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ISOTROPIC IMMERSIONS AND PARALLEL IMMERSIONS OF SPACE FORMS INTO SPACE FORMS

By

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Abstract. It is well-known that every parallel isometric immersion of space forms into space forms is isotropic in the sense of O'Neill. We characterize this parallel immersion from this point of view.

1. Introduction

Let $f: M^n \to \tilde{M}^{n+p}$ be an isometric immersion of an n-dimensional Riemannian manifold M^n into an $(n+p)$-dimensional Riemannian manifold $\tilde{M}^{n+p}$ with metric $\langle\,,\rangle$. We recall the notion of isotropic immersions ([O]): Let σ be the second fundamental form of f is said to be *isotropic* at $x \in M^n$ if $\|\sigma(X,X)\|/\|X\|^2$ is constant for all vectors $X(\neq 0)$ on M^n at x. If the immersion is isotropic at every point, then we find a function λ on M^n defined by $x(\in M^n) \mapsto \|\sigma(X,X)\|/\|X\|^2$ and the immersion f is said to be λ-isotropic or simply, isotropic.

On the other hand it is known that all parallel immersions of compact symmetric spaces M of rank one into space forms are isotropic. In this paper we pay attention to the case that the submanifold M is a space form.

Space forms $M^n(c)$ are Riemannian manifolds of constant curvature c which are locally isometric to either one of the standard spheres $S^n(c)$, Euclidean spaces $\boldsymbol{R}^n$ and hyperbolic spaces $\boldsymbol{H}^n(c)$.

We characterize all parallel immersions of space forms $M^n(c)$ into space forms $\tilde{M}^{n+p}(\tilde{c})$ by using the notion of isotropic and two inequalities related to the mean curvature $\mathrm{H}(:= \|\mathfrak{h}\|)$ of $M^n(c)$ in $\tilde{M}^{n+p}(\tilde{c})$, where $\mathfrak{h} := (1/n)$ trace σ.

The purpose of this paper is to prove the following:

THEOREM. *Let f be an isotropic immersion of an n-dimensional compact oriented space form $M^n(c)$ of curvature c into an $(n+p)$-dimensional space form $\tilde{M}^{n+p}(\tilde{c})$ of curvature $\tilde{c}$. Let Δ denote the Laplacian on $M^n(c)$. Suppose that*

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(i) $\mathrm{H}^2 \leq \dfrac{2(n+1)}{n}c - \tilde{c}$,

(ii) $0 \leq (1-n)\Delta \mathrm{H}^2 + n\langle \mathfrak{h}, \Delta \mathfrak{h}\rangle$.

Then $M^n(c)$ is a parallel submanifold of $\tilde{M}^{n+p}(\tilde{c})$. Moreover the immersion f is locally equivalent to one of the following:

(I) *f is a totally umbilic imbedding of $M^n(c)$ into $\tilde{M}^{n+p}(\tilde{c})$, where $c \geq \tilde{c}$. Here* $\mathrm{H}^2 = c - \tilde{c}$.

(II) $f = f_2 \circ f_1 : M^n(c) \xrightarrow{f_1} S^{n+n(n+1)/2-1}((2(n+1)/n)c) \xrightarrow{f_2} \tilde{M}^{n+p}(\tilde{c})$, *where f_1 is a minimal immersion, f_2 is a totally umbilic imbedding and* $(2(n+1)/n)c \geq \tilde{c}$. *Here* $\mathrm{H}^2 = (2(n+1)/n)c - \tilde{c}$.

Note that our theorem is no longer true if we omit the condition (ii) in the hypothesis (for details, see section 4).

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2. Basic Terminology

Here we recall terminology in this paper. Let M^n be an n-dimensional Riemannian submanifold of an $(n+p)$-dimensional Riemannian manifold $\tilde{M}^{n+p}$ with metric $\langle\,,\rangle$ via f. We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation of M^n (resp. $\tilde{M}^{n+p}$). Then the second fundamental form σ of f is defined by $\sigma(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where X and Y are vector fields tangent to M^n. The curvature tensor R of M^n is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, where X, Y and Z are vector fields tangent to M^n. For a vector field ξ normal to M^n, we write $\tilde{\nabla}_X\xi = -A_\xi X + D_X\xi$, where X is a vector field tangent to M^n and $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. the normal) component of $\tilde{\nabla}_X\xi$. The curvature tensor $R^\perp$ of the normal connection D on the normal bundle is defined by $R^\perp(X,Y)\xi = D_X D_Y\xi - D_Y D_X\xi - D_{[X,Y]}\xi$, where X and Y are vector fields tangent to M^n and ξ is a vector field normal to M^n. We define the covariant differentiation ∇' of the second fundamental form σ with respect to the connection in (tangent bundle) + (normal bundle) as follows:

$$(\nabla'_X\sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),$$

where X, Y and Z are vector fields tangent to M^n.

The second fundamental form σ is said to be *parallel* if $\nabla'\sigma = 0$, namely $(\nabla'_X\sigma)(Y,Z) = 0$ for all tangent vector fields X, Y and Z on M^n. We define the second covariant differentiation of the second fundamental form σ with respect to the connection in (tangent bundle) + (normal bundle) as follows:

$$(\nabla'_W\nabla'_X\sigma)(Y,Z) = D_W((\nabla'_X\sigma)(Y,Z)) - (\nabla'_X\sigma)(\nabla_W Y,Z) - (\nabla'_X\sigma)(Y,\nabla_W Z) - (\nabla'_{\nabla_W X}\sigma)(Y,Z),$$

where X, Y, Z and W are vector fields tangent to M^n.

We choose local orthonormal frames $\{e_1,\dots,e_n\}$ on M^n and define the mean curvature vector field $\mathfrak{h}$ as $\mathfrak{h} = \sum_{i=1}^n (1/n)\sigma(e_i,e_i)$. The mean curvature H of M^n is defined by $\mathrm{H} = \|\mathfrak{h}\|$. The mean curvature vector field $\mathfrak{h}$ is said to be *parallel* if $D\mathfrak{h} = 0$, namely $D_X\mathfrak{h} = 0$ for all tangent vector fields X on M^n. We denote by $\|\sigma\|$ the length of the second fundamental form σ, namely $\|\sigma\|^2 = \sum_{i,j=1}^n \langle\sigma(e_i,e_j),\sigma(e_i,e_j)\rangle$. $\|\nabla'\sigma\|$ is the length of the derivative of the second fundamental form σ, namely

$$\|\nabla'\sigma\|^2 = \sum_{i,j,k=1}^n \langle(\nabla'_{e_k}\sigma)(e_i,e_j),(\nabla'_{e_k}\sigma)(e_i,e_j)\rangle.$$

$\|D\mathfrak{h}\|$ denotes the length of the derivative of the mean curvature vector field $\mathfrak{h}$, namely

$$\|D\mathfrak{h}\|^2 = \sum_{i=1}^n \langle D_{e_i}\mathfrak{h}, D_{e_i}\mathfrak{h}\rangle.$$

3. Proof of Theorem

First of all we review the following lemma due to J. Simons [Si]. We here write the proof of the lemma in detail for readers.

LEMMA. *If M^n be an n-dimensional Riemannian submanifold of an $(n+p)$-dimensional space form $\tilde{M}^{n+p}(\tilde{c})$, then we can get the following equation:*

$$(3.1)\qquad \frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - \tilde{c}n^2\mathrm{H}^2 + \tilde{c}n\|\sigma\|^2 + \sum_{i,j,k=1}^n \langle D_{e_i}(D_{e_j}(\sigma(e_k,e_k))),\sigma(e_i,e_j)\rangle$$
$$+ \sum_{i,j,k,l=1}^n [2\langle\sigma(e_k,e_j),\sigma(e_i,e_l)\rangle\langle\sigma(e_l,e_k),\sigma(e_i,e_j)\rangle - 2\langle\sigma(e_k,e_j),\sigma(e_k,e_l)\rangle\langle\sigma(e_l,e_i),\sigma(e_i,e_j)\rangle$$
$$+ \langle\sigma(e_k,e_k),\sigma(e_i,e_l)\rangle\langle\sigma(e_l,e_j),\sigma(e_i,e_j)\rangle - \langle\sigma(e_i,e_j),\sigma(e_l,e_k)\rangle\langle\sigma(e_l,e_k),\sigma(e_i,e_j)\rangle],$$

where Δ *denotes the* Laplacian *on* M^n *and* $\{e_1, \dots, e_n\}$ *are local orthonormal frames on* M^n.

PROOF. We define local orthonormal frames $\{e_1, \dots, e_n\}$ around a point x of M^n by using parallel displacement for the vector e_i $(i = 1, \dots, n)$ at x along each geodesic with origin x. Then we easily find that $(\nabla_{e_i} e_j)(x) = 0$ $(i, j = 1, \dots, n)$. By direct computation, using the notation in section 2, we have

$$
\begin{aligned}
\frac{1}{2}\Delta\|\sigma\|^2 &= \frac{1}{2}\sum_{i,j,k=1}^{n} e_k(e_k(\langle\sigma(e_i, e_j), \sigma(e_i, e_j)\rangle)) \\
&= \sum_{i,j,k=1}^{n} e_k(\langle D_{e_k}(\sigma(e_i, e_j)), \sigma(e_i, e_j)\rangle) \\
&= \sum_{i,j,k=1}^{n} [\langle D_{e_k}(\sigma(e_i, e_j)), D_{e_k}(\sigma(e_i, e_j))\rangle + \langle D_{e_k}(D_{e_k}(\sigma(e_i, e_j))), \sigma(e_i, e_j)\rangle] \\
&= \|\nabla'\sigma\|^2 + \sum_{i,j,k=1}^{n} \langle D_{e_k}(D_{e_i}(\sigma(e_j, e_k))), \sigma(e_i, e_j)\rangle \\
&= \|\nabla'\sigma\|^2 + \sum_{i,j,k=1}^{n} [\langle D_{e_k}(D_{e_i}(\sigma(e_j, e_k))), \sigma(e_i, e_j)\rangle \\
&\quad - \langle D_{e_i}(D_{e_k}(\sigma(e_j, e_k))), \sigma(e_i, e_j)\rangle + \langle D_{e_i}(D_{e_k}(\sigma(e_j, e_k))), \sigma(e_i, e_j)\rangle] \\
&= \|\nabla'\sigma\|^2 + \sum_{i,j,k=1}^{n} [\langle\sigma(R(e_i, e_k)e_j, e_k), \sigma(e_i, e_j)\rangle \\
&\quad + \langle\sigma(e_j, R(e_i, e_k)e_k), \sigma(e_i, e_j)\rangle - \langle R^{\perp}(e_i, e_k)(\sigma(e_j, e_k)), \sigma(e_i, e_j)\rangle \\
&\quad + \langle D_{e_i}(D_{e_j}(\sigma(e_k, e_k))), \sigma(e_i, e_j)\rangle].
\end{aligned}
$$

Here we compute each term in the right-hand side in the this equation. We first get

$$
\begin{aligned}
&\sum_{i,j,k=1}^{n} \langle\sigma(R(e_i, e_k)e_j, e_k), \sigma(e_i, e_j)\rangle \\
&\qquad = \sum_{i,j,k=1}^{n} \langle\sigma(\tilde{c}(\delta_{kj}e_i - \delta_{ij}e_k) + A_{\sigma(e_k, e_j)}e_i - A_{\sigma(e_i, e_j)}e_k, e_k), \sigma(e_i, e_j)\rangle
\end{aligned}
$$

$$= \tilde{c}\|\sigma\|^2 - \tilde{c}n^2\mathrm{H}^2 + \sum_{i,j,k,l=1}^{n} [\langle\sigma(e_k, e_j), \sigma(e_i, e_l)\rangle\langle\sigma(e_l, e_k), \sigma(e_i, e_j)\rangle$$

$$- \langle\sigma(e_i, e_j), \sigma(e_k, e_l)\rangle\langle\sigma(e_l, e_k), \sigma(e_i, e_j)\rangle].$$

We compute similarly

$$\sum_{i,j,k=1}^{n} \langle\sigma(e_j, R(e_i, e_k)e_k), \sigma(e_i, e_j)\rangle$$

$$= \tilde{c}n\|\sigma\|^2 - \tilde{c}\|\sigma\|^2 + \sum_{i,j,k,l=1}^{n} [\langle\sigma(e_k, e_k), \sigma(e_i, e_l)\rangle\langle\sigma(e_l, e_j), \sigma(e_i, e_j)\rangle$$

$$- \langle\sigma(e_i, e_k), \sigma(e_k, e_l)\rangle\langle\sigma(e_l, e_j), \sigma(e_i, e_j)\rangle].$$

Next, we have

$$\sum_{i,j,k=1}^{n} \langle R^{\perp}(e_i, e_k)(\sigma(e_j, e_k)), \sigma(e_i, e_j)\rangle$$

$$= \sum_{i,j,k=1}^{n} [\langle D_{e_i}(D_{e_k}(\sigma(e_j, e_k))), \sigma(e_i, e_j)\rangle - \langle D_{e_k}(D_{e_i}(\sigma(e_j, e_k))), \sigma(e_i, e_j)\rangle]$$

$$= \sum_{i,j,k=1}^{n} [\langle\tilde{\nabla}_{e_i}(D_{e_k}(\sigma(e_j, e_k))), \sigma(e_i, e_j)\rangle - \langle\tilde{\nabla}_{e_k}(D_{e_i}(\sigma(e_j, e_k))), \sigma(e_i, e_j)\rangle]$$

$$= \sum_{i,j,k=1}^{n} [\langle\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_k}(\sigma(e_j, e_k)), \sigma(e_i, e_j)\rangle + \langle\tilde{\nabla}_{e_i}(A_{\sigma(e_j, e_k)}e_k), \sigma(e_i, e_j)\rangle$$

$$- \langle\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_i}(\sigma(e_j, e_k)), \sigma(e_i, e_j)\rangle - \langle\tilde{\nabla}_{e_k}(A_{\sigma(e_j, e_k)}e_i), \sigma(e_i, e_j)\rangle]$$

$$= \sum_{i,j,k=1}^{n} [\langle\tilde{R}(e_i, e_k)(\sigma(e_j, e_k)), \sigma(e_i, e_j)\rangle - \langle A_{\sigma(e_j, e_k)}e_k, \tilde{\nabla}_{e_i}(\sigma(e_i, e_j))\rangle$$

$$+ \langle A_{\sigma(e_j, e_k)}e_i, \tilde{\nabla}_{e_k}(\sigma(e_i, e_j))\rangle]$$

(where $\tilde{R}$ is the curvature tensor of $\tilde{M}^{n+p}(\tilde{c})$)

$$= \sum_{i,j,k=1}^{n} [\langle A_{\sigma(e_j, e_k)}e_k, A_{\sigma(e_i, e_j)}e_i\rangle - \langle A_{\sigma(e_j, e_k)}e_i, A_{\sigma(e_i, e_j)}e_k\rangle]$$

(because $\tilde{R}(e_i, e_k)(\sigma(e_j, e_k)) = \tilde{c}(\langle e_k, \sigma(e_j, e_k)\rangle e_i - \langle e_i, \sigma(e_j, e_k)\rangle e_k) = 0$)

$$= \sum_{i,j,k=1}^{n} [\langle\sigma(A_{\sigma(e_j, e_k)}e_k, e_i), \sigma(e_i, e_j)\rangle - \langle\sigma(A_{\sigma(e_j, e_k)}e_i, e_k), \sigma(e_i, e_j)\rangle]$$

$$= \sum_{i,j,k,l=1}^{n} [\langle\sigma(e_j, e_k), \sigma(e_l, e_k)\rangle\langle\sigma(e_l, e_i), \sigma(e_i, e_j)\rangle$$

$$- \langle\sigma(e_j, e_k), \sigma(e_l, e_i)\rangle\langle\sigma(e_l, e_k), \sigma(e_i, e_j)\rangle].$$

Therefore by virtue of the above computation, we obtain equation (3.1). Q.E.D.

In the following we study a λ-isotropically isometric immersion $f : M^n(c) \to \tilde{M}^{n+p}(\tilde{c})$. It is known that the following equation holds (for details, see page 46 in [Su]):

$$(3.2) \quad \langle\sigma(X, Y), \sigma(Z, W)\rangle$$
$$= \frac{c - \tilde{c}}{3}(2\langle X, Y\rangle\langle Z, W\rangle - \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle)$$
$$+ \frac{\lambda^2}{3}(\langle X, Y\rangle\langle Z, W\rangle + \langle X, Z\rangle\langle Y, W\rangle + \langle X, W\rangle\langle Y, Z\rangle),$$

where X, Y, Z and W are vector fields tangent to M^n.

Equation (3.2) yields the following:

$$(3.3) \qquad 3\|\sigma(X, Y)\|^2 + c - \tilde{c} = \lambda^2,$$

where X and Y are orthonormal vector fields tangent to M^n.

$$(3.4) \qquad \mathrm{H}^2 = \frac{2(n-1)(c - \tilde{c}) + \lambda^2(n+2)}{3n}.$$

$$(3.5) \qquad \|\sigma\|^2 = n^2\mathrm{H}^2 - n(n-1)(c - \tilde{c}).$$

Here we compute the fourth term of the right-hand side in our lemma. In order to compute this term easily we use again the condition that $\nabla e_i = 0$ at the point x, $i \in \{1, \dots, n\}$. It follows from the Codazzi equation $(\nabla'_X\sigma)(Y, Z) = (\nabla'_Y\sigma)(X, Z)$, (3.2) and (3.4) that

$$\sum_{i,j,k=1}^{n} \langle D_{e_i}(D_{e_j}(\sigma(e_k, e_k))), \sigma(e_i, e_j)\rangle$$

$$= \sum_{i,j,k=1}^{n} [e_i(\langle D_{e_j}(\sigma(e_k, e_k)), \sigma(e_i, e_j)\rangle) - \langle D_{e_j}(\sigma(e_k, e_k)), D_{e_i}(\sigma(e_i, e_j))\rangle]$$

$$= \sum_{i,j,k=1}^{n} [e_i(e_j(\langle\sigma(e_k,e_k),\sigma(e_i,e_j)\rangle)) - e_i(\langle\sigma(e_k,e_k),D_{e_j}(\sigma(e_i,e_j))\rangle)$$

$$- \langle D_{e_j}(\sigma(e_k,e_k)),D_{e_j}(\sigma(e_i,e_i))\rangle]$$

$$= \sum_{i,j,k=1}^{n} [e_i(e_j(\langle\sigma(e_k,e_k),\sigma(e_i,e_j)\rangle)) - \langle\sigma(e_k,e_k),D_{e_i}(D_{e_i}(\sigma(e_j,e_j)))\rangle$$

$$- 2\langle D_{e_j}(\sigma(e_k,e_k)),D_{e_j}(\sigma(e_i,e_i))\rangle]$$

$$= \sum_{i,j,k=1}^{n} \left[e_i\left(e_j\left[\frac{c-\tilde{c}}{3}(2\delta_{ij}-2\delta_{ki}\delta_{kj}) + \frac{3n\mathrm{H}^2-2(n-1)(c-\tilde{c})}{3(n+2)}(\delta_{ij}+2\delta_{ki}\delta_{kj})\right]\right)\right.$$

$$\left.- \langle\sigma(e_k,e_k),D_{e_i}(D_{e_i}(\sigma(e_j,e_j)))\rangle - 2\langle D_{e_j}(\sigma(e_k,e_k)),D_{e_j}(\sigma(e_i,e_i))\rangle\right]$$

$$= n\Delta\mathrm{H}^2 - n^2\langle\mathfrak{h},\Delta\mathfrak{h}\rangle - 2n^2\|D\mathfrak{h}\|^2$$

$$= n\Delta\mathrm{H}^2 + n^2\langle\mathfrak{h},\Delta\mathfrak{h}\rangle - n^2(2\|D\mathfrak{h}\|^2 + 2\langle\mathfrak{h},\Delta\mathfrak{h}\rangle)$$

$$= n(1-n)\Delta\mathrm{H}^2 + n^2\langle\mathfrak{h},\Delta\mathfrak{h}\rangle.$$

Therefore we can get the following equation:

$$\sum_{i,j,k=1}^{n} \langle D_{e_i}(D_{e_j}(\sigma(e_k,e_k))),\sigma(e_i,e_j)\rangle = n((1-n)\Delta\mathrm{H}^2 + n\langle\mathfrak{h},\Delta\mathfrak{h}\rangle). \tag{3.6}$$

Using our lemma, (3.2), (3.4), (3.5) and (3.6), we obtain the following equation:

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - \frac{n^3(n-1)}{n+2}(\mathrm{H}^2 - c + \tilde{c})\left(\mathrm{H}^2 - \frac{2(n+1)}{n}c + \tilde{c}\right)$$

$$+ n((1-n)\Delta\mathrm{H}^2 + n\langle\mathfrak{h},\Delta\mathfrak{h}\rangle).$$

Here it follows from (3.3) and (3.4) that

$$\mathrm{H}^2 - c + \tilde{c} = \frac{n+2}{n}(\lambda^2 - c + \tilde{c}) = \frac{n+2}{n}\|\sigma(X,Y)\|^2 \geq 0,$$

for each orthonormal pair of vectors X and Y.

This, together with the inequalities (i), (ii) in the assumption of our theorem and a well-known Hopf's lemma, yields that $\nabla'\sigma = 0$. Moreover we have $\mathrm{H}^2 \equiv c - \tilde{c}$ or $\mathrm{H}^2 \equiv (2(n+1)/n)c - \tilde{c}$. Therefore we get the conclusion ([F]).

4. Remarks

(1) We comment on the inequality $0 \leq (1-n)\Delta H^2 + n\langle \mathfrak{h}, \Delta\mathfrak{h}\rangle$. By easy computation we know that this inequality means that the mean curvature vector $\mathfrak{h}$ is parallel when the mean curvature H is constant.

In fact, when H is constant, from the inequality $0 \leq (1-n)\Delta \mathrm{H}^2 + n\langle \mathfrak{h}, \Delta\mathfrak{h}\rangle$ we know that $0 \leq \langle \mathfrak{h}, \Delta\mathfrak{h}\rangle$. Again, by using the condition that H is constant, we get $\|D\mathfrak{h}\|^2 = -\langle \mathfrak{h}, \Delta\mathfrak{h}\rangle$. It follows from these two inequalities that $D\mathfrak{h} = 0$.

(2) As an immediate consequence of our theorem we obtain the following:

COROLLARY. *Let f be an isotropic immersion of an n-dimensional compact oriented space form $M^n(c)$ of curvature c into an $(n+p)$-dimensional space form $\tilde{M}^{n+p}(\tilde{c})$ of curvature $\tilde{c}$. Suppose that*

(i) $\mathrm{H}^2 \leq \dfrac{2(n+1)}{n}c - \tilde{c}$,

(ii) *the mean curvature vector $\mathfrak{h}$ is parallel.*

Then $M^n(c)$ is a parallel submanifold of $\tilde{M}^{n+p}(\tilde{c})$. Moreover the immersion f is locally equivalent to one of the following:

(I) *f is a totally umbilic imbedding of $M^n(c)$ into $\tilde{M}^{n+p}(\tilde{c})$, where $c \geq \tilde{c}$. Here $\mathrm{H}^2 = c - \tilde{c}$.*

(II) *$f = f_2 \circ f_1 : M^n(c) \xrightarrow{f_1} S^{n+n(n+1)/2-1}((2(n+1)/n)c) \xrightarrow{f_2} \tilde{M}^{n+p}(\tilde{c})$, where f_1 is a minimal immersion, f_2 is a totally umbilic imbedding and $(2(n+1)/n)c \geq \tilde{c}$. Here $\mathrm{H}^2 = (2(n+1)/n)c - \tilde{c}$.*

(3) *We show that our theorem is no longer true if we omit the condition* (ii) *in the hypothesis. We recall the following example due to Maeda* ([M]).

EXAMPLE. Let $\chi_1 : S^n(n/(2(n+1))) \to S^{n+(n(n+1)/2)-1}(1)$ be the second standard minimal immersion and $\chi_2 : S^n(n/(2(n+1))) \to S^n(n/(2(n+1)))$ be the identity mapping. Using these minimal immersions, for $t \in (0, \pi/2)$ we define the following minimal immersion:

(4.1)
$$\chi_t(= (\chi_1, \chi_2)) : S^n\left(\frac{n}{2(n+1)}\right) \to S^{n+n(n+1)/2-1}\left(\frac{1}{\cos^2 t}\right) \times S^n\left(\frac{n}{2(n+1)\sin^2 t}\right).$$

Here the differential map $(\chi_t)_*$ of χ_t is given by $(\chi_t)_*X = (\cos t \cdot (\chi_1)_*X, \sin t \cdot (\chi_2)_*X)$ for each $X \in TS^n(n/(2(n+1)))$. The product space of spheres in (4.1) can be imbedded into a sphere as a Clifford hypersurface:

(4.2)
$$S^{n+n(n+1)/2-1}\left(\frac{1}{\cos^2 t}\right) \times S^n\left(\frac{n}{2(n+1)\sin^2 t}\right) \to S^{n+n(n+3)/2}\left(\frac{n}{n+(n+2)\sin^2 t}\right).$$

Combining (4.1) with (4.2), we obtain the following isometric immersion f_t:

(4.3)
$$f_t : S^n\left(\frac{n}{2(n+1)}\right) \to S^{n+n(n+3)/2}\left(\frac{n}{n+(n+2)\sin^2 t}\right).$$

By virtue of the result in [M], we obtain the following properties of f_t for each $t \in (0, \pi/2)$:

(a) the mean curvature H_t of f_t is given by

$$\mathrm{H}_t = \|\mathfrak{h}_t\| = \frac{(n+2)\sin t \cos t}{\sqrt{2(n+1)(n+(n+2)\sin^2 t)}} \neq 0,$$

(b) the mean curvature vector $\mathfrak{h}_t$ of f_t is not parallel. The length of the derivative of $\mathfrak{h}_t$ is given by:

$$\|D\mathfrak{h}_t\|^2 = \frac{n(n+2)^2}{4(n+1)^2}\sin^2 t \cos^2 t \neq 0.$$

(c) f_t is constant λ_t-isotropic. λ_t is given by

$$\lambda_t = \sqrt{\cos^4 t\frac{n-1}{n+1} + \frac{(\tilde{c}_1 \cos^2 t - \tilde{c}_2 \sin^2 t)^2}{\tilde{c}_1 + \tilde{c}_2}} \neq 0,$$

where $\tilde{c}_1 = 1/\cos^2 t$ and $\tilde{c}_2 = n/2(n+1)\sin^2 t$.

Now, in particular we set $\cos t = 1/\sqrt{n+1}$ and $\sin t = \sqrt{n/(n+1)}$. Then we have the following isometric immersion f:

(4.4)
$$f : S^n\left(\frac{n}{2(n+1)}\right) \to S^{n+n(n+1)/2-1}(n+1) \times S^n\left(\frac{1}{2}\right) \to S^{n+n(n+3)/2}\left(\frac{n+1}{2n+3}\right).$$

We shall show that this isometric immersion f given by (4.4) satisfies the inequality (i) but not the equality (ii) in the statement of our Theorem.

In fact, we have

(i)
$$\mathrm{H}^2 - \frac{2(n+1)}{n}c + \tilde{c} = \frac{(n+2)^2}{2(2n+3)(n+1)^2} - 1 + \frac{n+1}{2n+3}$$
$$= -\frac{n(n+2)}{2(n+1)^2} < 0,$$

(ii)
$$(1-n)\Delta \mathrm{H}^2 + n\langle \mathfrak{h}, \Delta\mathfrak{h}\rangle = n\langle \mathfrak{h}, \Delta\mathfrak{h}\rangle$$
$$= -n\|D\mathfrak{h}\|^2 = -\frac{n^3(n+2)^2}{4(n+1)^4} < 0.$$

This shows that our Theorem does not hold without the inequality (ii).

We finally note that the isometric immersion f given by (4.4) is a counterexample to theorem 5.1 in [IO].

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