

## A NOTE ON SPACES WITH A $\sigma$ -COMPACT-FINITE WEAK BASE\*

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**Abstract.** In this paper spaces with a  $\sigma$ -compact-finite weak base are discussed, and some characterizations of  $g$ -metrizable spaces are obtained by spaces with  $\sigma$ -compact-finite weak base and spaces with a  $\sigma$ -weakly hereditarily closure-preserved weak base.

In this paper all spaces are  $T_2$ . Readers may refer to [2] and [6] for unstated definitions.

Let  $\mathcal{P}$  be a family of subsets of a space  $X$ .  $\mathcal{P}$  is called *compact-finite* if any compact subset of  $X$  meets at most finitely many members of  $\mathcal{P}$ ;  $\mathcal{P}$  is called *closure-preserved* if  $\overline{\bigcup \mathcal{P}'} = \bigcup \{\bar{P} : P \in \mathcal{P}'\}$  for each  $\mathcal{P}' \subset \mathcal{P}$ ;  $\mathcal{P}$  is called *hereditarily closure-preserving* if a family  $\{H(P) : P \in \mathcal{P}\}$  is closure-preserving for each  $H(P) \subset P \in \mathcal{P}$ ;  $\mathcal{P}$  is called *weakly hereditarily closure-preserving* if a family  $\{p(P) : P \in \mathcal{P}\}$  is closure-preserving for each  $p(P) \in P \in \mathcal{P}$ .

Obviously, a locally finite family for a space is compact-finite and hereditarily closure-preserving, a hereditarily closure-preserving family is closure-preserving and weakly hereditarily closure-preserving. In a  $k$ -space, a compact-finite family is a weakly hereditarily closure-preserving family. In certain conditions spaces determined by hereditarily closure-preserving families have some similar properties with spaces determined by compact-finite families.

First, we discuss some properties of weakly hereditarily closure-preserving families. Let  $x \in P \subset X$ .  $P$  is called a *sequential neighborhood* of  $x$  in  $X$  if whenever  $\{x_n\}$  is a sequence converging to the point  $x$ , then  $\{x_n : n \geq m\} \subset P$  for some  $m \in \mathbb{N}$ .

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The following Lemmas can be checked directly.

LEMMA 1. *Let  $\mathcal{P}$  be a weakly hereditarily closure-preserving family of a space  $X$ . If  $\mathcal{P}$  is a family of sequential neighborhoods of a point  $x$  and there is a non-trivial sequence converging to  $x$  in  $X$ , then  $\mathcal{P}$  is finite.*  $\square$

LEMMA 2. *Every point-finite and weakly hereditarily closure-preserving family is compact-finite.*  $\square$

LEMMA 3. *Let  $\mathcal{P}$  be a weakly hereditarily closure-preserving family of a space  $X$ . Put  $D = \{x \in X : \mathcal{P} \text{ is not point-finite at } x\}$ . Then  $\{P \setminus D : P \in \mathcal{P}\} \cup \{\{x\} : x \in D\}$  is compact-finite.*

PROOF. Since  $\{P \setminus D : P \in \mathcal{P}\}$  is a point-finite and weakly hereditarily closure-preserving family of  $X$ , it is compact-finite by Lemma 2. If  $K \cap D$  is infinite for some compact subset  $K$  of  $X$ , there are an infinite subset  $\{x_i : i \in \mathbb{N}\}$  of  $K$  and a subset  $\{P_i : i \in \mathbb{N}\}$  of  $\mathcal{P}$  such that each  $x_i \in P_i$ , thus  $\{x_i : i \in \mathbb{N}\}$  is closed discrete in  $K$ , a contradiction. Therefore,  $\{P \setminus D : P \in \mathcal{P}\} \cup \{\{x\} : x \in D\}$  is compact-finite.  $\square$

If  $X$  is a  $k$ -space, then  $D$  in Lemma 3 is a closed discrete subset of  $X$ .

Let  $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$  be a cover of a space  $X$  such that for each  $x \in X$ ,

(1)  $\mathcal{P}_x$  is a *network* of  $x$  in  $X$ , i.e.,  $x \in \bigcap \mathcal{P}_x$  and for  $x \in U$  with  $U$  open in  $X$ ,  $P \subset U$  for some  $P \in \mathcal{P}_x$ .

(2) If  $U, V \in \mathcal{P}_x$ ,  $W \subset U \cap V$  for some  $W \in \mathcal{P}_x$ .

$\mathcal{P}$  is a *weak base* for  $X$  if whenever  $G \subset X$  satisfying for each  $x \in G$  there is a  $P \in \mathcal{P}_x$  with  $P \subset G$ , then  $G$  is open in  $X$ .  $\mathcal{P}$  is an *sn-network* [7] for  $X$  if each member of  $\mathcal{P}_x$  is a sequential neighborhood of  $x$  in  $X$  for each  $x \in X$ .

$\mathcal{P}_x$  above is called a *wn-network* and an *sn-network* of  $x$ , respectively. Every *wn-network* at  $x$  is an *sn-network* at  $x$  [6, Corollary 1.6.18]. A space  $X$  is called a *gf-countable space* if each point of  $X$  has a countable *wn-network*. A regular space with a  $\sigma$ -locally finite weak base is called a *g-metrizable space* [10].

Every *g-metrizable space* is a *gf-countable space*, every *gf-countable space* is a sequential space, and every sequential space is a *k-space*.

For a space  $X$ , denote  $I = \{x \in X : x \text{ is an isolated point of } X\}$ .

THEOREM 1. *The following are equivalent for a space  $X$ :*

- (1)  $X$  has a  $\sigma$ -compact-finite weak base.
- (2)  $X$  is a  $k$ -space with a  $\sigma$ -weakly hereditarily closure-preserving weak base.
- (3)  $X$  is a  $gf$ -countable space with a  $\sigma$ -weakly hereditarily closure-preserving weak base.

PROOF. We shall show that (2)  $\Rightarrow$  (3)  $\Rightarrow$  (1). Let  $X$  be a  $k$ -space with a  $\sigma$ -weakly hereditarily closure-preserving weak base.  $X$  has a  $\sigma$ -compact-finite network by Lemma 3, thus any compact subset of  $X$  has a countable network, hence any compact subset of  $X$  is metrizable [2, Theorem 3.1.19], and so  $X$  is a sequential space.  $X$  is  $gf$ -countable space by Lemma 1.

Let  $\mathcal{P} = \bigcup_{n \in N} \mathcal{P}_n$  be a  $\sigma$ -weakly hereditarily closure-preserving weak base for a  $gf$ -countable space  $X$ , here each  $\mathcal{P}_n$  is a weakly hereditarily closure-preserving family and  $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ . For each  $x \in X$  put  $\mathcal{H}_x = \{P \in \mathcal{P} : P \text{ is a sequential neighborhood of } x \text{ in } X\}$ . If  $x \in I$ , then  $\{x\}$  is open in  $X$ , thus  $\{x\} \in \mathcal{P}$ , so  $I$  is a  $\sigma$ -closed discrete subspace of  $X$ . For each  $n \in N$ , and  $P \in \mathcal{P}_n$ , put

$$D_n = \{x \in X : \mathcal{P}_n \text{ is not point-finite at } x\},$$

$$W_n(P) = (P \setminus D_n) \cup \{x \in X \setminus I : P \in \mathcal{H}_x\}.$$

Then  $W_n(P) \subset P$ . And put  $\mathcal{W}_n = \{W_n(P) : P \in \mathcal{P}_n\}$ . Then  $\mathcal{W}_n$  is point-finite. In fact, for each  $x \in X$  we can assume that  $x \in X \setminus I$  by the point-finiteness of the family  $\{P \setminus D_n : P \in \mathcal{P}_n\}$ ,  $\mathcal{H}_x \cap \mathcal{P}_n$  is finite by Lemma 1, thus  $\mathcal{W}_n$  is point-finite. And  $\mathcal{W}_n$  is compact-finite by Lemma 2.

For each  $x \in X$ , take  $\mathcal{B}_x = \{\{x\}\}$  if  $x \in I$ , take  $\mathcal{B}_x = \{W_n(P) : n \in N, P \in \mathcal{H}_x \cap \mathcal{P}_n\}$  if  $x \in X \setminus I$ , we shall show that the subset  $\bigcup_{x \in X} \mathcal{B}_x$  of  $\bigcup_{n \in N} \mathcal{W}_n \cup \{\{x\} : x \in I\}$  is a weak base for  $X$ . First, for each  $x \in X$  and any open neighborhood  $G$  of  $x$  in  $X$ , suppose that  $x \in X \setminus I$ , then there are an  $n \in N$  and a  $P \in \mathcal{H}_x \cap \mathcal{P}_n$  with  $P \subset G$ , thus  $x \in W_n(P) \subset P \subset G$ . Secondly, for each  $x \in X \setminus I$ , and  $U, V \in \mathcal{B}_x$ , there are  $n, m \in N$  and  $P \in \mathcal{H}_x \cap \mathcal{P}_n, Q \in \mathcal{H}_x \cap \mathcal{P}_m$  such that  $U = W_n(P), V = W_m(Q)$ , thus there are a  $k \geq \max\{n, m\}$  and  $R \in \mathcal{H}_x \cap \mathcal{P}_k$  with  $R \subset P \cap Q$ , hence  $W_k(R) \subset W_n(P) \cap W_m(Q)$ . Thirdly,  $\mathcal{B}_x$  is an  $sn$ -network of  $x$  in  $X$ . In fact, for each  $x \in X \setminus I, n \in N$  and  $P \in \mathcal{H}_x \cap \mathcal{P}_n$ , let  $\{x_i\}$  be a sequence converging to  $x$  in  $X$ , then  $\{x_i\}$  is eventually in  $P$ , so  $(\{x_i : i \in N\} \cup \{x\}) \cap D_n$  is finite by Lemma 3, hence  $\{x_i\}$  is eventually in  $(P \setminus D_n) \cup \{x\} \subset W_n(P)$ , therefore  $W_n(P)$  is a sequential neighborhood of  $x$  in  $X$ . Thus  $\mathcal{B}_x$  is an  $sn$ -network of  $x$  in  $X$ . Suppose that a subset  $G$  of  $X$  satisfies  $B \subset G$  for some  $B \in \mathcal{B}_x$  for each  $x \in G$ , then  $G$  is a sequentially neighborhood of each point in  $G$ , then  $G$  is open in  $X$  because  $X$  is a sequential space, so  $\mathcal{B}_x$  is a  $wn$ -network of  $x$  in  $X$ .

In a word,  $\bigcup_{x \in X} \mathcal{B}_x$  is a  $\sigma$ -compact-finite weak base for  $X$ .  $\square$

The main technique in the proof of Theorem 1 is the  $W_n(P)$  constructed, which generate directly a weak base for a space  $X$ . The  $\mathcal{H}_x$  in proof of Theorem is exactly a  $wn$ -network  $\mathcal{P}_x$  of  $x$  in  $X$ , it is convenient in proof by using the *sequential neighborhoods* instead of the usual *weak neighborhoods*. Next, we give a direct proof of some properties of  $g$ -metrizable spaces by the  $W_n(P)$ .

COROLLARY 1 [3, 6, 11]. *The following are equivalent for a regular space  $X$ :*

- (1)  $X$  is a  $g$ -metrizable space.
- (2)  $X$  is a  $k$ -space with a  $\sigma$ -hereditarily closure-preserving weak base.
- (3)  $X$  is a  $gf$ -countable space with a  $\sigma$ -hereditarily closure-preserving weak base.

PROOF. It only needs to show that (3)  $\Rightarrow$  (1). Let  $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$  be a  $\sigma$ -hereditarily closure-preserving weak base for a  $gf$ -countable space  $X$ , here each  $\mathcal{P}_n$  is a family of closed subsets of  $X$  by the regularity of  $X$  [6, Proposition 2.5.2]. For each  $n \in \mathbb{N}$  defined  $D_n, W_n(P)$  and  $\mathcal{W}_n$  as in proof of Theorem 1. To complete the proof, it suffices to show that  $\mathcal{W}_n$  is locally finite in  $X$  for each  $n \in \mathbb{N}$  by the proof of Theorem. For each  $P \in \mathcal{P}_n$  there is a subset  $D_n(P)$  of  $D_n$  such that  $W_n(P) = (P \setminus D_n) \cup D_n(P)$  because  $W_n(P) \subset P \subset (P \setminus D_n) \cup D_n$ . For each  $x \in X$ , if  $x \notin D_n$ , then  $\mathcal{P}_n$  is locally finite at  $x$ , thus  $\mathcal{W}_n$  is locally finite at  $x$ . If  $x \in D_n$ , there is at most finitely many sets  $\{P_i : i \leq m_1\}$  of  $\mathcal{P}_n$  such that  $x \in W_n(P_i)$  for  $\mathcal{W}_n$  is point-finite. Let  $\{H_k : k \in \mathbb{N}\}$  be a decreasing  $wn$ -network of  $x$  in  $X$ , there is a  $k \in \mathbb{N}$  such that at most finitely many members  $Q_j$  ( $j \leq m_2$ ) of  $\mathcal{P}_n$  with  $H_k \cap (Q_j \setminus \{x\}) \neq \emptyset$  as  $\mathcal{P}_n$  is hereditarily closure-preserving. Let  $U = X \setminus (\bigcup \{P \setminus \{x\} : P \in \mathcal{P}_n \setminus \{Q_j : j \leq m_2\}\}) \cup (D_n \setminus \{x\})$ . If  $x \in P \in \mathcal{P}_n \setminus \{Q_j : j \leq m_2\}$ , then  $H_k \cap P = \{x\}$ , thus  $P \setminus \{x\}$  is closed in  $X$  by the closeness of  $P$  and the definition of weak bases, and  $D_n \setminus \{x\}$  is closed in  $X$  by Lemma 3, so  $U$  is an open neighborhood of  $x$  in  $X$ . For each  $P \in \mathcal{P}_n$ , if  $U \cap W_n(P) \neq \emptyset$ , then  $U \cap (P \setminus D_n) \neq \emptyset$ , so  $U \cap (P \setminus \{x\}) \neq \emptyset$  or  $x \in W_n(P)$ , therefore  $P = Q_j$  for some  $j \leq m_2$  or  $P = P_i$  for some  $i \leq m_1$ , and  $\mathcal{W}_n$  is locally finite in  $X$ . Consequently,  $X$  has a  $\sigma$ -locally finite weak base.  $\square$

Y. Tanaka [11] proved that a Lindelöf space with a  $\sigma$ -hereditarily closure-preserving weak base has a countable weak base. The result is true for spaces with a  $\sigma$ -weakly hereditarily closure-preserving weak base.

COROLLARY 2. *Every Lindelöf space with a  $\sigma$ -weakly hereditarily closure-preserving weak base has a countable weak base.*

PROOF. Let  $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$  be a  $\sigma$ -weakly hereditarily closure-preserving weak base for a Lindelöf space  $X$ , here each  $\mathcal{P}_n$  is a weakly hereditarily closure-preserving family of  $X$ . First, we shall show that  $X$  is a  $gf$ -countable space. For each  $x \in X \setminus I$ , put  $\mathcal{H}_x = \{P \in \mathcal{P} : P \text{ is a sequential neighborhood of } x \text{ in } X\}$ . If there are an  $n \in \mathbb{N}$  and an uncountable subset  $\{B_x : \alpha < \omega_1\}$  of  $\mathcal{H}_x \cap \mathcal{P}_n$ , then for each  $\alpha < \omega_1$  and any open neighborhood  $U$  of  $x$  in  $X$ ,  $B_x \cap U \cap (X \setminus \{x\}) \neq \emptyset$  because  $X \setminus \{x\}$  is not closed in  $X$ . By the induction method, there is a subset  $\{x_\alpha : \alpha < \omega_1\}$  of  $X$  such that each  $x_\alpha \in B_x \cap (X \setminus \{x_\beta : \beta < \alpha\}) \cap (X \setminus \{x\})$ , then  $\{x_\alpha : \alpha < \omega_1\}$  is an uncountable and closed discrete subspace of  $X$ , a contradiction with Lindelöfness of  $X$ , thus  $\mathcal{H}_x \cap \mathcal{P}_n$  is a countable family for each  $n \in \mathbb{N}$ . Hence  $X$  is  $gf$ -countable. By Theorem 1,  $X$  has a  $\sigma$ -compact-finite weak base. To complete the proof, it is sufficient to show that every compact-finite family is countable in  $X$ . Let  $\mathcal{Q}$  be any compact-finite family of  $X$ , if  $\mathcal{Q}$  is not countable, then  $\mathcal{Q}$  contains an uncountable subset  $\{Q_\alpha : \alpha < \omega_1\}$ . For each  $\alpha < \omega_1$  take a  $q_\alpha \in Q_\alpha$ , thus  $\{q_\alpha : \alpha < \omega_1\}$  is countable because  $\mathcal{Q}$  is weakly hereditarily closure-preserving, so  $q$  is belong to uncountable many members of  $\{Q_\alpha : \alpha < \omega_1\}$  for some  $q \in X$ , hence  $\mathcal{Q}$  is not point-finite, a contradiction.  $\square$

Put  $S_1 = \{0\} \cup \{1/n : n \in \mathbb{N}\}$  with the usual topology. Next, spaces with a  $\sigma$ -compact-finite weak base are characterized by products.

THEOREM 2. *The following are equivalent for a space  $X$ :*

- (1)  *$X$  has a  $\sigma$ -compact-finite base.*
- (2)  *$X \times S_1$  has a  $\sigma$ -compact-finite weak base.*
- (3)  *$X \times S_1$  has a  $\sigma$ -weakly hereditarily closure-preserving weak base.*

PROOF. Put  $Z = X \times S_1$ .

(1)  $\Rightarrow$  (2). Suppose that  $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ ,  $\mathcal{Q} = \bigcup_{s \in S_1} \mathcal{Q}_s$  is a  $\sigma$ -compact-finite weak base of the space  $X$  and  $S_1$ , respectively. For each  $z = (x, s) \in Z$ , put  $\mathcal{H}_z = \{P \times Q : P \in \mathcal{P}_x, Q \in \mathcal{Q}_s\}$ , then  $\mathcal{H}_z$  is an  $sn$ -network of  $z$  in  $Z$ . Since  $X$  is a  $k$ -space and  $S_1$  is a locally compact space,  $Z$  is a  $k$ -space. And any compact subset of  $Z$  is metrizable, then  $Z$  is a sequential space, thus  $\mathcal{H}_z$  is a  $wn$ -network of  $z$  in  $Z$ . Hence  $\bigcup_{z \in Z} \mathcal{H}_z$  is a  $\sigma$ -compact-finite weak base of  $Z$ .

(2)  $\Rightarrow$  (3) is obvious. (3)  $\Rightarrow$  (1). Let  $\mathcal{P}$  be a  $\sigma$ -weakly hereditarily closure-

preserving weak base for a space  $Z$ . For each  $x \in X$ ,  $n \in N$ , put  $z_n = (x, 1/n)$ , then the sequence  $\{z_n\}$  converges to  $(x, 0)$  in  $Z$ , thus the family  $\{P \in \mathcal{P} : P \text{ is a sequential neighborhood of } (x, 0) \text{ in } Z\}$  is countable by Lemma 1, so the point  $(x, 0)$  is  $gf$ -countable in  $Z$ . Since  $X$  is homeomorphic to a closed subspaces  $X \times \{0\}$  of  $Z$ ,  $X$  is a  $gf$ -countable space with a  $\sigma$ -weakly hereditarily closure-preserving weak base,  $X$  has a  $\sigma$ -compact-finite weak base by Theorem 1.  $\square$

**COROLLARY 3.** *The following are equivalent for a regular space  $X$ :*

- (1)  $X$  is a  $g$ -metrizable space.
- (2)  $X \times S_1$  has a  $\sigma$ -locally-finite weak base.
- (3)  $X \times S_1$  has a  $\sigma$ -hereditarily closure-preserving weak base.  $\square$

**EXAMPLE.** There is a space  $X$  with a  $\sigma$ -weakly hereditarily closure-preserving weak base such that  $X$  does not any  $\sigma$ -compact-finite weak base or any  $\sigma$ -hereditarily closure-preserving weak base.

Let  $X$  be the non-metrizable, paracompact space with a  $\sigma$ -weakly hereditarily closure-preserving base in Example 9 in [1]. Then  $X$  has not any  $\sigma$ -hereditarily closure-preserving base by Theorem 5 in [1]. It has been shown that  $X$  is not a  $k$ -space in [1], thus  $X$  has not any  $\sigma$ -compact-finite weak base. By the construction of  $X$ ,  $X$  has a unique non-isolated point  $\bar{0}$ . If  $X$  has a  $\sigma$ -hereditarily closure-preserving weak base  $\mathcal{P}$ , for each  $\bar{0} \in P \in \mathcal{P}$ ,  $P$  is open by the definition of weak base, and for each  $x \in X \setminus \{\bar{0}\}$ ,  $\{x\} \in \mathcal{P}$  because  $\{x\}$  is open in  $X$ , thus  $X$  has a  $\sigma$ -hereditarily closure-preserving base, a contradiction. Hence  $X$  has not any  $\sigma$ -hereditarily closure-preserving weak base.  $\square$

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