A MUCH LARGER CLASS OF FRÖLICHER SPACES THAN THAT OF CONVENIENT VECTOR SPACES MAY EMBED INTO THE CAHIERS TOPOS

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Abstract

It is well known that the category of Frölicher spaces and smooth mappings is cartesian closed. The principal objective in this paper is to show that the full subcategory of Frölicher spaces that believe in fantasy that every Weil functor is really an exponentiation by the corresponding infinitesimal object is also cartesian closed. Under the assumption that a conjecture holds, it is shown that these Frölicher spaces embed into the Cahiers topos with the cartesian closed structure being preserved.

1. Introduction

It is widely held that the category of finite-dimensional smooth manifolds does not behave itself well. It is not left exact, though it barely allows transversal pullbacks. It is not cartesian closed, though it enjoys Weil functors so that it admits infinitesimal objects to exponentiate over smooth manifolds as the shadow of a shade. In order to develop really useful and meaningful infinite-dimensional differential geometry, we should emancipate ourselves from the regnant philosophy

of *manifolds*, which are obtained by pasting together some parts of a linear space. The prototype of the notion of manifold is traced back to Bernhart Riemann, and its modern definition should be attributed to Hassler Whitney.

It is not difficult, after the manner of the well-established differential geometry of finite-dimensional smooth manifolds, to introduce infinite-dimensional manifolds by using Banach spaces, Hilbert spaces, Fréchet spaces or more generally, convenient vector spaces in place of finite-dimensional vector spaces. However, the differential geometry of such infinite-dimensional manifolds is neither so interesting theoretically nor so useful practically. What is highly impressive, the last quarter of the 20th century witnessed that Frölicher, Kriegl and others finally arrived at the correct notion of a smooth space, which is often called a *Frölicher space* in respect to his fame (cf. [2], [3] and [5]). What is most important and surprising about Frölicher spaces is that they form a cartesian closed category, while infinite-dimensional manifolds of any kind do not.

The notion of a *Weil functor*, which is intended to stand for exponentiation by the infinitesimal object corresponding to a Weil algebra in fantasy, can be generalized easily to Frölicher spaces in general. The principal objective in this paper is to show that all the Frölicher spaces that believe that all Weil functors are really exponentiations by some adequate infinitesimal objects in imagination form a cartesian closed category. Such Frölicher spaces are called *Weil-exponentiable*. It is then shown, under the assumption that a conjecture holds, that the cartesian closed category of Weil-exponentiable Frölicher spaces embeds into the Cahiers topos with the cartesian closed structure being preserved. This is a direct generalization of Kock and Reyes' preceding result ([6] and [8]) that the category of convenient vector spaces embeds into the Cahiers topos with the cartesian closed structure being preserved. We hope that the paper will pave the way into a serious study on the intriguing class of such modest Frölicher spaces.

2. Preliminaries

2.1. Frölicher spaces

Frölicher and his followers have vigorously and consistently developed a general theory of smooth spaces, often called *Frölicher spaces* for his celebrity, which were intended to be the central object of study in infinite-dimensional differential geometry. A Frölicher space is an underlying set endowed with a class of real-valued functions on it (simply called *structure functions*) and a class of

mappings from $\mathbb{R}$ to the underlying set (called *structure curves*) subject to the condition that structure curves and structure functions should compose so as to yield smooth mappings from $\mathbb{R}$ to itself. It is required that the class of structure functions and that of structure curves should determine each other so that each of the two classes is maximal with respect to the other as far as they abide by the above condition. What is most important among many nice properties about the category **FS** of Frölicher spaces and smooth mappings is that it is cartesian closed, while the category of finite-dimensional smooth manifolds is not at all. It was Lawvere et al. [12] that established its cartesian closedness for the first time, though their original proof has been simplified considerably. For a standard reference on Frölicher spaces the reader is referred to [5].

2.2. Weil algebras

The notion of a *Weil algebra* was introduced by Weil himself in [17]. We denote by **W** the category of Weil algebras. Roughly speaking, each Weil algebra corresponds to an infinitesimal object in the shade. Although an infinitesimal object is undoubtedly imaginary in the real world, each Weil algebra yields a *Weil functor* on the category of manifolds of some kind to itself. Intuitively speaking, the Weil functor stands for the exponentiation by the infinitesimal object corresponding to the Weil algebra at issue. For Weil functors on the category of finite-dimensional smooth manifolds, the reader is referred to Section 35 of [9], while the reader can find a readable treatment of Weil functors on the category of smooth manifolds modeled on convenient vector spaces in Section 31 of [10].

2.3. Models of synthetic differential geometry

It is often forgotten that Newton, Leibniz, Euler and many other mathematicians in the 17th and 18th centuries developed differential calculus and analysis by using nilpotent infinitesimals. It is in the 19th century, in the midst of the industrial revolution, that nilpotent infinitesimals were overtaken by so-called $\varepsilon - \delta$ arguments. In the middle of the 20th century moribund nilpotent infinitesimals were revived by Grothendieck in algebraic geometry and by Lawvere in differential geometry. Differential geometry using nilpotent infinitesimals consistently is now called *synthetic differential geometry*, since it prefers synthetic arguments to dull calculations as in ancient Euclidean geometry. Model theory of synthetic differential geometry was developed vigorously by Dubuc and others around 1980 by using techniques of topos theory. For a standard reference on model theory of synthetic differential geometry the reader is referred to Kock's [7].

3. Weil Prolongation

The principal objective in this section is to assign, to each pair (X, W) of a Frölicher space X and a Weil algebra W, another Frölicher space $X \otimes W$ is called the *Weil prolongation of X with respect to W*, which is naturally extended to a bifunctor $\mathbf{FS} \times \mathbf{W} \to \mathbf{FS}$, and then to show that the functor $\cdot \otimes W : \mathbf{FS} \to \mathbf{FS}$ is product-preserving for any Weil algebra W.

We will first define $X \otimes W$ set-theoretically for any pair (X, W) of a Frölicher space X and a Weil algebra W. We define an equivalence relation $\equiv$ mod I on $\mathcal{C}^\infty(\mathbb{R}^n, X)$ to be

$$f \equiv g \mod I$$

iff

$$f(0, ..., 0) = g(0, ..., 0)$$

and

$$\chi \circ f - \chi \circ g \in I, \quad \text{for any} \ \chi \in \mathcal{C}^\infty(X, \mathbb{R}),$$

where $f, g \in \mathcal{C}^\infty(\mathbb{R}^n, X)$. The totality of equivalence classes with respect to $\equiv$ mod I is denoted by $X \otimes W$. This construction of $X \otimes W$ can naturally be extended to a functor $\cdot \otimes W : \mathbf{FS} \to \mathbf{Sets}$. We endow $X \otimes W$ with the initial smooth structure with respect to the mappings

$$X \otimes W \overset{\chi \otimes W}{\to} \mathbb{R} \otimes W,$$

where χ ranges over $\mathcal{C}^\infty(X, \mathbb{R})$. We note that the smooth structure on $\mathbb{R} \otimes W$ has already been discussed by Kock [6]. Indeed Kock [6] has indeed discussed how to endow $X \otimes W$ with a smooth structure in case that X is a convenient vector space. Apparently the set-theoretical constructions of $\mathbb{R} \otimes W$ in this paper and Kock's [6] should coincide above all. Therefore, we have to verify that

Proposition 1. *In case that X is a convenient vector space, our above definition of the smooth structure on $X \otimes W$ and that of Kock's [6] coincide.*

Proof. This follows readily from Theorem 1.3 of Kock's [6]. The details can safely be left to the reader.   □

Lemma 2. *Let* $f, g \in \mathcal{C}^{\infty}(\mathbb{R}^n, X)$ *and* $\varphi \in \mathcal{C}^{\infty}(X, Y)$. *Then*

$$f \equiv g \mod I$$

implies

$$\varphi \circ f \equiv \varphi \circ g \mod I.$$

Therefore, each $\varphi \in \mathcal{C}^{\infty}(X, Y)$ defines a mapping $X \otimes W \to Y \otimes W$ to be denoted by $\varphi \otimes W$, which is to be shown smooth.

Lemma 3. *The mapping* $\varphi \otimes W : X \otimes W \to Y \otimes W$ *is really smooth.*

Proof. This follows readily from the following commutative diagram:

$$
\begin{array}{ccc}
& \mathbb{R} \otimes W & \\
(\chi \circ \varphi) \otimes W \nearrow & & \nwarrow \chi \otimes W \\
X \otimes W & \overset{\varphi \otimes W}{\to} & Y \otimes W \\
\uparrow & & \uparrow \\
\mathcal{C}^{\infty}(\mathbb{R}^n, X) & \overset{\mathcal{C}^{\infty}(\mathbb{R}^n, \varphi)}{\to} & \mathcal{C}^{\infty}(\mathbb{R}^n, Y)
\end{array}
\quad ,
$$

where $\mathcal{C}^{\infty}(\mathbb{R}^n, X) \to X \otimes W$ and $\mathcal{C}^{\infty}(\mathbb{R}^n, Y) \to Y \otimes W$ are the canonical projections, and $\chi : Y \to \mathbb{R}$ is an arbitrary smooth mapping. $\square$

Let us assume that we are given two Weil algebras W_1 and W_2 of the forms $\mathcal{C}^{\infty}(\mathbb{R}^n)/I$ and $\mathcal{C}^{\infty}(\mathbb{R}^m)/J$ together with $\psi \in \hom_{\mathbf{w}}(W_1, W_2)$, which can be represented by a smooth map $\overline{\psi} : \mathbb{R}^m \to \mathbb{R}^n$ abiding by the conditions:

1. $\overline{\psi}(\underbrace{0, ..., 0}_{m}) = (\underbrace{0, ..., 0}_{n})$.

2. $\chi \in I$ implies $\chi \circ \overline{\psi} \in J$ for any $\chi \in \mathcal{C}^{\infty}(\mathbb{R}^n)$.

Then $\overline{\psi}$ naturally induces a mapping $X \otimes W_1 \to X \otimes W_2$, which is denoted by $X \otimes \psi$.

Lemma 4. *The mapping* $X \otimes \psi : X \otimes W_1 \to X \otimes W_2$ *is smooth.*

Proof. This follows readily from the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{R} \otimes W_1 & \xrightarrow{\mathbb{R} \otimes \psi} & & \mathbb{R} \otimes W_2 \\
\chi \otimes W_1 \uparrow & & & \uparrow \chi \otimes W_2 \\
X \otimes W_1 & \xrightarrow{X \otimes \psi} & & X \otimes W_2 \\
\uparrow & & & \uparrow \\
\mathcal{C}^\infty(\mathbb{R}^n, X) & \xrightarrow{\mathcal{C}^\infty(\overline{\psi}, X)} & & \mathcal{C}^\infty(\mathbb{R}^m, Y)
\end{array} \quad ,
$$

where $\mathcal{C}^\infty(\mathbb{R}^n, X) \to X \otimes W_1$ and $\mathcal{C}^\infty(\mathbb{R}^m, X) \to X \otimes W_2$ are the canonical projections, and $\chi : Y \to \mathbb{R}$ is an arbitrary smooth mapping. $\quad\square$

By combining Lemmas 3 and 4, we have

Proposition 5. *We have the bifunctor* $\otimes : \mathbf{FS} \times \mathbf{W} \to \mathbf{FS}$.

Now, we are going to show that the functor $\cdot \otimes W : \mathbf{FS} \to \mathbf{FS}$ is product-preserving for any Weil algebra W.

Lemma 6. *Let* $\pi_1 : X \times Y \to X$ *and* $\pi_2 : X \times Y \to Y$ *be the canonical projections. The functions on* $X \times Y$ *are generated by the compositions of* π_1 *followed by the functions on X and the compositions of* π_2 *followed by the functions on Y.*

Proof. The reader is referred to Proposition 1.1.4 of [5]. $\quad\square$

Corollary 7. *Let W be of the form* $\mathcal{C}^\infty(\mathbb{R}^n)/I$. *Let X and Y be Frölicher spaces. For any* $f, g \in \mathcal{C}^\infty(\mathbb{R}^n, X \times Y)$, *we have*

$$f \equiv g \bmod I$$

iff

$$\pi_1 \circ f \equiv \pi_1 \circ g \bmod I$$

and

$$\pi_2 \circ f \equiv \pi_2 \circ g \bmod I,$$

where $\pi_1 : X \times Y \to X$ *and* $\pi_2 : X \times Y \to Y$ *are the canonical projections.*

Proof. It suffices to note that the product $X \times Y$ in the category $\mathbf{FS}$ carries the initial structure induced by the mappings $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. More explicitly, the functions on $X \times Y$ are generated by the compositions of π_1 followed by functions on X and the compositions of π_2 followed by functions on Y, for which the reader is referred to Proposition 1.1.4 of [5]. $\quad\square$

Corollary 8. *Let* $U_{\mathbf{FS}} : \mathbf{FS} \to \mathbf{Sets}$ *be the forgetful functor. Then the functor* $U_{\mathbf{FS}} \circ (\cdot \otimes W) : \mathbf{FS} \to \mathbf{Sets}$ *is product-preserving, so that we have*

$$(X \times Y) \otimes W = (X \otimes W) \times (Y \otimes W)$$

set-theoretically for any Frölicher spaces X and Y.

Corollary 9. *The smooth structure of* $(X \times Y) \otimes W$ *is the initial structure with respect to the canonical projections* $(X \otimes W) \times (Y \otimes W) \to X \otimes W$ *and* $(X \otimes W) \times (Y \otimes W) \to Y \otimes W$.

Therefore, we have

Theorem 10. *Given a Weil algebra* W, *the functor* $\cdot \otimes W : \mathbf{FS} \to \mathbf{FS}$ *is product-preserving.*

4. Weil Exponentiability

A Frölicher space X is called *Weil-exponentiable* if

$$(X \otimes (W_1 \otimes_\infty W_2))^Y = (X \otimes W_1)^Y \otimes W_2 \tag{1}$$

holds naturally for any Frölicher space Y and any Weil algebras W_1 and W_2. If $Y = 1$, then (1) degenerates into

$$X \otimes (W_1 \otimes_\infty W_2) = (X \otimes W_1) \otimes W_2. \tag{2}$$

If $W_1 = \mathbb{R}$, then (1) degenerates into

$$(X \otimes W_2)^Y = X^Y \otimes W_2. \tag{3}$$

Proposition 11. *If X is a Weil-exponentiable Frölicher space, then so is* $X \otimes W$ *for any Weil algebra W.*

Proof. For any Frölicher space Y and any Weil algebras W_1 and W_2, we have

$$((X \otimes W) \otimes (W_1 \otimes_\infty W_2))^Y$$
$$= (X \otimes ((W \otimes_\infty W_1) \otimes_\infty W_2))^Y$$
$$= (X \otimes (W \otimes_\infty W_1))^Y \otimes W_2$$
$$= ((X \otimes W) \otimes W_1)^Y \otimes W_2. \qquad \square$$

Proposition 12. *If X and Y are Weil-exponentiable Frölicher spaces, then so is* $X \times Y$.

Proof. For any Frölicher space Z and any Weil algebras W_1 and W_2, we have

$$((X \times Y) \otimes (W_1 \otimes_\infty W_2))^Z$$

$$= \{(X \otimes (W_1 \otimes_\infty W_2)) \times (Y \otimes (W_1 \otimes_\infty W_2))\}^Z$$

$$= (X \otimes (W_1 \otimes_\infty W_2))^Z \times (Y \otimes (W_1 \otimes_\infty W_2))^Z$$

$$= ((X \otimes W_1)^Z \otimes W_2) \times ((Y \otimes W_1)^Z \otimes W_2)$$

$$= \{(X \otimes W_1)^Z \times (Y \otimes W_1)^Z\} \otimes W_2$$

$$= \{(X \otimes W_1) \times (Y \otimes W_1)\}^Z \otimes W_2$$

$$= \{(X \times Y) \otimes W_1\}^Z \otimes W_2. \qquad \square$$

Proposition 13. *If X is a Weil-exponentiable Frölicher space, then so is* X^Y *for any Frölicher space Y.*

Proof. For any Frölicher space Z and any Weil algebras W_1 and W_2, we have

$$(X^Y \otimes (W_1 \otimes_\infty W_2))^Z$$

$$= (X \otimes (W_1 \otimes_\infty W_2))^{Y \times Z}$$

$$= (X \otimes W_1)^{Y \times Z} \otimes W_2$$

$$= ((X \otimes W_1)^Y)^Z \otimes W_2$$

$$= (X^Y \otimes W_1)^Z \otimes W_2. \qquad \square$$

Theorem 14. *The full subcategory* $\mathbf{FS_{WE}}$ *of all Weil-exponentiable Frölicher spaces of the category* $\mathbf{FS}$ *of Frölicher spaces and smooth mappings is cartesian closed.*

Problem 15. Convenient vector spaces are all Weil-exponentiable. Find out a larger and interesting class consisting of Weil-exponentiable Frölicher spaces. If any, find out a Frölicher space which is not Weil-exponentiable.

5. The Embedding into the Cahiers Topos

Let $\mathbf{D}$ be the full subcategory of the category of $\mathcal{C}^\infty$-rings of form $\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W$ with a natural number n and a Weil algebra W. Now we would like to extend the Weil prolongation $\mathbf{FS_{WE}} \times \mathbf{W} \xrightarrow{\otimes} \mathbf{FS_{WE}}$ to a bifunctor $\mathbf{FS_{WE}} \times \mathbf{D} \xrightarrow{\otimes} \mathbf{FS_{WE}}$. On objects, we define

$$X \otimes C = \mathcal{C}^\infty(\mathbb{R}^n, X) \otimes W, \tag{4}$$

for any Weil-exponentiable Frölicher space X and any $C = \mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W$. By Proposition 13, $\mathcal{C}^\infty(\mathbb{R}^n, X)$ is Weil-exponentiable, so that $X \otimes C$ is Weil-exponentiable by Proposition 11. It is easy to see that the right hand of (4) is functorial in X, but we have not so far succeeded in establishing its functoriality in C. Therefore, we pose it as a conjecture.

Conjecture 16. *The right hand of* (4) *is functorial in C, so that we have a bifunctor* $\mathbf{FS_{WE}} \times \mathbf{D} \xrightarrow{\otimes} \mathbf{FS_{WE}}$.

In the following, we will assume that the conjecture is really true. We define the functor $\mathbf{J} : \mathbf{FS_{WE}} \to \mathbf{Sets^D}$ to be the exponential adjoint to the composite

$$\mathbf{FS_{WE}} \times \mathbf{D} \xrightarrow{\otimes} \mathbf{FS_{WE}} \to \mathbf{Sets},$$

where $\mathbf{FS_{WE}} \to \mathbf{Sets}$ is the underlying-set functor. Now, we have

Proposition 17. *For any Weil-exponentiable Frölicher space X and any object* $\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W$ *in* $\mathbf{D}$, *we have*

$$\mathbf{J}(X)^{\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W, \cdot)} = \mathbf{J}(X \otimes (\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W)).$$

Proof. Let $\mathcal{C}^\infty(\mathbb{R}^m) \otimes W'$ be an object in $\mathbf{D}$. Then

$$\mathbf{J}(X)^{\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W, \cdot)}(\mathcal{C}^\infty(\mathbb{R}^m) \otimes_\infty W')$$

$$= \hom_{\mathbf{Sets}^{\mathbf{D}}}\left(\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^m) \otimes_\infty W', \cdot), \mathbf{J}(X)^{\hom_{\mathbf{D}}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes_\infty W, \cdot)}\right)$$

$$= \hom_{\mathbf{Sets}^{\mathbf{D}}} \left(\hom_{\mathbf{D}}(\mathcal{C}^{\infty}(\mathbb{R}^{m}) \otimes_{\infty} W', \,\cdot\,) \times \hom_{\mathbf{D}}(\mathcal{C}^{\infty}(\mathbb{R}^{n}) \otimes_{\infty} W, \,\cdot\,), \mathbf{J}(X) \right)$$

$$= \hom_{\mathbf{Sets}^{\mathbf{D}}} \left(\hom_{\mathbf{D}}(\mathcal{C}^{\infty}(\mathbb{R}^{m+n}) \otimes_{\infty} W \otimes_{\infty} W', \,\cdot\,), \mathbf{J}(X) \right)$$

$$= \mathbf{J}(X)(\mathcal{C}^{\infty}(\mathbb{R}^{m+n}) \otimes_{\infty} W \otimes_{\infty} W')$$

$$= \mathcal{C}^{\infty}(\mathbb{R}^{m+n}, X) \otimes (W \otimes_{\infty} W')$$

$$= \mathcal{C}^{\infty}(\mathbb{R}^{m}, \mathcal{C}^{\infty}(\mathbb{R}^{n}, X) \otimes W) \otimes W'$$

$$= \mathbf{J}(X \otimes (\mathcal{C}^{\infty}(\mathbb{R}^{n}) \otimes_{\infty} W))(\mathcal{C}^{\infty}(\mathbb{R}^{m}) \otimes_{\infty} W'). \qquad \square$$

Proposition 18. *For any Weil-exponentiable Frölicher spaces X and Y, we have*

$$\mathbf{J}(X \times Y) = \mathbf{J}(X) \times \mathbf{J}(Y).$$

Proof. Let $\mathcal{C}^{\infty}(\mathbb{R}^{n}) \otimes W$ be an object in $\mathbf{D}$. Then

$$\mathbf{J}(X \times Y)(\mathcal{C}^{\infty}(\mathbb{R}^{n}) \otimes_{\infty} W)$$

$$= \mathcal{C}^{\infty}(\mathbb{R}^{n}, X \times Y) \otimes W$$

$$= \{\mathcal{C}^{\infty}(\mathbb{R}^{n}, X) \times \mathcal{C}^{\infty}(\mathbb{R}^{n}, Y)\} \otimes W$$

$$= \{\mathcal{C}^{\infty}(\mathbb{R}^{n}, X) \otimes W\} \times \{\mathcal{C}^{\infty}(\mathbb{R}^{n}, Y) \otimes W\}$$

$$= \mathbf{J}(X)(\mathcal{C}^{\infty}(\mathbb{R}^{n}) \otimes_{\infty} W) \times \mathbf{J}(Y)(\mathcal{C}^{\infty}(\mathbb{R}^{n}) \otimes_{\infty} W)$$

$$= (\mathbf{J}(X) \times \mathbf{J}(Y))(\mathcal{C}^{\infty}(\mathbb{R}^{n}) \otimes_{\infty} W). \qquad \square$$

Proposition 19. *For any Weil-exponentiable Frölicher spaces X and Y, we have the following isomorphism in* $\mathbf{Sets}^{\mathbf{D}}$:

$$\mathbf{J}(X^{Y}) = \mathbf{J}(X)^{\mathbf{J}(Y)}.$$

Proof. Let $\mathcal{C}^{\infty}(\mathbb{R}^{n}) \otimes W$ be an object in $\mathbf{D}$. Then

$$\mathbf{J}(X^{Y})(\mathcal{C}^{\infty}(\mathbb{R}^{n}) \otimes_{\infty} W)$$

$$= \mathcal{C}^{\infty}(\mathbb{R}^n, X^Y) \otimes W$$

$$= X^{Y \times \mathbb{R}^n} \otimes W$$

$$= (X^{\mathbb{R}^n} \otimes W)^Y.$$

On the other hand, we have

$$\mathbf{J}(X)^{\mathbf{J}(Y)}(\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W)$$

$$= \hom_{\mathbf{Sets}^{\mathbf{D}}} \left(\hom_{\mathbf{D}}(\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W, \cdot), \mathbf{J}(X)^{\mathbf{J}(Y)} \right)$$

$$= \hom_{\mathbf{Sets}^{\mathbf{D}}} \left(\hom_{\mathbf{D}}(\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W, \cdot) \times \mathbf{J}(Y), \mathbf{J}(X) \right)$$

$$= \hom_{\mathbf{Sets}^{\mathbf{D}}} \left(\mathbf{J}(Y), \mathbf{J}(X)^{\hom_{\mathbf{D}}(\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W, \cdot)} \right)$$

$$= \hom_{\mathbf{Sets}^{\mathbf{D}}} \left(\mathbf{J}(Y), \mathbf{J}(X \otimes (\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W)) \right)$$

$$= \hom_{\mathbf{FS}_{\mathbf{WE}}} \left(Y, X \otimes (\mathcal{C}^{\infty}(\mathbb{R}^n) \otimes_{\infty} W) \right). \qquad \square$$

Theorem 20. *The functor* $\mathbf{J} : \mathbf{FS}_{\mathbf{WE}} \to \mathbf{Sets}^{\mathbf{D}}$ *preserves the cartesian closed structure.* In other words, *it preserves finite products and exponentials. It is full and faithful. It sends the Weil prolongation to the exponentiation by the corresponding infinitesimal object.*

Proof. The first statement follows from Propositions 18 and 19. The second statement that it is full and faithful follows by the same token as in [6] and [8]. The final statement follows from Proposition 17. $\qquad \square$

The site of definition for the Cahiers topos $\mathcal{C}$ is the dual category $\mathbf{D}^{\mathrm{op}}$ of the category $\mathbf{D}$ together with the open-cover topology, so that we have the canonical embedding

$$\mathcal{C} \hookrightarrow \mathbf{Sets}^{\mathbf{D}}.$$

By the same token as in [6] and [8], we can see that the functor $\mathbf{J} : \mathbf{FS}_{\mathbf{WE}} \to \mathbf{Sets}^{\mathbf{D}}$ factors in the above embedding. The resulting functor is denoted by $\mathbf{J}_{\mathcal{C}}$. Since the above embedding preserves products and exponentials, Theorem 20 yields directly.

Theorem 21. *The functor* $\mathbf{J}_{\mathcal{C}} : \mathbf{FS_{WE}} \to \mathcal{C}$ *preserves the cartesian closed structure.* In other words, *it preserves finite products and exponentials. It is full and faithful. It sends the Weil prolongation to the exponentiation by the corresponding infinitesimal object.*

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