)-Hopf π-comodule W, we first show that $W \otimes A = \{(W \otimes A)_\alpha = W_\alpha \otimes A_\alpha\}_{\alpha\in\pi}$ endowed with the above structures is also an (H, A)-Hopf π-comodule. It is easy to see that each $W_\alpha \otimes A_\alpha$ is an A_α-module and to see that $(id_{(W\otimes A)_\alpha} \otimes \varepsilon)\rho_{\alpha,1}^{W\otimes A} = id_{(W\otimes A)_\alpha}$ for any $\alpha \in \pi$. Also, for any $\alpha, \beta, \gamma \in \pi$, $w \in W_{\alpha\beta\gamma}$ and $a \in A_{\alpha\beta\gamma}$, we have

$$\begin{aligned}
&(\rho_{\alpha,\beta}^{W\otimes A} \otimes id)\rho_{\alpha\beta,\gamma}^{W\otimes A}(w \otimes a) \\
&\quad = w_{(0,\alpha\beta)(0,\alpha)} \otimes a_{(0,\alpha\beta)(0,\alpha)} \otimes w_{(0,\alpha\beta)(1,\beta)}a_{(0,\alpha\beta)(1,\beta)} \otimes w_{(1,\gamma)}a_{(1,\gamma)} \\
&\quad = w_{(0,\alpha)} \otimes a_{(0,\alpha)} \otimes w_{(1,\beta\gamma)(1,\beta)}a_{(1,\beta\gamma)(1,\beta)} \otimes w_{(1,\beta\gamma)(2,\gamma)}a_{(1,\beta\gamma)(2,\gamma)} \\
&\quad = (id_{(W\otimes A)_\alpha} \otimes \Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma}^{W\otimes A},
\end{aligned}$$

and this proves that $W \otimes A$ is a π-comodule. The condition of the $\pi - H$-comodule algebra A will implies Eq. (2.13). Hence $W \otimes A$ is an object of $\mathcal{M}_A^{\pi-H}$.

Finally, by the argument similar to [D2, Theorem 2], it suffices to prove the *only if* part holds true since it is obvious that the *if* part holds based on $W \otimes A$ being a free A-module, i.e. each $W_\alpha \otimes A_\alpha$ is a free A_α-module for any $\alpha \in \pi$. Now for $\alpha \in \pi$, let $L_\alpha \subset P_\alpha$ be a subset that generates P_α as an A_α-module, and let W be the smallest $\pi - H$-subcomodule of P satisfying $W_\alpha \supset L_\alpha$, i.e., W_α is an R-submodule of P_α such that $\rho_{\alpha,\beta}^P(W_{\alpha\beta}) \subset W_\alpha \otimes H_\beta$ for any $\alpha, \beta \in \pi$. It is known that if L_α is a finite set then W_α is finite dimensional. Thus we build a $\pi - H$-comodule $W = \{W_\alpha\}_{\alpha\in\pi}$ and so $W \in \mathcal{M}_A^{\pi-H}$. Consider that the module structure map $\phi = \{\phi_\alpha : W_\alpha \otimes A_\alpha \to P_\alpha\}_{\alpha\in\pi}$, $w \otimes a \mapsto wa$ for any $w \in W_\alpha$, $a \in A_\alpha$, is a surjective morphism of $\mathcal{M}_A^{\pi-H}$. Now since P is A-projective and by Theorem 3.7, we have that P is an (H, A)-Hopf π-comodule direct summand of $W \otimes A$, i.e., each P_α is an A_α-module direct summand of $W \otimes A$ such that P is an (H, A)-Hopf π-comodule.

This concludes the proof of the theorem. ■

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Department of Mathematics, Southeast University
Nanjing, Jiangsu 210096, China and Department of Mathematics,
Henan Normal University, XinXiang
E-mail address: shuanhwang2002@yahoo.com