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ANALYTIC SMOOTHING EFFECTS FOR A CLASS OF DISPERSIVE EQUATIONS

By

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Abstract. We study the analytic smoothing effect for a class of dispersive equations. In this paper we consider the microlocal analytic smoothness for the solutions of a class of dispersive equations including not only the Schrödinger equation but also the linearized KdV equation. We make use of the Sjöstrand theory of the FBI transform as in Robbiano-Zuily's works in the case of Schrödinger equations.

1. Introduction and the Main Results

In this paper we consider the analytic smoothing properties for a class of dispersive operators. The typical example of these operators is the Schrödinger operator, and another example is the linearized KdV operator.

Let us describe our problem. Let m be an integer greater than or equal to 2. Let $P(y, D_y)$ be a linear differential operator of order m in $\boldsymbol{R}^n$,

$$P(y, D_y) = \sum_{|\alpha| \leq m} c_\alpha(y) D_y^\alpha. \tag{1.1}$$

We assume that $P(y, D_y)$ has analytic coefficients in $\boldsymbol{R}^n$ and a real principal symbol. We also assume that $P(y, D_y)$ is of real principal type (in a strong sense). For some integer j with $1 \leq j \leq n$ we have $\partial_{\eta_j} p_m(y, \eta) \neq 0$, where $p_m(y, \eta) = \sum_{|\alpha|=m} c_\alpha(y) \eta^\alpha$ is the principal symbol of $P(y, D_y)$.

We consider the initial value problem

$$\begin{cases} D_t u + P(y, D_y) u = 0, \\ u|_{t=0} = u_0(y). \end{cases} \tag{1.2}$$

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In this paper we restrict our problem to a simple case, where the dimension n is equal to one. In this simple case it suffices to treat the operator,

$$P(y, D_y)u = \sum_{0 \le l \le m} c_l(y) D_y^l, \tag{1.3}$$

where $c_m(y) = 1$ and $c_l(y)$ are analytic in $\boldsymbol{R}$, that is, the coefficient of the principal part is constant. We consider only the backward initial value problem (1.2) in order to simplify notation in the proofs.

We make the following assumptions:

One can find positive constants $C_0 > 0$, $R_0 > 0$, $K_0 > 0$ and $\sigma_0 \in (0, 1)$ such that for $y \in \boldsymbol{R}$ with $|y| > R_0$ and $k \in \boldsymbol{N} \cup \{0\}$,

$$\sum_{0 \le l \le m-1} |D_y^k c_l(y)| \le \frac{C_0 K_0^k k!}{|y|^{1+\sigma_0+k}}. \tag{1.4}$$

Let $\rho = (y, \eta) \in T^*\boldsymbol{R} \backslash 0$, and let $(Y(s; y, \eta), \Theta(s; y, \eta))$ be the solution to the equation

$$\begin{cases} \dfrac{d}{ds} Y(s) = \dfrac{\partial p_m}{\partial \eta}(Y(s), \Theta(s)), & Y(0) = y, \\ \dfrac{d}{ds} \Theta(s) = -\dfrac{\partial p_m}{\partial y}(Y(s), \Theta(s)), & \Theta(0) = \eta. \end{cases} \tag{1.5}$$

In our case $p_m(y, \eta) = p_m(\eta) = \eta^m$. Therefore

$$\begin{cases} Y(s) = y + ms\eta^{m-1}, \\ \Theta(s) = \Theta(0) = \eta. \end{cases} \tag{1.6}$$

We remark that for $\eta \neq 0$

$$\lim_{s \to \infty} |Y(s; y, \eta)| = +\infty. \tag{1.7}$$

Let $u(t, \cdot) \in C(\boldsymbol{R}, L^2(\boldsymbol{R}))$ be the solution of the initial value problem (1.2). Let us introduce the space of the initial data

$$X_{\rho_0}^+ = \{v \in L^2(\boldsymbol{R}); \exists \delta_0 > 0, e^{\delta_0 |y|^{1/(m-1)}} v(y) \in L^2(\Gamma_{\rho_0}^+)\}, \tag{1.8}$$

where

$$\Gamma_{\rho_0}^+ = \{Y(s; y_0, \eta_0) \in \boldsymbol{R}, s \ge 0\}. \tag{1.9}$$

Our main result about the microlocal smoothing effect is the following;

THEOREM 1.1. *Let $P(y, D_y)$ be defined in* (1.3) *satisfying* (1.4) *and $\rho_0 = (y_0, \eta_0) \in T^*\boldsymbol{R}\backslash 0$. Let $u_0 \in L^2(\boldsymbol{R})$ be in $X^+_{\rho_0}$. Then for all $t < 0$, ρ_0 does not belong to the analytic wave front set $WF_A[u(t, \cdot)]$ of the solution $u(t, \cdot)$ for* (1.2).

If one consider the Gevrey s wave front set, it suffices to change the weight of the decay of the initial data from $e^{\delta_0|y|^{1/(m-1)}}$ to $e^{\delta_0|y|^{1/s(m-1)}}$. From the theorem above we can easily get Corollary 1.1.

COROLLARY 1.1. *If the initial data $u_0 \in L^2(\boldsymbol{R})$ satisfies*

$$\int_{-\infty}^{\infty} e^{2\delta_0|y|^{1/(m-1)}}|u_0(y)|^2\, dy < \infty, \tag{1.10}$$

for some positive constant δ_0, then the solution to the initial value problem (1.2) *becomes analytic with respect to the space variable y for $t \neq 0$.*

The result of Theorem 1.1 in the case $m = 2$ was obtained by L. Robbiano and C. Zuily in [15]. In fact, they have studied Schrödinger operators with variable coefficients near the flat Laplacian. It was remarked in [15] that their method can be applied to the second order operators of real principal type. The purpose of the present paper is to show that their method is applicable to operators of a higher order. The difficulty in the higher order case comes from estimating the phase function globally (see the details in Lemma 3.2 below).

We remark that under different conditions on the initial data the properties of the analytic smoothing effects for the Schrödinger operators are obtained in [13], [16]. Smoothing effects in the dispersive operators, especially Schrödinger operators, have been studied by many authors (see [3]). Though there are many results even for Schrödinger equations with variable coefficients or nonlinear Schrödinger equations, there are few results in higher order cases in comparison with the second order case. For general order operators S. Doi studied the microlocal smoothing effects by using the commutator method in [5] and [6]. This method is completely different from ours. Recently T. Ōkaji has studied the smoothing properties by using the Wigner transformation in [14]. His method does not cover the linearized KdV operator which is treated in this paper. K. Kajitani and S. Wakabayashi studied the well-posedness and the smoothing effects for Schrödinger operators with analytic and Gevrey coefficients in [10] and [11] (cf. S. Tarama [17]). Their method uses an integral transformation. However their method is quite different from ours.

For nonlinear Schrödinger equations, the relationship between the properties of the initial data and the smoothness of solution has been studied, for example, in [7], [8] and [12]. Their results are based on the commutator method in terms of the linear theory. Recently in [1] and [2] H. Chihara has studied the case the nonlinear term includes the derivatives of the solution. We remark that S. Tarama in [18] studied the analytic smoothing properties for the KdV equation by means of the inverse scattering method instead of the commutator method, where the condition similar to (1.8) was given.

Our plan of this paper is as follows: In Section 2 we recall the relationship between the analytic wave front set and the FBI transform. In Section 3 we construct the phase function and the amplitude function in the FBI transform in order to transform the original operator $P(y, D_y)$ into the first order operator. To complete this transformation we solve the eikonal equation and the transport equations globally. In Section 4 we prove Theorem 1.1 by using the results in Section 3.

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2. The Analytic Wave Front Set and the FBI Transform

In this section we define the FBI transform and the analytic wave front set, which are referred to [15] and [19].

Let $\rho_0 = (y_0, \eta_0) \in T^*\boldsymbol{R}\backslash 0$. Let $\varphi(x, y)$ be a holomorphic function in a neighborhood $U_0 \times V_{y_0}$ of $(0, y_0)$ in $\boldsymbol{C} \times \boldsymbol{C}$ which satisfies

$$\frac{\partial \varphi}{\partial y}(0, y_0) = -\eta_0, \tag{2.1}$$

$$\operatorname{Im} \frac{\partial^2 \varphi}{\partial y^2}(0, y_0) > 0, \tag{2.2}$$

$$\frac{\partial^2 \varphi}{\partial x \partial y}(0, y_0) \neq 0. \tag{2.3}$$

For the phase function $\varphi(x, y)$ above we can define

$$\Phi(x) = \max_{y \in V_{y_0}|_{\boldsymbol{R}}} (-\operatorname{Im} \varphi(x, y)), \tag{2.4}$$

for $x \in U_0$. Let $f(x, y, \lambda) = \sum_{k \geq 0} f_k(x, y)\lambda^{-k}$ be an analytic symbol of order zero, elliptic in a neighborhood of $(0, y_0)$. Let $\chi \in C_0^\infty$ be a cutoff function with support in a neighborhood of y_0, $0 \leq \chi \leq 1$, and $\chi \equiv 1$ near y_0.

The FBI transform of a distribution $u \in \mathscr{D}'(\boldsymbol{R})$ is defined by

$$Tu(x, \lambda) = \langle \chi(\cdot)u, e^{i\lambda\varphi(x,\cdot)} f(x, \cdot, \lambda)\rangle, \quad \lambda > 1. \tag{2.5}$$

According to [19] we can characterize the analytic wave front set of $u \in \mathscr{D}'(\boldsymbol{R})$ by using the FBI transform. We have the following equivalence:

$$\rho_0 \notin WF_A[u]. \tag{2.6}$$

$$\exists C > 0,\ \exists \mu > 0,\ \exists \lambda_0 \geq 1 \quad \text{such that} \tag{2.7}$$

$$e^{-\lambda\Phi(x)}|Tu(x, \lambda)| \leq Ce^{-\mu\lambda} \quad \text{for } \forall x \in U_0,\ \forall \lambda \geq \lambda_0.$$

Assume that $u(t, \cdot)$ is a family of distributions on $\boldsymbol{R}$ depending on a real parameter t. Let $t_0 \in \boldsymbol{R}$. We shall say that a point $\rho_0 \in T^*\boldsymbol{R}\backslash 0$ does not belong to the locally uniform analytic wave front set $\widetilde{WF_A}[u(t_0, \cdot)]$ if there exist an FBI transform T, positive constants $C, \mu, \lambda_0, \varepsilon$ and a neighborhood U_0 of 0 such that

$$e^{-\lambda\Phi(x)}|Tu(x, \lambda)| \leq Ce^{-\mu\lambda} \tag{2.8}$$

$$\text{for } \forall x \in U_0,\ \forall \lambda \geq \lambda_0,\ \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

3. Construction of the Phase Function and the Amplitude Function

Let $R > 0$ be a sufficiently large positive constant with $R > R_0$, where R_0 is given in (1.4). Let $\rho_0 = (y_0, \eta_0) \in T^*\boldsymbol{R}\backslash 0$ be a point such that $|y_0| > 2R$ and $y_0\eta_0^{m-1} \geq 0$. We define the holomorphic function $\varphi_0(y) : \boldsymbol{C} \to \boldsymbol{C}$,

$$\varphi_0(y) = -\eta_0 y + \frac{i}{2}(y - y_0)^2. \tag{3.1}$$

We shall solve the eikonal equation by the geometrical way in order to get the global properties.

THEOREM 3.1. *There exist constants $\varepsilon_1, \varepsilon_2$ with $0 < \varepsilon_1 < \varepsilon_2$ and a holomorphic function $\varphi(x, z)$ in the set*

$$E = \{(x, z) \in \boldsymbol{C} \times \boldsymbol{C}; \operatorname{Re} x \geq -\varepsilon_1, |\operatorname{Im} x| < \varepsilon_1, \tag{3.2}$$

$$|z - Y(x; y_0, \eta_0)| < \varepsilon_2(1 + |x|)\},$$

such that

$$\frac{\partial\varphi}{\partial x}(x,z) = p_m\left(z, -\frac{\partial\varphi}{\partial z}(x,z)\right) \quad \textit{in } E, \tag{3.3}$$

$$\frac{\partial\varphi}{\partial z}(0, y_0) = -\eta_0, \tag{3.4}$$

$$\operatorname{Im}\frac{\partial^2\varphi}{\partial z^2}(0, y_0) > 0, \tag{3.5}$$

$$\frac{\partial^2\varphi}{\partial x\partial z}(0, y_0) \neq 0. \tag{3.6}$$

PROOF OF THEOREM 3.1. From the definition of $\varphi_0(y)$ we have

$$\frac{\partial\varphi_0}{\partial y}(y) = -\eta_0 + i(y - y_0).$$

We introduce the submanifold of $\boldsymbol{C}^4$ whose dimension is equal to 1,

$$\Lambda_0 = \left\{\left(0, y, p_m\left(y, -\frac{\partial\varphi_0}{\partial y}(y)\right), \frac{\partial\varphi_0}{\partial y}(y)\right) \in \boldsymbol{C}^4; |y - y_0| < \varepsilon_3\right\}, \tag{3.7}$$

where $\varepsilon_3 > 0$ will be determined later. For the 2-form $\sigma = d\xi \wedge dx + d\eta \wedge dy$ we have

$$\sigma|_{\Lambda_0} = 0.$$

We introduce the symbol

$$q(x, y, \xi, \eta) = \xi - {}^t p_m(y, \eta),$$

where ${}^t p_m(y, \eta) = p_m(y, -\eta) = (-\eta)^m$ is the principal symbol of the transposed operator ${}^t P(y, D)$ for $P(y, D)$.

Let $(X(s), F(s), \Xi(s), G(s))$ be the solution to the equations,

$$\begin{cases} \dfrac{d}{ds}X(s) = 1, & X(0) = 0, \\ \dfrac{d}{ds}F(s) = -\dfrac{\partial\, {}^t p_m}{\partial\eta}, & F(0) = y, \\ \dfrac{d}{ds}\Xi(s) = 0, & \Xi(0) = p_m\left(y, -\dfrac{\partial\varphi_0}{\partial y}(y)\right), \\ \dfrac{d}{ds}G(s) = \dfrac{\partial\, {}^t p_m}{\partial y}, & G(0) = \dfrac{\partial\varphi_0}{\partial y}(y). \end{cases} \tag{3.8}$$

We have

$$X(s) = s, \quad \Xi(s) = \Xi(0) = p_m\left(y, -\frac{\partial \varphi_0}{\partial y}(y)\right),$$

and $(F(s), G(s))$ is the solution of

$$\frac{d}{ds}F(s) = \frac{\partial p_m}{\partial \eta}(F(s), -G(s)) = m(-G(s))^{m-1},$$

$$\frac{d}{ds}G(s) = \frac{\partial p_m}{\partial y}(F(s), -G(s)) = 0,$$

$$F(0) = y, \quad G(0) = \frac{\partial \varphi_0}{\partial y}(y) \equiv -\eta.$$

That is,

$$F(s) = Y(s; y, \eta) = y + ms\eta^{m-1},$$

$$G(s) = -\Theta(s; y, \eta) = -\eta\left(= \frac{\partial \varphi_0}{\partial y}(y)\right).$$

By using these solutions we define

$$\begin{aligned}
\Lambda &= \bigcup_{s \in \boldsymbol{C},\, \mathrm{Re}\, s > -\varepsilon_1,\, |\mathrm{Im}\, s| < \varepsilon_1} e^{sH_q}\Lambda_0 \qquad (3.9)\\
&= \{(X(s), F(s), \Xi(s), G(s)) \in \boldsymbol{C}^4; (s, y) \in \mathcal{O}\}\\
&= \left\{\left(x, Y\left(x; y, -\frac{\partial \varphi_0}{\partial y}(y)\right), p_m\left(y, -\frac{\partial \varphi_0}{\partial y}(y)\right),\right.\right.\\
&\qquad \left.\left. -\Theta\left(x; y, -\frac{\partial \varphi_0}{\partial y}(y)\right)\right) \in \boldsymbol{C}^4; (x, y) \in \mathcal{O}\right\},
\end{aligned}$$

where we set $x = s$, and

$$\mathcal{O} = \{(x, y) \in \boldsymbol{C} \times \boldsymbol{C}; \mathrm{Re}\, x > -\varepsilon_1, |\mathrm{Im}\, x| < \varepsilon_1, |y - y_0| < \varepsilon_3\}.$$

Here $\varepsilon_1, \varepsilon_2$ and ε_3 are small positive constants which satisfy $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3$. Since H_q is transverse to Λ_0, the submanifold Λ of $\boldsymbol{C}^4$ is Lagrangian.

LEMMA 3.1. *There exists a holomorphic function $\varphi(x, z)$ in E such that*

$$\Lambda = \left\{\left(x, z, \frac{\partial \varphi}{\partial x}(x, z), \frac{\partial \varphi}{\partial z}(x, z)\right) \in \boldsymbol{C}^4; (x, z) \in E\right\}. \tag{3.10}$$

If Lemma 3.1 will be proved, the properties stated in Theorem 3.1 will follow from Lemma 3.1. The fact that

$$\frac{d}{ds}q(X(s),F(s),\Xi(s),G(s))=0,$$

implies that for $\forall s\in\boldsymbol{R}$

$$q(X(s),F(s),\Xi(s),G(s))=q(X(0),F(0),\Xi(0),G(0)),$$

that is,

$$\begin{aligned}\Xi(s)-{}^{t}p_m(F(s),G(s))&=\Xi(0)-{}^{t}p_m(F(0),G(0))\\&=p_m\left(y,-\frac{\partial\varphi_0}{\partial y}(y)\right)-p_m(y,\eta)\\&=0.\end{aligned}$$

Since $F(s)=z$, $\Xi(s)=(\partial\varphi/\partial x)(x,z)$ and $G(s)=G(x)=(\partial\varphi/\partial z)(x,z)$, we have the property (3.3)

$$\frac{\partial\varphi}{\partial x}(x,z)=p_m\left(z,-\frac{\partial\varphi}{\partial z}(x,z)\right).$$

It follows from the proof of Lemma 3.1

$$z=F(s)=Y\left(x;y,-\frac{\partial\varphi_0}{\partial y}(y)\right), \tag{3.11}$$

which shows the existence of a holomorphic function κ satisfying

$$y=\kappa(x,z),\quad y_0=\kappa(0,y_0). \tag{3.12}$$

The other three properties of Theorem 3.1 follow from the same argument as in the proof of Theorem 3.3 in [15].

PROOF OF LEMMA 3.1. Let π be the projection on the base space, $\pi(\lambda)=(x,Y(x;y,-(\partial\varphi_0/\partial y)(y)))$ for $\lambda\in\Lambda$.

First we show that the map $\pi:\Lambda\to E$ is bijective. When $m\geq 3$,

$$
\begin{aligned}
&Y(x; y, \eta) - Y(x; y_0, \eta_0) \\
&\quad = (y + mx\eta^{m-1}) - (y_0 + mx\eta_0^{m-1}) \\
&\quad = (y - y_0) + mx(\eta^{m-1} - \eta_0^{m-1}) \\
&\quad = (y - y_0) + mx[\{\eta_0 - i(y - y_0)\}^{m-1} - \eta_0^{m-1}] \\
&\quad = (y - y_0) + mx\sum_{j=1}^{m-1}\binom{m-1}{j}\eta_0^{m-j-1}(-i)^j(y - y_0)^j \\
&\quad = \{1 - im(m-1)\eta_0^{m-2}x\}(y - y_0) \\
&\qquad + mx\sum_{j=2}^{m-1}\binom{m-1}{j}\eta_0^{m-j-1}(-i)^j(y - y_0)^j.
\end{aligned}
$$

It suffices to show that for a fixed $x \in \boldsymbol{C}$ with $\operatorname{Re} x > -\varepsilon_1$, $|\operatorname{Im} x| < \varepsilon_1$ and $z \in \boldsymbol{C}$ with $|z - Y(x; y, -(\partial\varphi_0/\partial y)(y))| < \varepsilon_2(1 + |x|)$ there exists a unique $y \in \boldsymbol{C}$ with $|y - y_0| < \varepsilon_3$ such that

$$
Y\left(x; y, -\frac{\partial\varphi_0}{\partial y}(y)\right) - Y(x; y_0, \eta_0) = z - Y(x; y_0, \eta_0). \tag{3.13}
$$

We define

$$
\begin{aligned}
H(y) = y_0 &+ \frac{1}{1 - im(m-1)\eta_0^{m-2}x} \\
&\times \left\{ z - Y(x, y_0, \eta_0) - mx\sum_{j=2}^{m-1}\binom{m-1}{j}\eta_0^{m-j-1}(-i)^j(y - y_0)^j \right\}.
\end{aligned} \tag{3.14}
$$

If $\varepsilon_1 = \varepsilon_1(m, |\eta_0|) > 0$ is small enough, we have

$$
|1 - im(m-1)\eta_0^{m-2}x| \geq c_m^{-1}(1 + |x|), \quad \text{in } \boldsymbol{C}.
$$

For $y \in \boldsymbol{C}$ with $|y - y_0| < \varepsilon_3$,

$$
\begin{aligned}
|H(y) - y_0| &\leq \frac{c_m}{1 + |x|}(|z - Y(x; y_0, \eta_0)| + |x|c_m\varepsilon_3^2) \\
&\leq c_m\varepsilon_2 + c_m'\varepsilon_3^2\frac{|x|}{1 + |x|} \\
&\leq \varepsilon_3, \quad (0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 \ll 1).
\end{aligned}
$$

Hence $H(y)$ is a map from the ball $B(y_0, \varepsilon_3)$ into itself. On the other hand

$$\begin{aligned}
&|H(y_1) - H(y_2)| \\
&= \frac{m|x|}{|1 - im(m-1)\eta_0^{m-2}x|} \\
&\left|\sum_{j=2}^{m-1} \binom{m-1}{j} \eta_0^{m-j-1}(-i)^j\{(y_1 - y_0)^j - (y_2 - y_0)^j\}\right| \\
&\leq \frac{c_m|x|}{1+|x|}\varepsilon_3|y_1 - y_2| \\
&\leq k|y_1 - y_2|, \quad 0 < k < 1.
\end{aligned}$$

Then $H(y)$ is also a contraction map in $B(y_0, \varepsilon_3)$.

It means that (3.14) has a unique solution, in other words, the map $\pi : \Lambda \to E$ is surjective. When $m = 2$, the projection map of $\pi : \Lambda \to E$ is linear, so we can also define $H(y)$ and get the same conclusion.

We shall prove that the differential map $d\pi(\lambda) : T_\lambda\Lambda \to T_{\pi(\lambda)}E$ is surjective. Let us consider the map $F : \mathcal{O} \to \Lambda$, $(x, y) \mapsto (x, Y(\cdot), \Theta(\cdot), -\Theta(\cdot))$. It suffices to prove that the map $\pi \circ F : \mathcal{O} \to E$ is surjective ($\pi \circ F(x, y) = (x, Y(x; y, \eta))$). In fact, we have

$$\left|\frac{\partial Y}{\partial y}\right| \geq C(1 + |x|), \quad (x, y) \in \mathcal{O}, \tag{3.15}$$

because

$$Y(x; y, \eta(y)) = y + mx\{\eta_0 - i(y - y_0)\}^{m-1}.$$

Now we obtain the desired function $\varphi(x, z)$ as in the proof of Corollary 3.6 in [15]. ■

As stated after Lemma 3.1, we have completed the proof of Theorem 3.1.

We remark on the choices of the constants $\varepsilon_1, \varepsilon_2$ and ε_3. When we consider the contraction map $H(y)$ in the proof of Lemma 3.1, we can choose $\varepsilon_1 = \varepsilon_1(m, |\eta_0|)$ and $\varepsilon_3 = \varepsilon_3(m)$ independently. The choice of ε_2 depends on ε_3 as $2c_m\varepsilon_2 \leq \varepsilon_3$. So we can choose ε_2 which satisfies $c\varepsilon_3^2 \leq \varepsilon_2 \leq (1/2c_m)\varepsilon_3$. These choices are important when we consider the global properties of the phase function $\varphi = \varphi(x, z)$.

Here we present the global properties of the phase function φ in order to construct the amplitude function f along the set E globally.

In the proof of Lemma 3.1, the projection map π is a diffeomorphism from $\pi^{-1}(E)(\subset \Lambda)$ to E. For the map $z = Y(x; y, -(\partial\varphi_0/\partial y)(y))$ there exists the inverse map $\kappa : E \to \mathbf{C}$ such that $y = \kappa(x, z)$. The function κ is holomorphic in E and $\kappa(0, y_0) = y_0$. We have seen

$$|\kappa(x, z) - y_0| < \varepsilon_3, \quad (x, z) \in E. \tag{3.16}$$

By the identification of (3.9) with (3.10) we have

$$\frac{\partial\varphi}{\partial z}(x, z) = -\Theta(x; y, \eta) = -\eta_0 + i(y - y_0)|_{y=\kappa(x,z)}. \tag{3.17}$$

From (3.14)

$$\begin{aligned}\kappa(x, z) - y_0 = \frac{1}{1 - im(m-1)\eta_0^{m-2}x}\Bigg\{ & z - Y(x; y_0, \eta_0) \\ & - mx\sum_{j=2}^{m-1}\binom{m-1}{j}\eta_0^{m-j-1}(-i)^j(\kappa(x, z) - y_0)^j\Bigg\}.\end{aligned} \tag{3.18}$$

We have

$$\begin{aligned}\frac{\partial\kappa}{\partial z}(x, z) = \frac{1}{1 - im(m-1)\eta_0^{m-2}x}\Bigg\{ & 1 - mx\sum_{j=2}^{m-1}\binom{m-1}{j} \\ & \eta_0^{m-j-1}(-i)^j j(\kappa(x, z) - y_0)^{j-1}\frac{\partial\kappa}{\partial z}(x, z)\Bigg\}.\end{aligned} \tag{3.19}$$

Since $\left|\frac{1}{1 - im(m-1)\eta_0^{m-2}x}\right| \leq \frac{C}{1+|x|}$, we have

$$\left|\frac{\partial\kappa}{\partial z}(x, z)\right| \leq \frac{C}{1+|x|}\left\{1 + m|x|\sum_{j=2}^{m-1}\binom{m-1}{j}|\eta_0|^{m-j-1}j\varepsilon_3^{j-1}\left|\frac{\partial\kappa}{\partial z}(x, z)\right|\right\}.$$

Since ε_3 is small enough, the second term can be absorbed in the left hand side. We get

$$\left|\frac{\partial\kappa}{\partial z}\right| \leq \frac{C}{1+|x|}, \quad (x, z) \in E.$$

In the same way we have

$$\left|\frac{\partial^2\kappa}{\partial z^2}\right| \leq \frac{C}{(1+|x|)^2}, \quad (x, z) \in E.$$

It follows from these estimates that

$$\left|\frac{\partial \varphi}{\partial z}(x,z)\right| \le C, \quad \left|\frac{\partial^2 \varphi}{\partial z^2}(x,z)\right| \le \frac{C}{(1+|x|)}, \quad (x,z) \in E. \tag{3.20}$$

Now we shall state the key lemma of the main theorem. It should be noted that the inequality (3.21) is valid only in the region where z is a real variable, which is different from Lemma 3.7 in [15].

LEMMA 3.2. *Let us define*

$$\tilde{E} = \left\{ (x,z) \in \boldsymbol{C} \times \boldsymbol{R};\, \mathrm{Re}\, x > -\varepsilon_1, |\mathrm{Im}\, x| < \varepsilon_1, \right.$$
$$\left. |z - Y(x; y_0, \eta_0)| < \frac{1}{2}\varepsilon_2(1+|x|) \right\}.$$

Then we have

$$\mathrm{Im}\, \frac{\partial^2 \varphi}{\partial z^2}(x,z) \ge \frac{C}{(1+|x|)^2}, \quad (x,z) \in \tilde{E}. \tag{3.21}$$

PROOF OF LEMMA 3.2. Since $y = \kappa(x,z)$ is the inverse function of

$$z = Y\left(x; y, -\frac{\partial \varphi_0}{\partial y}(y)\right)$$
$$= y + mx\{\eta_0 - i(y - y_0)\}^{m-1},$$

we have

$$z = \kappa(x,z) + mx\{\eta_0 - i(\kappa(x,z) - y_0)\}^{m-1}.$$

By setting

$$X(x,z) = \eta_0 - i(y - y_0)|_{y=\kappa(x,z)}, \quad (x,z) \in E(\subset \boldsymbol{C} \times \boldsymbol{C}),$$

this function $X(x,z)$ is the simple root for the equation

$$mx\{X(x,z)\}^{m-1} + iX(x,z) + y_0 - i\eta_0 - z = 0, \quad (x,z) \in E, \tag{3.22}$$

which satisfies $|X(x,z) - \eta_0| < \varepsilon_3$. By taking the differentiation with respect to z in (3.22), we have

$$\{m(m-1)x\{X(x,z)\}^{m-2} + i\}\frac{\partial X}{\partial z}(x,z) - 1 = 0.$$

Since $\operatorname{Re} x > -\varepsilon_1$, $|\operatorname{Im} x| < \varepsilon_1$ and $|X(x,z) - \eta_0| < \varepsilon_3$ in E, we have

$$m(m-1)x\{X(x,z)\}^{m-2} + i \neq 0,$$

for $(x,z) \in E$. On the other hand we have from (3.17)

$$\begin{aligned}\frac{\partial\varphi}{\partial z}(x,z) &= -\Theta\left(x; y, -\frac{\partial\varphi_0}{\partial y}(y)\right)\Big|_{y=\kappa(x,z)}\\ &= \frac{\partial\varphi_0}{\partial y}(y)\Big|_{y=\kappa(x,z)}\\ &= -\eta_0 + i(y-y_0)|_{y=\kappa(x,z)}\\ &= -X(x,z)\end{aligned}$$

Using these properties we have

$$\begin{aligned}\operatorname{Im}\frac{\partial^2\varphi}{\partial z^2}(x,z) &= -\operatorname{Im}\frac{\partial X}{\partial z}(x,z)\\ &= -\operatorname{Im}\frac{1}{m(m-1)x\{X(x,z)\}^{m-2} + i}.\end{aligned}$$

First we consider the case that $|x|$ is large enough. Let us introduce the set $\tilde{E}_1$ which satisfies $\tilde{E} \subset \tilde{E}_1$ as

$$\begin{aligned}\tilde{E}_1 = \bigcup_{r_1 \in \boldsymbol{R},\, r_0 \in \boldsymbol{C},\, |r_1| < \varepsilon_2,\, |r_0| < \varepsilon_2} &\{(x,z) \in \boldsymbol{C} \times \boldsymbol{R};\\ &\operatorname{Re} x > -\varepsilon_1, |\operatorname{Im} x| < \varepsilon_1, z = Y(x; y_0, \eta_0) + r_1 x + r_0\}.\end{aligned}$$

We write

$$\begin{aligned}z &= Y(x; y_0, \eta_0) + r_1 x + r_0\\ &= y_0 + mx\eta_0^{m-1} + r_1 x + r_0\\ &= (m\eta_0^{m-1} + r_1)x + y_0 + r_0\\ &= f(x).\end{aligned}$$

Since $f(x)$ is a holomorphic function and $|z - Y(x; y_0, \eta_0)| < \varepsilon_2(1 + |x|)$ for (r_0, r_1), we can define the holomorphic function $\tilde{X}(x) = X(x, f(x))$ for $x \in \boldsymbol{C}$ with $\operatorname{Re} x > -\varepsilon_1$, $|\operatorname{Im} x| < \varepsilon_1$. From (3.22) $\tilde{X}(x)$ is the simple root of the equation

$$mxW^{m-1} + iW - (m\eta_0^{m-1} + r_1)x - (r_0 + i\eta_0) = 0, \tag{3.23}$$

where $|\tilde{X}(x) - \eta_0| < \varepsilon_3$. We change the variable $t \in \boldsymbol{C}$ with $t = 1/x$. Instead of (3.23) we consider the equation

$$P(t, r_0, r_1, V) = V^{m-1} + \frac{i}{m} tV - \left(\eta_0^{m-1} + \frac{r_1}{m}\right) - \frac{1}{m}(r_0 + i\eta_0)t = 0. \tag{3.24}$$

This equation is well defined for $t \in \boldsymbol{C}$, $r_0 \in \boldsymbol{C}$ and $r_1 \in \boldsymbol{R}$. By Rouché's theorem there exists a positive constant μ_0 independent of ε_2 such that the equation (3.24) has the simple root $V(t) = V(t, r_0, r_1)$ near η_0 for t and (r_0, r_1) with $|t| < \mu_0$, $r_0 < \mu_0$ and $r_1 < \mu_0$. So $V(t)$ is given by the residue theorem as

$$V(t) = \int_\gamma \frac{V \partial_V P(t, r_0, r_1, V)}{P(t, r_0, r_1, V)} dV,$$

where γ is a simple closed path enclosing the distinct root $V(t)$. We note that we can choose the same γ for t and (r_0, r_1) with $|t| \leq \mu_0$, $|r_0| \leq \mu_0$ and $|r_1| \leq \mu_0$. Since $P(t, r_0, r_1, V) \neq 0$ on γ and $P(t, r_0, r_1, V)$ is an analytic function of t, it follows from the expression of $V(t)$ that $V(t)$ is also an analytic function of t. We write

$$V(t) = c_0 + c_1 t + \sum_{j=2}^{\infty} c_j t^j, \quad \text{for } |t| \leq R,$$

where $R = R(\mu_0) > 0$ does not decrease when μ_0 tends to zero. Defining the constant

$$D = \max_{0 \leq j \leq 2} \sup_{|t| < R, |r_0| \leq \mu_0, |r_1| \leq \mu_0} \left| \frac{d^j V}{dt^j}(t) \right|,$$

we also have the estimate

$$|V(t) - c_0 - c_1 t| \leq Dt^2, \quad \text{for } |t| \leq R.$$

Therefore $\tilde{X}(x)$ can be written as

$$\tilde{X}(x) = c_0 + c_1 \frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{3.25}$$

for $x \in \boldsymbol{C}$ with $\operatorname{Re} x > -\varepsilon_1$, $|\operatorname{Im} x| < \varepsilon_1$ and $|x| > M$, where $M = 1/R > 0$ is a large enough constant and $c_0 = c_0(r_0, r_1)$, $c_1 = c_1(r_0, r_1)$ are constants. We note that the remainder term in $\mathcal{O}(1/x^2)$ is estimated uniformly with respect to (r_0, r_1) as above. By using (3.25),

$$\{\tilde{X}(x)\}^{m-1} = c_0^{m-1}\left\{1 + (m-1)\frac{c_1}{c_0}\frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)\right\},$$

and (3.23), we get

$$\begin{aligned} &mxc_0^{m-1}\left\{1 + (m-1)\frac{c_1}{c_0}\frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)\right\} + i\left\{c_0 + c_1\frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)\right\} \\ &\quad - (m\eta_0^{m-1} + r_1)x - (r_0 + i\eta_0) = 0. \end{aligned}$$

From this expansion we obtain

$$\begin{aligned} mc_0^{m-1} - m\eta_0^{m-1} - r_1 &= 0, \\ m(m-1)c_0^{m-2}c_1 + ic_0 - i\eta_0 - r_0 &= 0. \end{aligned}$$

Since $r_1 \in \boldsymbol{R}$ and $|r_1| < \varepsilon_2$, c_0 is the real root of the first equation near η_0. Otherwise the other choices of c_0 contradict the fact $|X(x, f(x)) - \eta_0| \leq \varepsilon_3$. Clearly

$$\begin{aligned} c_0 &= \left\{\eta_0^{m-1} + \frac{r_1}{m}\right\}^{1/(m-1)} = \eta_0\left\{1 + \frac{r_1}{m\eta_0^{m-1}}\right\}^{1/(m-1)}, \\ c_1 &= -\frac{i(c_0 - \eta_0) - r_0}{m(m-1)c_0^{m-2}}. \end{aligned}$$

The choice of c_0 and the fact $r_0 \in \boldsymbol{C}$ and $|r_0| < \varepsilon_2$ imply that $c_1 \in \boldsymbol{C}$ satisfies $|c_1| < C\varepsilon_2$. If (x, z) is restricted to the set $\tilde{E}_1$, we have

$$\begin{aligned} &m(m-1)xX(x,z)^{m-2} + i \\ &\quad = m(m-1)xc_0^{m-2}\left\{1 + (m-2)\frac{c_1}{c_0}\frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)\right\} + i, \end{aligned}$$

We note that

$$\operatorname{Im}\frac{-1}{A + iB} = \frac{B}{A^2 + B^2}, \quad \text{for } A, B \in \boldsymbol{R}.$$

In our case

$$A(x) = m(m-1)c_0^{m-2}\operatorname{Re} x + m(m-1)(m-2)c_0^{m-3}\operatorname{Re} c_1 + \operatorname{Re}\mathcal{O}\left(\frac{1}{x}\right),$$

$$B(x) = 1 + m(m-1)c_0^{m-2}\operatorname{Im} x + m(m-1)(m-2)c_0^{m-3}\operatorname{Im} c_1 + \operatorname{Im}\mathcal{O}\left(\frac{1}{x}\right).$$

The smallness of $|\mathrm{Im}\, x|$ and $|\mathrm{Im}\, c_1|$ implies $B(x) \geq 1/2$. Therefore we have

$$\mathrm{Im}\, \frac{\partial^2 \varphi}{\partial z^2}(x, f(x)) \geq \frac{C}{(1+|x|)^2}, \tag{3.26}$$

for $x \in \boldsymbol{C}$ with $\mathrm{Re}\, x \geq M$, $|\mathrm{Im}\, x| < \varepsilon_1$, where $M = M(\mu_0) > 0$ is independent of ε_1 and ε_2.

Next we consider the case $|x| \leq M$. We note that we can choose the constant ε_1 independently of $\varepsilon_2, \varepsilon_3$ and M. For $(x, z) \in \tilde{E}$ we have $|X(x, z) - \eta_0| < \varepsilon_3$. This shows $|\{X(x, z)\}^{m-2} - \eta_0^{m-2}| < C\varepsilon_3$. If we choose ε_1 and ε_3 small enough, we have

$$|\mathrm{Im}(m(m-1)x\{X(x, z)\}^{m-2})| \leq \frac{1}{2},$$

for $(x, z) \in \tilde{E}$ with $|x| \leq M$, $\mathrm{Re}\, x > -\varepsilon_1$ and $|\mathrm{Im}\, x| < \varepsilon_1$. We have

$$\mathrm{Im}\, \frac{\partial^2 \varphi}{\partial z^2}(x, z) \geq C, \tag{3.27}$$

for $(x, z) \in \tilde{E}$ with $|x| \leq M$.

From (3.26) and (3.27) we have

$$\mathrm{Im}\, \frac{\partial^2 \varphi}{\partial z^2}(x, z) \geq \frac{C}{(1+|x|)^2}, \tag{3.28}$$

for $(x, z) \in \tilde{E}$. This completes the proof of Lemma 3.2. ■

Let $x \in \boldsymbol{C}$ be such that $\mathrm{Re}\, x > -\varepsilon_1$, $|\mathrm{Im}\, x| < \varepsilon_1$, and let $z \in \boldsymbol{R}$, $|z - Y(\mathrm{Re}\, x; y_0, \eta_0)| < (1/2)\varepsilon_2(1+|x|)$. We set

$$g(x, z) = \mathrm{Im}\, \frac{\partial \varphi}{\partial z}(x, z).$$

Since

$$g(\mathrm{Re}\, x, Y(\mathrm{Re}\, x; y_0, \eta_0)) = \mathrm{Im}\, \frac{\partial \varphi}{\partial z}(\mathrm{Re}\, x, Y(\mathrm{Re}\, x; y_0, \eta_0)) = \mathrm{Im}(-\eta_0) = 0,$$

it follows from Lemma 3.2 that there exists a function $z(x) : \boldsymbol{C} \to \boldsymbol{R}$ such that

$$g(x, z(x)) = \mathrm{Im}\, \frac{\partial \varphi}{\partial z}(x, z(x)) = 0,$$

$$z(\mathrm{Re}\, x) = Y(\mathrm{Re}\, x; y_0, \eta_0).$$

Let us check the position of $z(x)$, when x varies in the region mentioned above. In fact the repeated use of the implicit function theorem shows the existence of a unique real point $z(x)$ if we check

$$|z(x) - Y(\operatorname{Re} x; y_0, \eta_0)| < \frac{1}{2}\varepsilon_2(1 + |x|). \tag{3.29}$$

for x belonging to the region specified above. The point $z = z(x)$ is characterized by

$$\begin{aligned}\operatorname{Im} \frac{\partial\varphi}{\partial z}(x, z(x)) &= \operatorname{Im}\{-\eta_0 + i(\kappa(x, z(x)) - y_0)\} \\ &= \operatorname{Re}(\kappa(x, z(x)) - y_0) = 0.\end{aligned}$$

Since $|\kappa(x, z) - y_0| < \varepsilon_3$, we can put $\kappa(x, z(x)) - y_0 = i\delta$, $\delta \in \boldsymbol{R}$ with $|\delta| < \varepsilon_3$. By (3.18)

$$\begin{aligned}&[\{1 + m(m-1)\eta^{m-2} \operatorname{Im} x\}^2 + m^2(m-1)^2\eta_0^{2(m-2)}(\operatorname{Re} x)^2] \\ &\quad\times (\kappa(x, z(x)) - y_0) \\ &= [\{1 + m(m-1)\eta^{m-2} \operatorname{Im} x\} + im(m-1)\eta_0^{m-2} \operatorname{Re} x] \\ &\quad\times \Bigg[z(x) - (y_0 + m\eta_0^{m-1}(\operatorname{Re} x)) - im\eta_0^{m-1}(\operatorname{Im} x) \\ &\qquad - m(\operatorname{Re} x + i \operatorname{Im} x) \sum_{j=2}^{m-1} \binom{m-1}{j} \eta_0^{m-j-1}(-i)^j(i\delta)^j\Bigg].\end{aligned}$$

Since $\operatorname{Re}\{\kappa(x, z(x)) - y_0\} = 0$, we have

$$z(x) - Y(\operatorname{Re} x; y_0, \eta_0) = d_1\delta^2 \operatorname{Re} x + \frac{1}{1 + d_2 \operatorname{Im} x}\{d_3(\operatorname{Im} x)(\operatorname{Re} x)\},$$

where d_j $(1 \le j \le 3)$ are real constants depending on η_0 and m. We have (3.24), because $|\operatorname{Im} x| < \varepsilon_1 < (1/8)\varepsilon_2$ and $d_1\delta^2 < d_1\varepsilon_3^2 < (1/4)\varepsilon_2$.

For $x \in \boldsymbol{C}$ with $\operatorname{Re} x > -\varepsilon_1$, $|\operatorname{Im} x| < \varepsilon_1$ let us set

$$\Phi(x) = -\operatorname{Im} \varphi(x, z(x)). \tag{3.30}$$

As proved in [15] (see Lemma 3.8 in [15]) we have

$$\frac{\partial}{\partial \operatorname{Re} x}\Phi(x) = 0. \tag{3.31}$$

In this paper we don't repeat the proof of this property. We have obtained the same global properties of the phase function φ as those in [15] except for (3.21), in fact (3.21) is valid in the case z is restricted in $\boldsymbol{R}$. This restriction is harmless because (3.21) will not be used in constructing the amplitude function, but only in Section 4.

Let $u(t,z)$ be the solution of the initial value problem

$$\begin{cases} [D_t + P(z,D_z)]u(t,z) = 0, \\ u|_{t=0} = u_0(z). \end{cases} \tag{3.32}$$

Let $\chi \in C_0^\infty(\boldsymbol{R})$ with $0 \le \chi \le 1$ and

$$\chi(r) = \begin{cases} 1, & |r| \le \frac{1}{4}\varepsilon_2, \\ 0, & |r| \ge \frac{1}{2}\varepsilon_2. \end{cases}$$

We set

$$\begin{aligned} &Su(t,x,\lambda) \\ &\quad = \int_{\boldsymbol{R}} e^{i\lambda\varphi(x,z)} f(x,z,\lambda)\chi\left(\frac{z - Y(\operatorname{Re} x; y_0, \eta_0)}{1+|x|}\right) u(t,z)\, dz, \end{aligned} \tag{3.33}$$

where $f = f(x,z,\lambda) = \sum_{k=0}^\infty f_k(x,z)\lambda^{-k}$ is the analytic symbol of order 0, elliptic near the support of the cutoff function. We have

$$\left(\frac{\partial}{\partial t} + \lambda^{m-1}\frac{\partial}{\partial x}\right) Su(t,x,\lambda) = i\lambda^m\left(\frac{1}{\lambda}D_x Su - \frac{1}{\lambda^m}SPu\right). \tag{3.34}$$

We set

$$\begin{aligned} I(t,x,\lambda) &= \frac{1}{\lambda}D_x Su - \frac{1}{\lambda^m}SPu \\ &= \int_{\boldsymbol{R}} \left(\frac{1}{\lambda}D_x - \frac{1}{\lambda^m}\,{}^tP(z,D_z)\right)(e^{i\lambda\varphi} f\chi) u(t,z)\, dz, \end{aligned} \tag{3.35}$$

where

$${}^tP(z,D_z)w = \sum_{l=0}^m (-D_z)^l (a_l(z)w(z)) = (-D_z)^m w + \sum_{l=0}^{m-1} b_l(z) D_z^l.$$

The coefficients also satisfy the condition (1.4). We define

$$F(x,z,\lambda) = \left(\frac{1}{\lambda}D_x - \frac{1}{\lambda^m}\,{}^tP(z,D_z)\right)(e^{i\lambda\varphi} f). \tag{3.36}$$

We have

$$
\begin{aligned}
&e^{-i\lambda\varphi}F(x,z,\lambda)\\
&=\frac{1}{\lambda}\left(D_x+\lambda\frac{\partial\varphi}{\partial x}\right)f-\frac{1}{\lambda^m}{}^tP\left(z,D_z+\lambda\frac{\partial\varphi}{\partial z}\right)f\\
&=\frac{\partial\varphi}{\partial x}f+\frac{1}{i\lambda}\frac{\partial f}{\partial x}-\frac{1}{\lambda^m}\left\{\left(-D_z-\lambda\frac{\partial\varphi}{\partial z}\right)^m f\right.\\
&\quad\left.+b_{m-1}(z)\left(D_z+\lambda\frac{\partial\varphi}{\partial z}\right)^{m-1}f+\sum_{l=0}^{m-2}b_l(z)\left(D_z+\lambda\frac{\partial\varphi}{\partial z}\right)^l f\right\}\\
&=\left\{\frac{\partial\varphi}{\partial x}-\left(-\frac{\partial\varphi}{\partial z}\right)^m\right\}f+\frac{1}{\lambda}\left\{\frac{1}{i}\frac{\partial}{\partial x}f\right.\\
&\quad\left.-\sum_{j=0}^{m-1}\left(-\frac{\partial\varphi}{\partial z}\right)^{m-j-1}(-D_z)\left(\left(-\frac{\partial\varphi}{\partial z}\right)^j f\right)-b_{m-1}(z)\left(\frac{\partial\varphi}{\partial z}\right)^{m-1}f\right\}\\
&\quad+\frac{1}{\lambda^2}(\cdots)+\cdots+\frac{1}{\lambda^m}{}^tP(z,D_z)f\\
&=\left\{\frac{\partial\varphi}{\partial x}-p_m\left(z,-\frac{\partial\varphi}{\partial z}\right)\right\}f+\frac{1}{i\lambda}\left\{\frac{\partial f}{\partial x}-(-1)^m m\left(\frac{\partial\varphi}{\partial z}\right)^{m-1}\frac{\partial f}{\partial z}\right.\\
&\quad\left.-(-1)^m\frac{1}{2}m(m-1)\left(\frac{\partial\varphi}{\partial z}\right)^{m-2}\left(\frac{\partial^2\varphi}{\partial z^2}\right)f-ib_{m-1}(z)\left(\frac{\partial\varphi}{\partial z}\right)^{m-1}f\right\}\\
&\quad+\frac{1}{\lambda^2}(\cdots)f+\cdots+\frac{1}{\lambda^m}{}^tP(z,D_z)f.
\end{aligned}
$$

By Lemma 3.1 we have

$$
\frac{\partial\varphi}{\partial x}-p_m\left(z,-\frac{\partial\varphi}{\partial z}\right)=0.
$$

We define the change of variables $\Psi:\mathcal{C}\to E$ by

$$
z=z(x,y)=Y\left(x;y,-\frac{\partial\varphi_0}{\partial y}(y)\right).
$$

We have

$$\frac{\partial}{\partial x}(f \circ \Psi) = \frac{\partial}{\partial x}(f(x, z(x, y)))$$
$$= \left(\frac{\partial}{\partial x} f - (-1)^m m \left(\frac{\partial \varphi}{\partial z}\right)^{m-1} \frac{\partial}{\partial z} f\right) \circ \Psi.$$

This calculus, properties of φ_0, (3.20) and (1.4) imply

$$i\lambda e^{-i\lambda\varphi} F \circ \Psi = \left(\frac{\partial}{\partial x} + d(x, y)\right) g - \sum_{l=1}^{m-1} \frac{1}{\lambda^l} Q_l(x, y, D_y) g, \tag{3.37}$$

where $g(x, y, \lambda) = f \circ \Psi = \sum_{k \geq 0} \lambda^{-k} g_k(x, y)$ and

$$\begin{cases} |d(x, y)| \leq \dfrac{C}{1 + |x|}, \\ Q_l(x, y, D_y) = \displaystyle\sum_{j=0}^{l+1} q_j^l(x, y) D_y^j, \\ |q_j^l(x, y)| \leq \dfrac{C_{l,j}}{(1 + |x|)^{1+\sigma_0}}. \end{cases} \tag{3.38}$$

From this expression, we construct a nice amplitude function.

LEMMA 3.3. *There exists an analytic symbol f of order zero defined in E such that*

$$|F(x, z, \lambda)| \leq C e^{\lambda \Phi(x) - \mu_0 \lambda} (1 + |x|)^{N_0}, \tag{3.39}$$

where $\mu_0 > 0$ and N_0 is an integer.

PROOF OF LEMMA 3.3. From (3.37) we shall construct $\{g_k\}$ inductively

$$\begin{cases} \dfrac{\partial g_0}{\partial x} + d g_0 = 0, \quad g_0|_{x=0} = 1, \\ \dfrac{\partial g_k}{\partial x} + d g_k = \displaystyle\sum_{l=1}^{M} Q_l g_{k-l}, \quad g_k|_{x=0} = 0, \quad (k \geq 1), \end{cases} \tag{3.40}$$

where

$$M = \begin{cases} k, & (k \leq m - 1), \\ m - 1, & (k \geq m). \end{cases}$$

We set

$$A(x, y) = \partial_x^{-1} d(x, y) = x \int_0^1 d(tx, y)\, dt.$$

From the fact that $A(x, y)$ is holomorphic with respect to y and (3.33), we get

$$|\partial_y^l A(x, y)| \leq C \log(e + |x|),$$
$$(x, y) \in \mathcal{O}_1 \subset\subset \mathcal{O},\ \forall l \in \boldsymbol{N},$$

where

$$\mathcal{O}_1 = \left\{ (x, y) \in \boldsymbol{C} \times \boldsymbol{C};\ \mathrm{Re}\, x > -\frac{1}{2}\varepsilon_1, |\mathrm{Im}\, x| < \frac{1}{2}\varepsilon_1, |y - y_0| < \frac{1}{2}\varepsilon_3 \right\}.$$

We set

$$h_k = e^{A(x,y)} g_k.$$

Then the analytic symbol h should be constructed by the equations

$$\text{(3.41)} \qquad \begin{cases} \dfrac{\partial h_0}{\partial x} = 0, \quad h_0|_{x=0} = 1, \\ \dfrac{\partial h_k}{\partial x} = \displaystyle\sum_{l=1}^{M} W_l h_{k-l}, \quad h_k|_{x=0} = 0, \quad (k \geq 1), \end{cases}$$

where

$$\begin{cases} W_l = e^A Q_l e^{-A} = \displaystyle\sum_{j=0}^{l+1} w_j^l(x, y) D_y^j, \\ |w_j^l(x, y)| \leq \dfrac{C_{j,l}}{(1 + |x|)^{1+\sigma_0}} \log(e + |x|). \end{cases}$$

The solutions of the equations above are given by

$$\text{(3.42)} \qquad \begin{cases} h_0(x, y) = 1, \\ h_k(x, y) = x \displaystyle\int_0^1 \sum_{l=1}^{M} W_l h_{k-l}(tx, y)\, dt, \quad (k \geq 1). \end{cases}$$

The existence and boundedness of the solution in $\mathcal{O}_1$ are not so difficult. However it is not clear that we have the estimates

$$|g_k(x, y)| \leq C_0 C_1^k k^k. \tag{3.43}$$

To prove these estimates we make use of the technique, "nested open set", as described in [15] and [19]. For $j \in (0 \cup \boldsymbol{N})$ we define the sequences $\{s_j\}, \{R_j\}, \{r_j\}$ by

$$\begin{cases} s_j = 2^j, \quad R_j = 2^{j(1+(1/2)\sigma_0)}, \quad j \geq 1, \\ r_j = r_{j-1} - \dfrac{r_{j-1}}{R_{j-1}}(s_j - s_{j-1}), \quad j \geq 1, \\ s_0 = 0, \quad R_0 = 2^{(1/4)\sigma_0}, \quad r_0 \text{ given in } (0, \varepsilon_1). \end{cases} \tag{3.44}$$

The monotone decreasing sequence $\{r_j\}_{j=0}^{\infty}$ converges to $r_\infty > 0$. We define the open set

$$\Omega_t^j = \{(x, y) \in \boldsymbol{C} \times \boldsymbol{C}; |y - y_0| + M_j(x) < r_j - t, \operatorname{Re} x \geq s_j\}, \tag{3.45}$$

where $t \in (0, r_j]$ and

$$M_j(x) = \frac{r_j}{R_j}|\operatorname{Re} x - s_j| + |\operatorname{Im} x|.$$

We note that $(s_j, y) \in \Omega_t^j$ implies $(s_j, y) \in \Omega_t^{j-1}$ for $j \geq 1$. Let ρ be a positive number. We denote by $A_{\rho,j}$ the space of formal analytic symbols $h = \sum_{k \geq 0} \lambda^{-k} h_k$ such that

$$\sup_{\Omega_t^j} |h_k| \leq f_{k,j}(h) k^k t^{-k}, \quad 0 < t \leq r_j, \tag{3.46}$$

where $f_{k,j}(h)$ is the best constant and the series $\sum_{k=0}^{\infty} f_{k,j}(h)\rho^k$ is convergent. Then $\|h\|_{\rho,j} = \sum_{k=0}^{\infty} f_{k,j}(h)\rho^k$ is a norm on $A_{\rho,j}$.

The solution of (3.41) given by (3.42) gives rise to the formal symbol h which is a solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial h}{\partial x} - \displaystyle\sum_{l=1}^{m-1} \lambda^{-l} W_l h = 0, \quad \operatorname{Re} x \geq s_j, \\ h|_{x=s_j} = h(s_j, y). \end{cases} \tag{3.47}$$

We denote by h^j the value of the solution in Ω_0^j. h^j satisfies

$$\begin{cases} \dfrac{\partial h^j}{\partial x} - \displaystyle\sum_{l=1}^{m-1} \lambda^{-l} W_l h^j = 0, \\ h^j|_{x=s_j} = h^{j-1}|_{x=s_j}, \end{cases}$$

where we define $h^{-1}|_{x=0} = 1$. We set $\beta^j = h^j - h^{j-1}|_{x=s_j}$. The system can be written

$$\begin{cases} (\mathrm{Id} - B)\beta^j = B(h^{j-1}|_{x=s_j}), \\ \beta^j|_{x=s_j} = 0, \end{cases} \tag{3.48}$$

where

$$B = \partial_x^{-1} \sum_{l=1}^{m-1} \lambda^{-l} W_l = \lambda \partial_x^{-1} \sum_{l=1}^{m-1} \lambda^{-(l+1)} W_l, \tag{3.49}$$

and $\partial_x^{-1} v = \int_{s_j}^{x} v(\tau)\, d\tau$.

We shall show that B is a contraction map in $A_{\rho,j}$ for small $\rho > 0$.

LEMMA 3.4. *There exists a positive constant C such that for $j \geq 1$ and $\beta \in A_{\rho,j}$ we have*

$$\|B\beta\|_{\rho,j} \leq C \frac{\log(1+s_j)}{(1+s_j)^{1+\sigma_0}} \frac{R_j}{r_j} \rho \|\beta\|_{\rho,j}. \tag{3.50}$$

PROOF OF LEMMA 3.4. First we shall prove

$$\|\lambda \partial_x^{-1} \gamma\|_{\rho,j} \leq \frac{C}{\rho} \frac{R_j}{r_j} \|\gamma\|_{\rho,j}, \tag{3.51}$$

$$\text{for } \gamma = \sum_{k=1}^{\infty} \lambda^{-(k+1)} \gamma_{k+1} \in A_{\rho,j}.$$

Indeed

$$\partial_x^{-1} \gamma_{k+1}(x,y) = (x - s_j) \int_0^1 \gamma_{k+1}(s_j + \sigma(x - s_j), y)\, d\sigma.$$

If $(x,y) \in \Omega_t^j$, then

$$\begin{aligned}&|y-y_0|+M_j(s_j+\sigma(x-s_j))\\&\quad=|y-y_0|+\sigma M_j(x)\\&\quad<r_j-(t+(1-\sigma)M_j(x)),\end{aligned}$$

that is, $(s_j+\sigma(x-s_j),y)\in\Omega_T^j$ where $T=t+(1-\sigma)M_j(x)$. We note that $T\le t+M_j(x)<r_j$ for $(x,y)\in\Omega_t^j$, and

$$\begin{aligned}|x-s_j|&\le|\mathrm{Re}\,x-s_j|+|\mathrm{Im}\,x|\\&\le 2\frac{R_j}{r_j}M_j(x),\end{aligned}$$

because $M_j(x)=(r_j/R_j)|\mathrm{Re}\,x|+|\mathrm{Im}\,x|$ and $R_j/r_j\ge 1$. From the definition of $f_{k,j}(h)$ we have

$$f_{k,j}(h)=\frac{1}{k^k}\sup_{0<t\le r_j}\left\{t^k\sup_{(x,y)\in\Omega_t^j}|h_k(x,y)|\right\}.$$

For $(x,y)\in\Omega_t^j$ and $0<t\le r_j$, we have

$$\begin{aligned}&|\partial_x^{-1}\gamma_{k+1}(x,y)|\\&\quad\le|x-s_j|\int_0^1|\gamma_{k+1}(s_j+\sigma(x-s_j),y)|\,d\sigma\\&\quad\le|x-s_j|\int_0^1\sup_{(z,y)\in\Omega_T^j}|\gamma_{k+1}(z,y)|\,d\sigma\\&\quad\le|x-s_j|f_{k+1,j}(\gamma)(k+1)^{k+1}\int_0^1T^{-(k+1)}\,d\sigma\\&\quad=|x-s_j|f_{k+1,j}(\gamma)(k+1)^{k+1}\int_0^1\{t+(1-\sigma)M_j(x)\}^{-(k+1)}\,d\sigma\\&\quad=|x-s_j|f_{k+1,j}(\gamma)(k+1)^{k+1}\left[\frac{1}{kM_j(x)}\{t+(1-\sigma)M_j(x)\}^{-k}\right]_{\sigma=0}^{\sigma=1}\\&\quad\le|x-s_j|f_{k+1,j}(\gamma)(k+1)^{k+1}\frac{1}{kM_j(x)}t^{-k}\\&\quad\le 2\frac{R_j}{r_j}f_{k+1,j}(\gamma)\frac{(k+1)^{k+1}}{k}t^{-k}.\end{aligned}$$

We obtain

$$\begin{aligned}
\|\lambda \partial_x^{-1} \gamma\|_{\rho,j} &= \left\| \sum_{k=0}^{\infty} \lambda^{-k} \partial_x^{-1} \gamma_{k+1} \right\|_{\rho,j} \\
&= \sum_{k=1}^{\infty} \frac{\rho^k}{k^k} \sup_{0<t\le r_j} \left\{ t^k \sup_{(x,y)\in\Omega_t^j} |\partial_x^{-1} \gamma_{k+1}(x,y)| \right\} \\
&\le \sum_{k=1}^{\infty} \frac{\rho^k}{k^k} \sup_{0<t\le r_j} \left\{ t^k \cdot 2 \frac{R_j}{r_j} f_{k+1,j}(\gamma) \frac{(k+1)^{k+1}}{k} t^{-k} \right\} \\
&= \sum_{k=1}^{\infty} 2 \frac{R_j}{r_j} f_{k+1,j}(\gamma) \left(1 + \frac{1}{k}\right)^k \left(1 + \frac{1}{k}\right) \rho^k \\
&\le \frac{C}{\rho} \frac{R_j}{r_j} \sum_{k=1}^{\infty} f_{k+1,j}(\gamma) \rho^{k+1} \\
&\le \frac{C}{\rho} \frac{R_j}{r_j} \|\gamma\|_{\rho,j}.
\end{aligned}$$

This completes the estimate (3.51).

Next we shall prove

$$\left\| \frac{1}{\lambda} \frac{\partial}{\partial y} \gamma \right\|_{\rho,j} \le C\rho \|\gamma\|_{\rho,j}, \quad \text{for } \gamma \in A_{\rho,j}. \tag{3.52}$$

If $(x,y) \in \Omega_t^j$ and $|z-y| = t - t_1$, then

$$\begin{aligned}
|z - y_0| + M_j(x) &\le |z-y| + |y-y_0| + M_j(x) \\
&< (t - t_1) + (r_j - t) = r_j - t_1,
\end{aligned}$$

that is, $(x,z) \in \Omega_{t_1}^j$. For $(x,y) \in \Omega_t^j$ it follows from Cauchy's formula

$$\begin{aligned}
&\left| \frac{\partial}{\partial y} \gamma_{k-1}(x,y) \right| \\
&\quad = \left| \frac{1}{2\pi i} \int_{|z-y|=|t-t_1|} \frac{\gamma_{k-1}(x,z)}{(y-z)^2} \, dz \right| \\
&\quad \le \frac{1}{2\pi} \cdot \frac{1}{|t-t_1|^2} \sup_{(x,z)\in\Omega_{t_1}^j} |\gamma_{k-1}(x,y)| \int_{|z-y|=|t-t_1|} |dz| \\
&\quad \le \frac{1}{|t-t_1|} \sup_{(x,z)\in\Omega_{t_1}^j} |\gamma_{k-1}(x,y)|.
\end{aligned}$$

For

$$\frac{1}{\lambda}\frac{\partial}{\partial y}\gamma = \frac{1}{\lambda}\frac{\partial}{\partial y}\sum_{k=0}^{\infty}\frac{1}{\lambda^k}\gamma_k = \sum_{k=1}^{\infty}\frac{1}{\lambda^k}\delta_k = \delta,$$

where $\delta_k = \partial_y \gamma_{k-1}$, we have

$$\begin{aligned}
\|\delta\|_{\rho,j} &= \sum_{k=1}^{\infty} f_{k,j}(\delta)\rho^k \\
&= \sum_{k=1}^{\infty}\frac{\rho^k}{k^k}\sup_{0<t\le r_j}\left\{t^k \sup_{(x,y)\in\Omega_t^j}|\delta_k(x,y)|\right\} \\
&\le \sum_{k=1}^{\infty}\frac{\rho^k}{k^k}\sup_{0<t\le r_j}\left\{\frac{t^k}{|t-t_1|}\sup_{(x,z)\in\Omega_{t_1}^j}|\gamma_{k-1}(x,z)|\right\} \\
&= \rho\sup_{0<t\le r_j}\left\{\frac{t}{|t-t_1|}\sup_{(x,z)\in\Omega_{t_1}^j}|\gamma_0(x,z)|\right\} \\
&\quad + \sum_{k=2}^{\infty}\frac{\rho^k}{k^k}\sup_{0<t\le r_j}\left\{\frac{t^k}{|t-t_1|}\cdot\frac{1}{t_1^{k-1}}(k-1)^{k-1}\right. \\
&\quad \left.\times\frac{1}{(k-1)^{k-1}}t_1^{k-1}\sup_{(x,z)\in\Omega_{t_1}^j}|\gamma_{k-1}(x,z)|\right\}.
\end{aligned}$$

For $k=1$ we choose t_1 as $t_1 = (1/3)t$. It follows from $t_1 < (1/3)t \le r_j$ that

$$\begin{aligned}
&\rho\sup_{0<t\le r_j}\left\{\frac{3t_1}{|3t_1-t_1|}\sup_{(x,z)\in\Omega_{t_1}^j}|\gamma_0(x,z)|\right\} \\
&\quad\le \frac{3}{2}\rho\sup_{0<t\le r_j}\left\{\sup_{(x,z)\in\Omega_{t_1}^j}|\gamma_0(x,z)|\right\} = \frac{3}{2}\rho f_{0,j}(\gamma).
\end{aligned}$$

For $k\ge 2$ we choose t_1 as

$$t_1 = \frac{k-1}{k}t = \left(1-\frac{1}{k}\right)t,$$

then we have

$$\frac{t^k}{|t-t_1|}\cdot\frac{1}{t_1^{k-1}}(k-1)^{k-1}=\frac{t^k}{\frac{t}{k}}\cdot\frac{k^{k-1}}{(k-1)^{k-1}t^{k-1}}\cdot(k-1)^{k-1}=k^k.$$

We have

$$\begin{aligned}
\|\delta\|_{\rho,j} &\le \frac{3}{2}\rho f_{0,j}(\gamma)+\sum_{k=2}^{\infty}\frac{\rho^k}{k^k}k^k\sup_{0<t\le r_j}\left\{\frac{1}{(k-1)^{k-1}}t_1^{k-1}\sup_{(x,z)\in\Omega_{t_1}^j}|\gamma_{k-1}(x,z)|\right\}\\
&\le \frac{3}{2}\rho f_{0,j}(\gamma)+\rho\sum_{k=2}^{\infty}\frac{\rho^{k-1}}{(k-1)^{k-1}}\sup_{0<t_1\le r_j}\left\{t_1^{k-1}\sup_{(x,z)\in\Omega_{t_1}^j}|\gamma_{k-1}(x,z)|\right\}\\
&\le C\rho\sum_{k=1}^{\infty}f_{k-1,j}(\gamma)\rho^{k-1}\\
&\le C\rho\|\gamma\|_{\rho,j}.
\end{aligned}$$

This implies (3.52).

In the same way we have

$$\left\|\frac{1}{\lambda}\gamma\right\|_{\rho,j}\le C\rho\|\gamma\|_{\rho,j},\quad \text{for } \gamma\in A_{\rho,j}. \tag{3.53}$$

In our case

$$B=\lambda\partial_x^{-1}\frac{1}{\lambda}\sum_{l=1}^{m-1}\frac{1}{\lambda^l}W_l,$$

where

$$\begin{cases}W_l=\displaystyle\sum_{\alpha=0}^{l+1}w_\alpha^l(x,y)D_y^\alpha,\\[2mm] |w_\alpha^l(x,y)|\le \dfrac{C_{l,\alpha}}{(1+s_j)^{1+\sigma_0}}\log(1+s_j),\quad \text{for } (x,y)\in\Omega_t^j.\end{cases}$$

It follows from (3.52), (3.53) and the expressions above that we have

$$\left\|\sum_{l=1}^{m-1}\frac{1}{\lambda^{l+1}}W_l\beta\right\|_{\rho,j}\le C\frac{\log(1+s_j)}{(1+s_j)^{1+\sigma_0}}\rho^2\|\beta\|_{\rho,j}. \tag{3.54}$$

The estimates (3.51) and (3.54) give the required result of Lemma 3.4. ■

The same calculation shows

$$\|B\beta\|_{\rho,0} \leq C_0\rho\|\beta\|_{\rho,0}.$$

For $j \geq 1$ we set

$$K_j = C\frac{\log(1+s_j)}{(1+s_j)^{1+\sigma_0}}\frac{R_j}{r_j}\rho.$$

If we choose a small enough ρ, we have

$$\begin{cases} K_j \leq \dfrac{C\rho}{r_\infty} < 1, \\ K_j \leq C2^{-(1/3)\sigma_0 j}, \end{cases}$$

for $j \geq 1$ and $K_0 = C_0\rho < 1$. From (3.48) we have

$$\|\beta^j\|_{\rho,j} \leq K_j\|\beta^j\|_{\rho,j} + K_j\|(h^{j-1}|_{x=s_j})\|_{\rho,j},$$

so

$$\|\beta^j\|_{\rho,j} \leq \frac{K_j}{1-K_j}\|(h^{j-1}|_{x=s_j})\|_{\rho,j} \quad \text{for } j \geq 0. \tag{3.55}$$

Since $(s_j, y) \in \Omega_t^j$ implies $(s_j, y) \in \Omega_t^{j-1}$, we have

$$|h_k^{j-1}(s_j, y)| \leq \sup_{(x,y)\in\Omega_t^{j-1}} |h_k^{j-1}(x,y)| \leq f_{k,j-1}(h^{j-1})k^k t^{-k},$$

for $0 < t \leq r_j (< r_{j-1})$, and

$$\begin{aligned} \|(h^{j-1}|_{x=s_j})\|_{\rho,j} &= \sum_{k=0}^{\infty}\frac{\rho^k}{k^k}\sup_{0<t\leq r_j}\left\{t^k \sup_{(x,y)\in\Omega_t^j}|h_k^{j-1}(s_j,y)|\right\} \\ &\leq \sum_{k=0}^{\infty}\frac{\rho^k}{k^k}\sup_{0<t\leq r_j}\{t^k f_{k,j-1}(h^{j-1})k^k t^{-k}\} \\ &= \sum_{k=0}^{\infty}\rho^k f_{k,j-1}(h^{j-1}) \\ &= \|h^{j-1}\|_{\rho,j-1}. \end{aligned}$$

Since $\beta^j = h^j - h^{j-1}|_{x=s_j}$, we obtain

$$\|h^j\|_{\rho,j} \le \frac{1}{1-K_j}\|h^{j-1}\|_{\rho,j-1}.$$

and hence

(3.56) $$\|h^j\|_{\rho,j} \le \left(\prod_{l=1}^{j}\frac{1}{1-K_l}\right)\|h^0\|_{\rho,0} \le C\|h^0\|_{\rho,0}.$$

Since $h^{-1}|_{x=0} = 1$, it follows from (3.55), (3.56) and the definition of the norm that

$$\sup_{\Omega_t^j}|h_k^j| \le C\rho^{-k}k^k t^{-k}, \quad 0 < t \le r_j.$$

Since h_k^j is the restriction of the solution h_k to Ω_t^j we have

$$|h_k(x,y)| \le \sup_{\Omega^j_{(1/2)r_\infty}}|h_k| \le C\rho^{-k}k^k\left(\frac{r_\infty}{2}\right)^{-k}.$$

So the system (3.41) has a solution $h = \sum_{k\ge 0}\lambda^{-k}h_k$ such that for $k \ge 0$ we have

$$|h_k(x,y)| \le C^k k^k,$$

in the set $\mathcal{O}_2 = \{(x,y); |y-y_0| + |\mathrm{Im}\, x| \le (1/2)r_\infty, \mathrm{Re}\, x \ge 0\}$.

It follows from the definition $A(x,y)$, h_k and the estimates above that the system (3.40) has a solution $g = \sum_{k\ge 0}\lambda^{-k}g_k$ with

(3.57) $$|g_k(x,y)| \le C^k k^k(1+|x|)^{N_1}, \quad (x,y) \in \mathcal{O}_2,$$

where N_1 is a fixed integer.

Continuation of the proof of Lemma 3.3. Let us take $g = \sum_{k=0}^K g_k$ where g_k have been found in (3.40), and $K \in \boldsymbol{N}$ is chosen later. Then by (3.37), (3.57) and Cauchy's formula we have

(3.58) $$\begin{aligned}|i\lambda e^{-i\lambda\varphi}F\circ\Psi| &\le \frac{1}{\lambda^{K+1}}C^K K^K(1+|x|)^{N_1}\\ &\le \frac{1}{\lambda}\left(\frac{CK}{\lambda}\right)^K(1+|x|)^{N_1}.\end{aligned}$$

We take k_0 and λ_0 with $0 < k_0 \leq 1/2C$ and $\lambda_0 \geq 2/k_0$ respectively. Defining the integer $K = K(\lambda)$ with $k_0\lambda_0 - 1 < K = [k_0\lambda] \leq k_0\lambda$ for $\lambda \geq \lambda_0$, we have

$$\begin{aligned}\log\left(\frac{CK}{\lambda}\right) &= \log\left(\frac{C}{\lambda}[k_0\lambda]\right) \leq \log\left(\frac{C}{\lambda}k_0\lambda\right) \\ &\leq \log\left(\frac{1}{2}\right) = -\mu_1 < 0.\end{aligned}$$

We get

$$\begin{aligned}|i\lambda e^{-i\lambda\varphi} F \circ \Psi| &\leq \frac{1}{\lambda} e^{K \log(CK/\lambda)}(1 + |x|)^{N_1} \\ &\leq \frac{1}{\lambda} e^{-\mu_1[k_0\lambda]}(1 + |x|)^{N_1} \\ &\leq \frac{1}{\lambda} e^{-\mu_1(k_0\lambda - 1)}(1 + |x|)^{N_1} \\ &\leq \frac{e^{\mu_1}}{\lambda} e^{-\mu_1 k_0\lambda}(1 + |x|)^{N_1}.\end{aligned} \tag{3.59}$$

The proof of Lemma 3.3 is completed by setting a positive number $\mu_0 = \mu_1 k_0$. ■

4. Proof of the Main Theorem

We shall prove the main theorem. Thanks to Proposition 4.1 and (1.7) it suffices to prove Theorem 1.1 for a special point $\rho_0 = (y_0, \eta_0) \in T^*\boldsymbol{R}\backslash 0$.

PROPOSITION 4.1 (Theorem 6.1. in [15]). *Let $\rho_0 = (y_0, \eta_0) \in T^*\boldsymbol{R}\backslash 0$ and γ_{ρ_0} be the bicharacteristic of p_m passing through ρ_0. Let $t_0 \in \boldsymbol{R}$ and $u \in C(\boldsymbol{R}; L^2(\boldsymbol{R}))$ be the solution of* (1.2).

$$\text{If } \rho_0 \notin \widetilde{WF_A}[u(t_0, \cdot)], \quad \text{then } \widetilde{WF_A}[u(t_0, \cdot)] \cap \gamma_{\rho_0} = \varnothing. \tag{4.1}$$

This theorem is a local one. The difference between the orders of the operators does not appear as long as we consider the problems locally. Indeed we can solve the eikonal equation $(\partial\varphi/\partial x)(x, z) = p_m(z, -(\partial\varphi/\partial z)(x, z))$ and the transport equations locally. So the proof of Theorem 6.1. in [15] works well even in our cases $m \geq 3$.

It is enough to show Theorem 1.1 in the case that ρ_0 is the outgoing point, that is, $\rho_0 = (y_0, \eta_0) \in T^*\boldsymbol{R}\backslash 0$ with $y_0 > 2R_0$, and $y_0\eta_0^{m-1} \geq 0$. We shall show

$\rho_0 \notin \widetilde{WF_A}[u(t_0,\cdot)]$ for any $t_0 < 0$ under the hypotheses of Theorem 1.1. We define the set

$$E_0 = \{(x,z) \in \boldsymbol{C} \times \boldsymbol{R}; \operatorname{Re} x \geq -\varepsilon_0, |\operatorname{Im} x| < \varepsilon_0, \\ |z - Y(\operatorname{Re} x; y_0, \eta_0)| < \varepsilon_0(1 + |x|)\}, \tag{4.2}$$

for $\varepsilon_0 > 0$. Let us take ε_0 such that $\varepsilon_0 \ll (1/2)\varepsilon_1 < (1/4)\varepsilon_2$, $\varepsilon_0 < (1/2)r_\infty$ and the initial data $u_0 \in X^+_{\rho_0}$. If ε_0 is small enough, then we have $E_0 \subset \tilde{E} \subset E$. Let $\chi \in C_0^\infty(\boldsymbol{R})$ with $0 \leq \chi \leq 1$ and

$$\chi(r) = \begin{cases} 1, & |r| \leq \frac{1}{2}\varepsilon_0, \\ 0, & |r| \geq \varepsilon_0. \end{cases}$$

Let $u(t,\cdot)$ be the solution of (1.2). Let us apply the transformation S introduced in (3.33), where the phase function $\varphi(x,y)$ and the amplitude function $f = \sum_{k=0}^K \lambda^{-k} f_k$ are given in Lemma 3.1 and 3.3, respectively. Then we have

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \lambda^{m-1}\dfrac{\partial}{\partial x}\right) Su(t,x,\lambda) = i\lambda^m I(t,x,\lambda), & t < 0, \\ Su(0,x,\lambda) = Su_0(x,\lambda). \end{cases} \tag{4.3}$$

We obtain

$$Su(t,x,\lambda) = Su(0,x - \lambda^{m-1}t,\lambda) \\ + i\lambda^m \int_0^t I(\tau, x + \lambda^{m-1}(\tau - t),\lambda)\, d\tau. \tag{4.4}$$

We have

$$\begin{aligned} &Su(0,x - \lambda^{m-1}t,\lambda) \\ &\quad = Su_0(x - \lambda^{m-1}t,\lambda) \\ &\quad = \int_{\boldsymbol{R}} e^{i\lambda\varphi(x - \lambda^{m-1}t,z)} f(x - \lambda^{m-1}t,z,\lambda) \\ &\qquad \chi\left(\frac{z - Y(\operatorname{Re} x - \lambda^{m-1}t; y_0,\eta_0)}{1 + |x - \lambda^{m-1}t|}\right) e^{-(\delta_0/4)|z|^{1/(m-1)}} e^{(\delta_0/4)|z|^{1/(m-1)}} u_0(z)\, dz. \end{aligned}$$

If ε_0 and $|t - t_0|$ are small enough, we get $|z| \geq (1/4)|\eta_0|^{m-1}|t_0|\lambda^{m-1}$ on the support of χ. Then we have

$$|Su_0(x - \lambda^{m-1}t, \lambda)| \tag{4.5}$$
$$\le Ce^{\lambda\Phi(x)-(1/16)|\eta_0||t_0|^{1/(m-1)}\delta_0\lambda} \int_{\boldsymbol{R}} |\chi(\cdots)e^{(\delta_0/4)|z|^{1/(m-1)}} u_0(z)|\, dz$$
$$\le Ce^{\lambda\Phi(x)-(1/16)|\eta_0||t_0|^{1/(m-1)}\delta_0\lambda} \|e^{\delta_0|z|^{1/(m-1)}} u_0\|_{L^2(\Gamma_{\rho_0}^+)}.$$

On the other hand we write

$$I(t,x,\lambda) = I_1(t,x,\lambda) + I_2(t,x,\lambda),$$

where

$$I_k(t,x,\lambda) = \int_{\boldsymbol{R}} J_k(x,z,\lambda)u(t,z)\, dz, \quad (k = 1,2), \tag{4.6}$$

and

$$J_1(x,z,\lambda) = \left\{\left(\frac{1}{\lambda}D_x - \frac{1}{\lambda^m}{}^tP(z,D_z)\right)(e^{i\lambda\varphi}f)\right\}\chi, \tag{4.7}$$

$$J_2(x,z,\lambda) = \left[\left(\frac{1}{\lambda}D_x - \frac{1}{\lambda^m}{}^tP(z,D_z)\right),\chi\right](e^{i\lambda\varphi}f), \tag{4.8}$$

where $[P,Q] = PQ - QP$. To estimate I_1 we use Lemma 3.3,

$$|I_1(t,x,\lambda)| \tag{4.9}$$
$$\le Ce^{\lambda\Phi(x)-\mu_0\lambda}(1+|x|)^{N_0} \int \chi\left(\frac{z - Y(\mathrm{Re}\ x; y_0, \eta_0)}{1+|x|}\right)|u(t,z)|\, dz,$$
$$\le Ce^{\lambda\Phi(x)-\mu_0\lambda}(1+|x|)^{N_0+(1/2)}\|u(t,x)\|_{L^2(\Gamma_{\rho_0}^+)}.$$

The term I_2 can be estimated by Lemma 3.2. For $x \in \boldsymbol{C}$ and $z \in \boldsymbol{R}$ we have the Taylor's expansion with respect to z,

$$\mathrm{Re}(i\varphi(x,z)) \tag{4.10}$$
$$= -\mathrm{Im}\ \varphi(x,z(x)) - \mathrm{Im}\ \frac{\partial\varphi}{\partial z}(x,z(x))(z - z(x))$$
$$-\frac{1}{2}(z - z(x))^2 \int_0^1 (1-\theta)\ \mathrm{Im}\ \frac{\partial^2\varphi}{\partial z^2}(x, z(x) + \theta(z - z(x)))\, d\theta,$$

where $\partial/\partial z$ is the real differentiation and $z(x)$ is the real valued function given in Section 3. On the support of χ Lemma 3.2 implies

$$\mathrm{Re}(i\varphi(x,z)) \leq \Phi(x) - \frac{C}{(1+|x|)^2}(z-z(x))^2. \tag{4.11}$$

Since we have

$$|z - z(x)| \geq \frac{1}{4}\varepsilon_0(1+|x|), \tag{4.12}$$

on the support of a derivative of χ, there exist $\delta > 0$ and $\mu_1 > 0$ such that for $|t - t_0| < \delta$ we have

$$\begin{aligned} &\left|\lambda^m \int_0^t I(\tau, x + \lambda^{m-1}(\tau - t), \lambda)\, d\tau\right| \\ &\qquad \leq Ce^{\lambda\Phi(x)-\mu_1\lambda} \sup_{|t-t_0|<\delta} \|u(t,\cdot)\|_{L^2(\Gamma_{\rho_0}^+)}. \end{aligned} \tag{4.13}$$

It follows from (4.4), (4.5) and (4.13) that there exist $C > 0$ and $\mu_2 > 0$ such that

$$|Su(t,x,\lambda)| \leq Ce^{\lambda\Phi(x)-\mu_2\lambda}. \tag{4.14}$$

Let us set

$$Tu(t,x,\lambda) = \int_R e^{i\lambda\varphi(x,z)} f(x,z,\lambda)\chi(z-y_0)u(t,z)\, dz. \tag{4.15}$$

In view of Proposition 5.3 in [15], it follows from (4.14) that

THEOREM 4.1. *Let $t_0 < 0$. Assume $u_0 \in X_{\rho_0}^+$, then there exist $C > 0$, $\varepsilon > 0$, $\delta > 0$ and $\mu > 0$ such that for $|x| < \varepsilon$ and $|t - t_0| < \delta$, we have*

$$|Tu(t,x,\lambda)| \leq Ce^{\lambda\Phi(z)-\mu\lambda}. \tag{4.16}$$

Now the proof of Theorem 1.1 is completed because of (4.15) and the definition of the uniform analytic wave front set.

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