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GORENSTEIN INJECTIVE MODULES AND EXT

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Abstract. The aim of this paper is to characterize n-Gorenstein rings in terms of Gorenstein injective modules and the Ext functor. We will show that if R is a left and right noetherian ring and n is a positive integer, then R is n-Gorenstein if and only if M being Gorenstein injective means that $\mathrm{Ext}^1(L,M)=0$ for all countably generated R-modules L of projective dimension at most n. In particular, if R is n-Gorenstein, then an R-module M is Gorenstein injective if and only if it is U-Gorenstein injective whenever U is a free R-module with a countable base.

1 Introduction

Throughout this paper, R will denote an associate ring with 1 and all modules are unitary. By an R-module, we shall mean a left R-module.

An R-module M is said to be *Gorenstein injective* if there exists an exact sequence

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

of injective R-modules with $M = \mathrm{Ker}(E^0 \to E^1)$ such that $\mathrm{Hom}_R(E,-)$ leaves the sequence exact whenever E is an injective R-module. These modules were first introduced in [1]. Clearly, every injective R-module is Gorenstein injective and a Gorenstein injective R-module has finite injective dimension if and only if it is injective.

It is also easy to see that if M is Gorenstein injective, then $\mathrm{Ext}^1_R(L,M)=0$ for all R-modules L of finite projective dimension. Furthermore, the converse holds if R is *n-Gorenstein* (or *Iwanaga-Gorenstein*), that is, if R is left and right noetherian and has injective dimension at most n on either side (see [2, Prop-

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osition 1.11]). In this paper, we will show that in fact for a positive integer n, a ring R is n-Gorenstein if and only if $\mathrm{Ext}_R^1(L, M) = 0$ for all R-modules L of projective dimension at most n implies M is Gorenstein injective, and that the same holds true if we replace the modules L by countably generated R-modules of projective dimension at most n (Theorem 2.5).

Now let U be an R-module. Then we will say that an R-module M is *U-Gorenstein injective* if $\mathrm{Hom}(U, M) \to \mathrm{Hom}(K, M) \to 0$ is exact for all submodules $K \subset U$ such that $\mathrm{pd}\, U/K < \infty$. We will argue that an R-module M is U-Gorenstein injective for all R-modules U if and only if $\mathrm{Ext}_R^1(L, M) = 0$ for all R-modules L of finite projective dimension (Proposition 3.2). So in particular, over Iwanaga-Gorenstein rings, an R-module M is Gorenstein injective if and only if M is U-Gorenstein injective for all R-modules U and if and only if M is U-Gorenstein injective whenever $U = R \oplus R \oplus R \oplus \cdots$ (that is, U is free with a countable base) (Theorem 3.3).

As usual, we will let pd denote the projective dimension.

2 Main Result

We start with the following two well known results and we include proofs here for completeness.

LEMMA 2.1. *If P is a projective R-module, then $P \oplus F$ is free for some free R-module F.*

PROOF. Since P is a projective R-module, $Q \oplus P$ is free for some R-module Q. So if we let $F = Q \oplus P \oplus Q \oplus P \oplus \cdots$, then F is a free module and the module $P \oplus F$ is also free. This is called the Eilenberg trick. □

REMARK 2.2. We note that Kaplansky [4, Theorem 180] proved this result for projective modules with a finite free resolution.

LEMMA 2.3. *Let n be a positive integer. If $\mathrm{pd}\, L = n$, then L has a free resolution $0 \to F_n \to F_{n-1} \cdots \to F_1 \to F_0 \to L \to 0$.*

PROOF. Since $\mathrm{pd}\, L = n$, $n \geq 1$, L has a projective resolution $0 \to P_n \to F_{n-1} \cdots \to F_1 \to F_0 \to L \to 0$ with each F_i free and P_n projective. But then there is a free R-module Q such that $P_n \oplus Q$ is free by the lemma above. Thus $0 \to P_n \oplus Q \to F_{n-1} \oplus Q \cdots \to F_1 \to F_0 \to L \to 0$ is a free resolution of L. □

We now prove the following result that is analogous to Proposition 7.4.5 of [3].

PROPOSITION 2.4. *Let n be a positive integer. If* $\operatorname{pd} L = n$, *then every countably generated submodule $S \subset L$ is contained in a countably generated submodule L' of L with* $\operatorname{pd} L' \le n$.

PROOF. By Lemma 2.3 above, L has a resolution $0 \to F_n \xrightarrow{\partial_n} \cdots \to F_0 \xrightarrow{\partial_0} L \to 0$ of L with each of $F_n, \ldots, F_0$ free. Let X_i be a base of F_i for each i. Our aim is to choose countable subsets $Y_i \subset X_i$ such that $0 \to \langle Y_n \rangle \to \cdots \to \langle Y_0 \rangle$ is an exact subcomplex of $0 \to F_n \to \cdots \to F_0$. This will give the desired L'.

We start by choosing a countably subset $Z_0 \subset X_0$ such that $\partial_0(\langle Z_0 \rangle) \supset S$. Then we choose a countable subset $Z_1 \subset X_1$ so that $\partial_1(\langle Z_1 \rangle) \supset \operatorname{Ker}(\partial_0 | \langle Z_0 \rangle)$. We then choose a countable subset $Z_2 \subset X_2$ so that $\partial_2(\langle Z_2 \rangle) \supset \operatorname{Ker}(\partial_1 | \langle Z_1 \rangle)$. We repeat this procedure until we have a countable subset $Z_n \subset X_n$ with $\partial_n(\langle Z_n \rangle) \supset \operatorname{Ker}(\partial_{n-1} | \langle Z_{n-1} \rangle)$. We now enlarge Z_{n-1} to a countable subset Z'_{n-1} in such a way that $\partial_n(\langle Z_n \rangle) \subset \langle Z'_{n-1} \rangle$. Then we enlarge Z_{n-2} to a countable Z'_{n-2} so that $\partial_{n-1}(\langle Z'_{n-1} \rangle) \subset \langle Z'_{n-2} \rangle$. Continuing in this manner, we construct countable sets $Z'_n, Z'_{n-1}, \ldots, Z'_0$ satisfying the obvious conditions. Now we start over and enlarge Z'_1 to a countable Z''_1 so that $\partial_1(\langle Z''_1 \rangle) \supset \operatorname{Ker}(\partial_0 | \langle Z'_0 \rangle)$. We then enlarge Z'_2 to Z''_2 and so on. We then continue with this zig-zag procedure and eventually let $Y_i \subset X_i$ be the union of all the countable subsets of X_i we chose at each stage of the procedure.

Then the sequence $0 \to \langle Y_n \rangle \to \cdots \to \langle Y_0 \rangle$ is exact and each Y_i is countable. So we let $L' = \partial_0(\langle Y_0 \rangle)$. □

We are now in a position to prove the following.

THEOREM 2.5. *The following are equivalent for a left and right noetherian ring R and positive integer n.*

1) *R is n-Gorenstein.*
2) *An R-module M is Gorenstein injective if and only if* $\operatorname{Ext}^1(L, M) = 0$ *for all R-modules L with* $\operatorname{pd} L \le n$.
3) *An R-module M is Gorenstein injective if and only if* $\operatorname{Ext}^1(L, M) = 0$ *for all countably generated R-modules L with* $\operatorname{pd} L \le n$.

PROOF. $1 \Rightarrow 2$ follows from Enochs-Jenda [2, Proposition 1.11] noting that if R is n-Gorenstein then $\operatorname{pd} L < \infty$ if and only if $\operatorname{pd} L \le n$.

$2 \Rightarrow 1$. Let N be an R-module and $0 \to N \to E^0 \to E^1 \to \cdots \to E^{n-1} \to C^n \to 0$ be an exact sequence with each E^i injective. Then $\mathrm{Ext}^1(L, C^n) \cong \mathrm{Ext}^{n+1}(L, N)$ for any R-module L. Hence $\mathrm{Ext}^1(L, C^n) = 0$ for all R-modules L with $\mathrm{pd}\, L \leq n$. But then C^n is Gorenstein injective by assumption. So every nth cosyzygy is Gorenstein injective and thus R is n-Gorenstein by [2, Theorem 3.2].

$2 \Rightarrow 3$. Again by definition, if M is Gorenstein injective, then $\mathrm{Ext}^1(L, M) = 0$ for all countably generated R-modules L with $\mathrm{pd}\, L \leq n$.

For the converse, let L be an R-module of finite projective dimension and let $0 \to S \to P \to L \to 0$ be exact with P projective. Set $S_0 = S$. Then there is a countably generated submodule S_1/S_0 of P/S_0 such that $\mathrm{pd}\, S_1/S_0 < \infty$ by the proposition above.

But then $\mathrm{pd}\, P/S_1 < \infty$ since $P/S_1 \cong (P/S_0)/(S_1/S_0)$ and $P/S_0 = L$ and S_1/S_0 have finite projective dimension. Then there is a countably generated submodule S_2/S_1 of P/S_1 with $\mathrm{pd}\, S_2/S_1 < \infty$. We repeat the process to construct a continuous chain of submodules

$$S_0 = S \subset S_1 \subset S_2 \subset \cdots \subset S_\omega = \bigcup_{i=0}^{\infty} S_i \subset S_{\omega+1} \subset \cdots$$

of P such that $S_{\alpha+1}/S_\alpha$ is countably generated and has finite projective dimension. We note that $S_{\alpha+1}/S_\alpha \cong (S_{\alpha+1}/S_0)/(S_\alpha/S_0)$ and so $S_{\alpha+1}/S_0$ is countably generated since $S_{\alpha+1}/S_\alpha$ and S_α/S_0 are. We also note that to construct $S_{\alpha+1}$ we need $\mathrm{pd}\, P/S_\alpha < \infty$ which we establish by transfinite induction. For $P/S_\alpha \cong (P/S_0)/(S_\alpha/S_0)$ and S_α/S_0 is a direct limit of submodules of finite projective dimension and so $\mathrm{pd}\, S_\alpha/S_0 < \infty$ since R is Iwanaga-Gorenstein by $2 \Rightarrow 1$. So indeed $\mathrm{pd}\, P/S_\alpha < \infty$.

Thus given any linear map $f : S \to M$ and the continuous chain $S_0 = S \subset S_1 \subset S_2 \subset \cdots \subset S_\omega \subset S_{\omega+1} \subset \cdots \subset P$ constructed above, we see that f can be extended to $S_1 \to M$ since $\mathrm{Ext}^1(S_1/S_0, M) = 0$ by assumption. Then $S_1 \to M$ can be extended to $S_2 \to M$ and so on to get that f can be extended to $P \to M$. Hence $\mathrm{Ext}^1(L, M) = 0$ for all R-modules L of finite projective dimension. So M is Gorenstein injective by (2) again noting that over n-Gorenstein rings, $\mathrm{pd}\, L < \infty$ if and only if $\mathrm{pd}\, L \leq n$.

$3 \Rightarrow 2$ is trivial. □

3 U-Gorenstein Injective Modules

We start with the following easy observation.

LEMMA 3.1. *Suppose A is an R-submodule of B with* $\mathrm{pd}\, B/A < \infty$. *If an*

R-module M is B-Gorenstein injective, then M is also A and B/A-Gorenstein injective.

Proof. Let $K \subset A$ be such that $\operatorname{pd} A/K < \infty$. Then $\operatorname{pd} B/K < \infty$ and so $\operatorname{Hom}(B, M) \to \operatorname{Hom}(K, M) \to 0$ is exact. But then $\operatorname{Hom}(A, M) \to \operatorname{Hom}(K, M) \to 0$ is exact and so M is A-Gorenstein injective.

Now let $L' \subset B/A$ be such that $\operatorname{pd}(B/A)/L' < \infty$. Then $L' \cong L/A$ for some submodule $L \subset B$ and so we have the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & L/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & B/A \arrow[r] \arrow[d] & 0 \\
 & & B/L \arrow[r, equal] \arrow[d] & B/L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

But B/A has finite projective dimension. So we have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(B/A, M) \arrow[r] \arrow[d] & \operatorname{Hom}(B, M) \arrow[r] \arrow[d] & \operatorname{Hom}(A, M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Hom}(L/A, M) \arrow[r] & \operatorname{Hom}(L, M) \arrow[r] & \operatorname{Hom}(A, M) &
\end{tikzcd}$$

But B/L also has finite projective dimension. So the middle vertical map is surjective. Thus $\operatorname{Hom}(B/A, M) \to \operatorname{Hom}(L/A, M) \to 0$ is exact and so we are done. □

Proposition 3.2. *The following are equivalent for an R-module M.*

1) *M is U-Gorenstein injective for all R-modules U.*
2) *M is P-Gorenstein injective for all projective R-modules P.*
3) *M is F-Gorenstein injective for all free R-modules F.*

4) *M is U-Gorenstein injective for all R-modules U with* $\operatorname{pd} U < \infty$.
5) $\operatorname{Ext}^1(L, M) = 0$ *for all R-modules L with* $\operatorname{pd} L < \infty$.

PROOF. $1 \Rightarrow 2 \Rightarrow 3$ is trivial.

$3 \Leftrightarrow 4$. $3 \Rightarrow 4$ easily follows from the lemma above, and the converse is trivial.

$3 \Rightarrow 5$. Let $0 \to L' \to F \to L \to 0$ be exact with $\operatorname{pd} L < \infty$ and F free. Then $\operatorname{Hom}(F, M) \to \operatorname{Hom}(L', M) \to 0$ is exact by assumption. So $\operatorname{Ext}^1(L, M) = 0$.

$5 \Rightarrow 1$. Let $K \subset U$ be such that $\operatorname{pd} U/K < \infty$. Then $\operatorname{Ext}^1(U/K, M) = 0$ and so we are done. □

We note that the proposition above still holds if we replace "for all" by "for all countably generated". We are now in a position to state the following.

THEOREM 3.3. *Let R be n-Gorenstein. Then the following are equivalent for an R-module M.*

1) *M is Gorenstein injective.*
2) *M is U-Gorenstein injective for all R-modules U.*
3) *M is U-Gorenstein injective for all R-modules U of projective dimension at most n.*
4) *M is U-Gorenstein injective whenever* $U = R \oplus R \oplus R \oplus \cdots$ (*that is, U is free with a countable base*).

PROOF. $1 \Leftrightarrow 2 \Leftrightarrow 3$ follows from Proposition 3.2 and Theorem 2.5 above.

$3 \Rightarrow 4$ is trivial.

$4 \Rightarrow 1$. This follows from the remark above and we state the argument here for completeness. Let L be a countably generated R-module of finite projective dimension. Then L has an exact sequence $0 \to L' \to F \to L \to 0$ where F is a free R-module with a countable base. So $\operatorname{Hom}(F, M) \to \operatorname{Hom}(L', M) \to 0$ is exact by assumption. Thus $\operatorname{Ext}^1(L, M) = 0$. That is, $\operatorname{Ext}^1(L, M) = 0$ for all countably generated R-modules L of finite projective dimension. Hence M is Gorenstein injective by Theorem 2.5. □

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