

EXISTENCE OF ALGEBRAIC MINIMAL SURFACES FOR AN ARBITRARY PUNCTURE SET

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ABSTRACT. We will show that any punctured Riemann surface can be conformally immersed into a Euclidean 3-space as a branched complete minimal surface of finite total curvature called an algebraic minimal surface.

1. INTRODUCTION

The purpose of this paper is to discuss the puncture set problem for algebraic minimal surfaces. The account of the problem is given in the book [3] in the case of unbranched algebraic minimal surfaces.

An *algebraic minimal surface* in \mathbb{R}^3 is a branched complete conformal minimal immersion $X: M \rightarrow \mathbb{R}^3$ of finite total curvature. In this case, we can consider M as a compact Riemann surface \bar{M} with finitely many points, called *puncture points*, removed. We will call the set of distinct points Σ on \bar{M} such that there exists an algebraic minimal surface $X: \bar{M} \setminus \Sigma \rightarrow \mathbb{R}^3$ the *puncture set* of \bar{M} .

We will state the Puncture Set Problem as follows:

The Puncture Set Problem. *Given a compact Riemann surface \bar{M} , determine the set of puncture sets on \bar{M} .*

We have known some results on the Puncture Set Problem. Let \bar{M}_ℓ be a compact Riemann surface of genus ℓ and $\text{PS}_0^r(\bar{M}_\ell)$ the set of r distinct points Σ_r on \bar{M}_ℓ such that there exists an *unbranched* algebraic minimal surface $X: \bar{M}_\ell \setminus \Sigma_r \rightarrow \mathbb{R}^3$. Let $\mathcal{S}(\bar{M}_\ell; n, m, m') = \{(p_1, \dots, p_n, q_1, \dots, q_{m'})\}$ be the set of $n + m'$ distinct points on \bar{M}_ℓ such that $\{p_1, \dots, p_n\} = \text{supp}(F)_-$, $\{q_1, \dots, q_m\} = \text{supp}(dF)_+$ and $\{q_1, \dots, q_m, \dots, q_{m'}\} = \text{supp}(G)_+$, where F and G are meromorphic functions on \bar{M}_ℓ such that $(G)_+ \geq (dF)_+$. Then

- The set $\text{PS}_0^r(\mathbb{C}P^1)$ consists of all r distinct points on $\mathbb{C}P^1$ ([3, pp. 95–100]).
- The set $\mathcal{S}(\bar{M}_\ell; n, m, m')$ is contained in $\text{PS}_0^{n+m'}(\bar{M}_\ell)$ for any $\ell \geq 1$ ([3, p. 127, Proposition 25 and Proposition 26]).

We will consider *branched* algebraic minimal surfaces, too. Let $\text{PS}^r(\bar{M}_\ell)$ be the set of puncture sets consisting of r distinct points on \bar{M}_ℓ . Then, we will prove the following theorem:

Theorem 1.1. *The set $\text{PS}^r(\bar{M}_\ell)$ consists of all r distinct points on \bar{M}_ℓ .*

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In other words, any punctured Riemann surface can be conformally immersed into \mathbb{R}^3 as an algebraic minimal surface.

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2. PRELIMINARIES

In this section, we will recall the Weierstrass representation formula for minimal surfaces. For more details, see [3].

Let \bar{M} be a compact Riemann surface. For a divisor D on \bar{M} , we will denote by $\text{mult}_p D$ the multiplicity of D at p , $\deg D$ the degree of D , $\text{supp } D$ the support of D and D_+ and D_- the effective divisors such that $D = D_+ - D_-$:

$$\begin{aligned} D &= \sum_{p \in \bar{M}} (\text{mult}_p D) \cdot p, \quad \deg D = \sum_{p \in \bar{M}} \text{mult}_p D, \\ \text{supp } D &= \{p \in \bar{M} \mid \text{mult}_p D \neq 0\}, \\ D_+ &= \sum_{p \in \bar{M}} \max\{\text{mult}_p D, 0\} \cdot p, \quad D_- = \sum_{p \in \bar{M}} \max\{-\text{mult}_p D, 0\} \cdot p. \end{aligned}$$

For a meromorphic function g on \bar{M} and a meromorphic one-form ω on \bar{M} , let $\text{ord}_p g$ and $\text{ord}_p \omega$ be the order of g at p and the order of ω at p , respectively. Then, the divisor (g) of g and the divisor (ω) of ω are defined by

$$(g) := \sum_{p \in \bar{M}} (\text{ord}_p g) \cdot p, \quad (\omega) := \sum_{p \in \bar{M}} (\text{ord}_p \omega) \cdot p,$$

respectively.

Let M be an open Riemann surface, $X: M \rightarrow \mathbb{R}^3$ a branched complete conformal minimal immersion and Φ_i the holomorphic one-form on M defined by $\Phi_i = (\partial X_i / \partial z) dz$, where z is a local holomorphic coordinate of M ($i = 1, 2, 3$). The triplet $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ satisfies the following relation:

$$(2.1) \quad (\Phi_1 \otimes \Phi_1) + (\Phi_2 \otimes \Phi_2) + (\Phi_3 \otimes \Phi_3) = 0.$$

We will call the holomorphic map $[\Phi]: M \rightarrow \mathbb{C}P^2$ defined by $[\Phi] := [\Phi_1, \Phi_2, \Phi_3]$ in homogenous coordinates on $\mathbb{C}P^2$ the *generalized Gauss map* of X . The generalized Gauss map is called an *algebraic Gauss map* if M is biholomorphic to a compact Riemann surface with a finite set of points removed and $[\Phi]$ extends holomorphically on M . The removed points are called the *puncture points* of X .

Definition 2.1. We will call a branched complete conformal minimal immersion $X: M \rightarrow \mathbb{R}^3$ an *algebraic minimal surface* if the generalized Gauss map is algebraic.

We can see that the following lemma holds ([1]).

Lemma 2.2. *Let $X: M \rightarrow \mathbb{R}^3$ be a branched complete conformal minimal immersion. Then, the total curvature of X is finite if and only if the generalized Gauss map is algebraic.*

From the above lemma, we can see the following:

Corollary 2.3. *Let $X: M \rightarrow \mathbb{R}^3$ be a branched complete conformal minimal immersion. Then, X is an algebraic minimal surface if and only if the holomorphic one-form Φ_i on M extends meromorphically on a compact Riemann surface \bar{M} ($i = 1, 2, 3$).*

In the following, we will assume that X is an algebraic minimal surface and that M is biholomorphic to a compact Riemann surface \bar{M}_ℓ of genus ℓ with puncture points $\{p_1, \dots, p_r\}$ removed.

Definition 2.4. For $\Phi = (\Phi_1, \Phi_2, \Phi_3)$, we will define the divisor (Φ) of Φ by

$$(\Phi) := \sum_{p \in \bar{M}_\ell} \left(\min_{i=1,2,3} \text{ord}_p \Phi_i \right) \cdot p.$$

Then, we can see that $\text{supp}(\Phi)_+$ is the set of all the branch points of X and $\text{supp}(\Phi)_-$ is the set of all the puncture points of X . From the completeness of X , the multiplicity of $(\Phi)_-$ at each puncture point is equal to or greater than 2:

$$(2.2) \quad \text{supp}(\Phi)_- = \{p_1, \dots, p_r\}, \quad \text{mult}_{p_i}(\Phi)_- \geq 2 \quad (i = 1, \dots, r).$$

Conversely, from a triplet Φ of meromorphic one-forms on \bar{M}_ℓ satisfying the condition (2.1) and (2.2), we can obtain a multi-valued algebraic minimal surface $X: M \rightarrow \mathbb{R}^3$ by integration

$$(2.3) \quad X(p) := \text{Re} \int_{p_0}^p \Phi,$$

where p and p_0 are non-puncture points.

Definition 2.5. We will call the condition (2.2) the *divisor condition*.

If we choose another base point in the integral, then the image of the map shifts by a translation in \mathbb{R}^3 .

We can see that the multi-valued algebraic minimal surface X can be reduced to a single-valued algebraic minimal surface if and only if

$$(2.4) \quad \text{Re} \int_{\gamma_i} \Phi = (0, 0, 0) \quad (i = 1, \dots, 2\ell + r - 1),$$

where $\gamma = \{\gamma_1, \dots, \gamma_{2\ell+r-1}\}$ is a basis of the first homology group $H_1(M, \mathbb{Z})$ of M .

Definition 2.6. We will call the condition (2.4) the *period condition*.

We can obtain a pair (g, ω) consisting of a meromorphic function g on \bar{M}_ℓ and a meromorphic one-form ω on \bar{M}_ℓ which is equivalent to Φ by

$$(g, \omega) = \left(\frac{\Phi_3}{\Phi_1 - \sqrt{-1}\Phi_2}, \Phi_1 - \sqrt{-1}\Phi_2 \right),$$

$$(\Phi_1, \Phi_2, \Phi_3) = (1 - g^2, \sqrt{-1}(1 + g^2), 2g) \frac{\omega}{2}.$$

When the triplet Φ corresponds to a single-valued minimal surface $X: M \rightarrow \mathbb{R}^3$ by (2.3), the pair (g, ω) is called the *Weierstrass data* of X . It is well-known that g is the stereographic projection of the Gauss map of X from the north pole.

We can see that the following relation holds among (g) , (ω) , and (Φ) :

$$(\Phi) = -2(g)_- + (\omega).$$

Then, the divisor condition (2.2) becomes as follows:

$$(2.5) \quad \begin{aligned} \text{supp}(-2(g)_- + (\omega))_- &= \{p_1, \dots, p_r\}, \\ \text{mult}_{p_i}(-2(g)_- + (\omega))_- &\geq 2 \quad (i = 1, \dots, r). \end{aligned}$$

3. PROOF OF THEOREM 1.1

In this section, we will prove Theorem 1.1.

Let $\text{Div}(\bar{M}_\ell)$ be the set of divisors on \bar{M}_ℓ and $\text{Div}_+^d(\bar{M}_\ell) = \{D \in \text{Div}(\bar{M}_\ell) \mid D > 0, \deg D = d\}$. For a divisor $D \in \text{Div}(\bar{M}_\ell)$, let $\mathcal{I}(D)$ be the complex vector space of meromorphic one-forms on \bar{M}_ℓ defined by $\mathcal{I}(D) := \{\xi \mid (\xi) \geq D\}$ and $i(D)$ the complex dimension of $\mathcal{I}(D)$.

We will assume the following lemmas:

Lemma 3.1 ([3, Proposition 3]). *Let ω be a meromorphic one-form on \bar{M}_ℓ . Then, we have $\deg(\omega) = 2\ell - 2$.*

Lemma 3.2 ([3, p. 35]). *Let $-D \in \text{Div}_+^d(\bar{M}_\ell)$. Then, we have $i(D) = \ell + d - 1$.*

Lemma 3.3 ([3, Proposition 10]). *If $D \in \text{Div}_+^d(\bar{M}_\ell)$, $d \geq 2\ell$, then there exists a meromorphic function g such that $(g)_- = D$.*

Then, we can prove the following lemmas:

Lemma 3.4. *For any distinct points $\{p_1, \dots, p_r\} \subset \bar{M}_\ell$, there exists a meromorphic function g on \bar{M}_ℓ such that $\text{supp}(g)_- = \{p_1, \dots, p_r\}$.*

Proof. Let $d \geq 2\ell$ and $D \in \text{Div}_+^d(\bar{M}_\ell)$ such that $\text{supp} D = \{p_1, \dots, p_r\}$. Then, there exists a meromorphic function g such that $(g)_- = D$ by Lemma 3.3. Then, $\text{supp}(g)_- = \text{supp} D = \{p_1, \dots, p_r\}$. \square

Lemma 3.5. *Let $-D \in \text{Div}_+^d(\bar{M}_\ell)$ with $\text{supp} D = \{p_1, \dots, p_r\}$. If g is a meromorphic function on \bar{M}_ℓ such that $\text{supp}(g)_- = \{p_1, \dots, p_r\}$ and $d \leq 2(\text{mult}_{p_i}(g)_- - \ell)$ ($i = 1, \dots, r$), then the pair (g, ω) satisfies the condition (2.5) for any $\omega \in \mathcal{I}(D) \setminus \{0\}$.*

Proof. Let $\omega \in \mathcal{I}(D) \setminus \{0\}$. It is clear that if $p \notin \{p_1, \dots, p_r\}$, then

$$\text{mult}_p(-2(g)_- + (\omega))_- = 0.$$

If $p \in \{p_1, \dots, p_r\}$, then

$$\begin{aligned} \text{mult}_p(-2(g)_- + (\omega))_- &\geq \text{mult}_p(-2(g)_- + (\omega)_+)_- \\ &= \max\{\text{mult}_p(2(g)_- - (\omega)_+), 0\} \\ &\geq \max\{(2\ell + d) - (2\ell + d - 2), 0\} = 2, \end{aligned}$$

since $\deg(\omega)_+ = 2\ell - 2 + \deg(\omega)_- \leq 2\ell + d - 2$ by Lemma 3.1 and the assumption of Lemma 3.5. Hence, the pair (g, ω) satisfies (2.5). \square

Remark 3.6. From the assumption in Lemma 3.5, we can see that $r \leq d$.

Let (g, ω) be a pair satisfying (2.5) and let $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{3(2\ell+r-1)}) : \mathcal{I}(D) \rightarrow \mathbb{R}^{3(2\ell+r-1)}$ be a real linear map defined by

$$\begin{aligned} \mathcal{P}_i(\omega) &= \text{Re} \int_{\gamma_i} (1 - g^2) \omega, \\ \mathcal{P}_{(2\ell+r-1)+i}(\omega) &= \text{Re} \int_{\gamma_i} \sqrt{-1} (1 + g^2) \omega, \\ \mathcal{P}_{2(2\ell+r-1)+i}(\omega) &= \text{Re} \int_{\gamma_i} g \omega \end{aligned}$$

($i = 1, \dots, 2\ell + r - 1$), where $\{\gamma_1, \dots, \gamma_{2\ell+r-1}\}$ is a basis for $H_1(M, \mathbb{Z})$. We will denote by $\ker \mathcal{P}$ the kernel of \mathcal{P} and $\dim_{\mathbb{R}} \ker \mathcal{P}$ the real dimension of $\ker \mathcal{P}$.

Lemma 3.7. *If $2d > 4\ell + 3r - 1$, then $\dim_{\mathbb{R}} \ker \mathcal{P} > 0$.*

Proof. By Lemma 3.2, the real dimension of $\mathcal{I}(D)$ is equal to $2(\ell + d - 1)$. Hence, if $2(\ell + d - 1) > 3(2\ell + r - 1)$, that is, $2d > 4\ell + 3r - 1$, then $\dim_{\mathbb{R}} \ker \mathcal{P} \neq 0$. Thus, $\dim_{\mathbb{R}} \ker \mathcal{P} > 0$. \square

Proof of Theorem 1.1. Let $\{p_1, \dots, p_r\}$ be a set of distinct points on \bar{M}_ℓ and d a positive integer such that $d \geq r$ and $2d > 4\ell + 3r - 1$. By Lemma 3.4, we can see that there exists a meromorphic function g such that $\text{supp}(g)_- = \{p_1, \dots, p_r\}$. We may assume that $2 \text{mult}_{p_i}(g)_- \geq 2\ell + d$ ($i = 1, \dots, r$). Let $-D \in \text{Div}_+^d(\bar{M}_\ell)$ such that $\text{supp } D = \{p_1, \dots, p_r\}$. By Lemma 3.5, the pair (g, ω) satisfies the condition (2.5) for any $\omega \in \mathcal{I}(D) \setminus \{0\}$. We can see that $\ker \mathcal{P} \setminus \{0\}$ is a set of Weierstrass data for algebraic minimal surfaces from $\bar{M}_\ell \setminus \{p_1, \dots, p_r\}$ into \mathbb{R}^3 . By Lemma 3.7, $\dim_{\mathbb{R}} \ker \mathcal{P} > 0$. Hence, $\ker \mathcal{P} \setminus \{0\} \neq \emptyset$. \square

Remark 3.8. When we consider that algebraic minimal surfaces are similar to meromorphic functions on a compact Riemann surface, we may consider that Theorem 1.1 is similar to Lemma 3.4. We will discuss deformations of unbranched algebraic minimal surfaces from this point of view in [2].

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