GENERALIZED q-BOSON ALGEBRAS AND THEIR INTEGRABLE MODULES

AKIRA MASUOKA

ABSTRACT. We define the generalized q-boson algebra $\mathcal{B}$ associated to a pair of Nichols algebras and a skew pairing. We study integrable $\mathcal{B}$-modules, generalizing results by M. Kashiwara and T. Nakashima on integrable modules over a q-boson (Kashiwara) algebra.

0. INTRODUCTION

As was realized by Andruskiewitsch and Schneider (see [AS]), the notion of Nichols algebras gives a sophisticated, 'coordinate-free' viewpoint to study quantized enveloping algebras and associated objects. This paper aims to clarify from that viewpoint what happens to q-boson (Kashiwara) algebras and their modules.

Let U_q denote the quantized enveloping algebra associated to a symmetrizable generalized Cartan matrix $\mathbb{A}$, and assume that q is transcendental over $\mathbb{Q}$. To study crystal bases of U_q, Kashiwara [K] introduced the q-boson (Kashiwara) algebra B_q, and proved that the minus part U_q^- is naturally an integrable left B_q-module (the Verma module), and is in fact simple as a B_q-module. He also announced without proof that every integrable left B_q-modules is isomorphic to the direct sum of some copies of U_q^-; see [K, Remark 3.4.10]. A proof of this fact was given later by Nakashima [N], who introduced the *extremal projector* for the purpose. Tan [Tan, Proposition 3.2] formulated the same result in the generalized situation when $\mathbb{A}$ is a symmetrizable Borcherds-Cartan matrix, and gave a proof. But, his proof and even formulation are incomplete, I think; see Remark 4.6.

As was shown in [M1], U_q can be constructed as a cocycle deformation of a simpler, graded Hopf algebra. In [M2], this construction was generalized in the context of (pre-)*Nichols algebras*, and as its special case, the construction of U_q by generalized quantum doubles, as was given by Joseph [J], was explained; see also [RS]. Recall that this last construction works, when we are given data R, S, τ. Here, $R = \bigoplus_{n=0}^{\infty} R(n)$, $S = \bigoplus_{n=0}^{\infty} S(n)$ are Nichols algebras (see [AS]), a sort of braided graded Hopf algebras, in the braided tensor categories ${}^J_J\mathcal{YD}$, ${}^K_K\mathcal{YD}$, respectively, of Yetter-Drinfeld modules, where J, K are (ordinary) Hopf algebras with bijective antipode. As their bosonizations, we have the graded Hopf algebras $R \rtimes J$, $S \rtimes K$. The τ above is a skew pairing $(R \rtimes J) \otimes (S \rtimes K) \to k$ such that $\tau(R(n) \otimes J, S(m) \otimes K) = 0$ if $n \neq m$. In this paper, given such R, S, τ, we define the *generalized q-boson algebra* $\mathcal{B}$ (see Definition 3.1), and prove for this, the result by

Kashiwara [K] and Nakashima [N] cited above, under the assumptions that $R(1)$, $S(1)$ are finite-dimensional, and τ is non-degenerate, restricted to $R(1) \otimes S(1)$. We formulate the result as a category equivalence

$$\mathcal{O}(\mathcal{B}) \approx \mathcal{V}ec$$

between the integrable left $\mathcal{B}$-modules and the vector spaces; see Theorem 3.13. The original result is recovered if we suppose that $R \rtimes J = U_q^{\leq 0}$, $S \rtimes K = U_q^{\geq 0}$, and τ is the Killing form given by Tanisaki [Ts]. It is a key for us to identify $\mathcal{B}$ with the generalized smash product $R \# S$ associated to τ, which enables us to define a natural representation $\rho : \mathcal{B} = R \# S \rightarrow \operatorname{End}(R)$ of $\mathcal{B}$ on R; see Proposition 3.4. It is not difficult to prove that ρ is injective, and its image is dense in $\operatorname{End}(R)$. The latter immediately implies that R, regarded as a left $\mathcal{B}$-module (the Verma module) by ρ, is simple; see Theorem 3.6. As its completion, ρ extends to an isomorphism $\hat{\mathcal{B}} \xrightarrow{\simeq} \operatorname{End}(R)$ of (complete) topological algebras. We will define the *extremal projector* in $\hat{\mathcal{B}}$ (see Definition 3.15), which will play an important role to complete the proof of our main result.

Our main result cannot apply directly to Tan's situation, in which the matrix $\mathbb{A}$ may not be finite, because $R(1)$, $S(1)$ then are not necessarily finite-dimensional. But, we will refine Tan's result cited above, by slightly modifying the definition of $\mathcal{O}(\mathcal{B})$ and the proof of our main result; see Theorem 4.1. The main result will also apply to the special situation in which $\mathbb{A}$ is of finite type, q is a root of 1, and U_q is replaced by a finite-dimensional quotient, u_q, called the *Frobenius-Lusztig kernel*. The associated $\mathcal{B}$ is then seen to be a finite-dimensional simple algebra which is isomorphic to $\operatorname{End}(R)$ via $\rho(= \hat{\rho})$; see Section 4.4.

Preceding Sections 3, 4 whose contents were roughly described above, Sections 1, 2 are devoted to showing preliminary results on braided Hopf algebras and skew pairings.

1. Preliminaries on braided Hopf algebras

Throughout this paper we work over a fixed field k, whose characteristic may be arbitrary except in the last Section 4.

1.1. Let J be a Hopf algebra. We denote the coalgebra structure by

$$(1.1) \qquad \Delta = \Delta_J : J \longrightarrow J \otimes J, \quad \Delta(a) = a_1 \otimes a_2; \quad \varepsilon : J \longrightarrow k,$$

and the antipode by $\mathcal{S} = \mathcal{S}_J$ in script. We assume that $\mathcal{S}$ is bijective, and denote the composite-inverse by $\overline{\mathcal{S}}$, which will be called the *pode* of J. Let $^J_J \mathcal{YD}$ denote the braided tensor category of Yetter-Drinfeld modules over J; see [Mo, p.213]. Let $V \in {}^J_J \mathcal{YD}$. Thus, V is a left J-module, whose action will be denoted by $a \rightharpoonup v$ $(a \in J, v \in V)$. It is at the same time a left J-comodule, whose structure will be denoted by

$$(1.2) \qquad \theta : V \longrightarrow J \otimes V, \quad \theta(v) = v_J \otimes v_V,$$

and satisfies

$$\theta(a \rightharpoonup v) = a_1 v_J \mathcal{S}(a_3) \otimes (a_2 \rightharpoonup v_V) \quad (a \in J, v \in V).$$

The category $^J_J \mathcal{YD}$ has the obvious tensor product and the braiding given by

$$(1.3) \qquad c = c_{V,W} : V \otimes W \xrightarrow{\simeq} W \otimes V, \quad c(v \otimes w) = (v_J \rightharpoonup w) \otimes v_V,$$

where W is another object in ${}_J^J\mathcal{YD}$. The inverse of c is given by

$$(1.4) \qquad c^{-1}(w \otimes v) = v_V \otimes (\overline{\mathcal{S}}(v_J) \rightharpoonup w).$$

1.2. Since ${}_J^J\mathcal{YD}$ is braided, (graded) bialgebras or Hopf algebras in ${}_J^J\mathcal{YD}$ are defined in the natural manner, which are called with the adjective 'braided' added. Let $V \in {}_J^J\mathcal{YD}$, as above. The tensor algebra $T(V)$ of V turns uniquely into a braided bialgebra in ${}_J^J\mathcal{YD}$, if each element $v \in V$ is supposed to be primitive so that $\Delta(v) = v \otimes 1 + 1 \otimes v$. In fact, $T(V)$ is a braided graded Hopf algebra in ${}_J^J\mathcal{YD}$ with respect to the obvious grading. Here and in what follows, gradings mean those by $\mathbb{N} = \{0, 1, 2, \cdots\}$ unless otherwise stated. A *pre-Nichols algebra* [M2] of V is a quotient $T(V)/I$ of $T(V)$ by some homogeneous bi-ideal (necessarily, Hopf ideal) I in ${}_J^J\mathcal{YD}$ such that $I \cap V = 0$. In other words, it is a braided graded bialgebra $R = \bigoplus_{n=0}^{\infty} R(n)$ such that

$$(1.5) \qquad\qquad R(0) = k,$$

$$(1.6) \qquad\qquad R(1) = V,$$

$$(1.7) \qquad\qquad R \text{ is generated by } R(1),$$

The condition (1.5) implies

$$(V =)R(1) \subset P(R) := \{\text{all primitives in } R\},$$

and that R is a braided (graded) Hopf algebra. A pre-Nichols algebra $T(V)/I = R$ of V is said to be the *Nichols algebra* [AS] of V, if I is the largest possible, or equivalently if $R(1) = P(R)$.

1.3. Let $R = \bigoplus_{n=0}^{\infty} R(n)$ be such a braided graded Hopf algebra in ${}_J^J\mathcal{YD}$ that satisfies (1.5). To distinguish from the ordinary (or trivially braided) situation, we denote the coproduct of R by

$$\Delta_R(r) = r_{(1)} \otimes r_{(2)} \quad (r \in R); \text{ cf. } (1.1).$$

By Radford's biproduct construction (or bosonization), we have an ordinary Hopf algebra,

$$\mathcal{A} = R \rtimes J.$$

This is denoted by $R \times J$ in [R], and by $R\#J$ in [AS] and others; our notation is due to Shahn Majid. As a vector space, $\mathcal{A}$ equals $R \otimes J$; an element $r \otimes a$ in $\mathcal{A}$ will be denoted simply by ra. $\mathcal{A}$ has the unit $1 \cdot 1(= 1 \otimes 1)$ and the counit $\varepsilon \otimes \varepsilon$, while its product and coproduct are given respectively by

$$\begin{aligned} (ra)(sb) &= r(a_1 \rightharpoonup s)(a_2 b) \\ \Delta(ra) &= r_{(1)}(r_{(2)J}a_1) \otimes r_{(2)R}a_2, \end{aligned}$$

where $r, s \in R, a, b \in J$. Especially, if $v \in R(1)$, then $v \in P(R)$, whence

$$(1.8) \qquad\qquad \Delta_{\mathcal{A}}(v) = v \otimes 1 + v_J \otimes v_R.$$

Notice that $\mathcal{A}$ is a graded Hopf algebra with $\mathcal{A}(n) = R(n) \otimes J$. Since by (1.5), the coradical of $\mathcal{A}$ is included in $J = \mathcal{A}(0)$, $\mathcal{A}$ has a bijective antipode.

Let $\mathcal{A}^{\mathrm{cop}}$ denote the algebra $\mathcal{A}$ which is given the coproduct $\Delta^{\mathrm{cop}}(\alpha) = \alpha_2 \otimes \alpha_1$ ($\alpha \in \mathcal{A}$) opposite to the original one. By the original counit and grading, $\mathcal{A}^{\mathrm{cop}}$ is a graded Hopf algebra with bijective antipode $\overline{\mathcal{S}}_{\mathcal{A}}$. The original Hopf algebra

projection $\varepsilon \otimes \mathrm{id} : \mathcal{A} = R \otimes J \to J$ gives a Hopf algebra projection $\mathcal{A}^{\mathrm{cop}} \to J^{\mathrm{cop}}$, which we call π. Let

$$(1.9) \qquad \overline{R} = \{\alpha \in \mathcal{A}^{\mathrm{cop}} \mid (\mathrm{id} \otimes \pi) \circ \Delta^{\mathrm{cop}}(\alpha) = \alpha \otimes 1\}$$

denote the (left coideal) subalgebra of $\mathcal{A}^{\mathrm{cop}}$ consisting of all right J^{cop}-coinvariants along π. By Radford [R], $\overline{R}$ forms a braided graded Hopf algebra in $_{J^{\mathrm{cop}}}^{J^{\mathrm{cop}}}\mathcal{YD}$ such that $\overline{R} \rtimes J^{\mathrm{cop}} = \mathcal{A}^{\mathrm{cop}}$.

Proposition 1.1. *Let $\mathcal{S}, \overline{\mathcal{S}}$ denote the antipode and the pode of $\mathcal{A}$, respectively.*

(1) $\overline{R} = \mathcal{S}(R) = \overline{\mathcal{S}}(R)$.

(2) $\overline{R}$ *is in* $_{J^{\mathrm{cop}}}^{J^{\mathrm{cop}}}\mathcal{YD}$ *with respect to the structure given by*

$$a^{\mathrm{cop}} \rightharpoonup \overline{\mathcal{S}}(r) := \overline{\mathcal{S}}(a \rightharpoonup r),$$
$$\overline{\mathcal{S}}(r) \mapsto \overline{\mathcal{S}}(r_J) \otimes \overline{\mathcal{S}}(r_R),$$

where $r \in R$, and $a \in J$ with copy $a^{\mathrm{cop}} \in J^{\mathrm{cop}}$.

(3) *The coproduct of $\overline{R}$ is given by*

$$\Delta_{\overline{R}}(\overline{\mathcal{S}}(r)) = \overline{\mathcal{S}}(\overline{\mathcal{S}}(r_{(2)J}) \rightharpoonup r_{(1)}) \otimes \overline{\mathcal{S}}(r_{(2)R}).$$

where $r \in R$.

Proof. It is easy to prove (1). The remaining (2), (3) follow from [R, Theorem3]. To see (3), notice that the coproduct $\Delta_{\mathcal{A}}(\overline{\mathcal{S}}(r))$ of $\mathcal{A}$ is given by

$$(1.10) \qquad \Delta_{\mathcal{A}}(\overline{\mathcal{S}}(r)) = \overline{\mathcal{S}}(r_{(2)R}) \otimes \overline{\mathcal{S}}(\overline{\mathcal{S}}(r_{(2)J1}) \rightharpoonup r_{(1)})\overline{\mathcal{S}}(r_{(2)J2}).$$

$\square$

Proposition 1.2. *With the same notation as above, set $V = R(1)$ in $_J^J\mathcal{YD}$, and $\overline{V} = \overline{R}(1)$ in $_{J^{\mathrm{cop}}}^{J^{\mathrm{cop}}}\mathcal{YD}$.*

(1) $\overline{V} = \mathcal{S}(V) = \overline{\mathcal{S}}(V)$.

(2) R *is a pre-Nichols algebra (resp., the Nichols algebra) of V if and only if $\overline{R}$ is a pre-Nichols algebra (resp., the Nichols algebra) of $\overline{V}$.*

Proof. Part (1) and the assertion on 'pre-Nichols' in (2) follow by Proposition 1.1, while the assertion on 'Nichols' follows since we see directly that $\overline{\mathcal{S}}$ induces an isomorphism

$$(1.11) \qquad P(R) \overset{\simeq}{\longrightarrow} P(\overline{R}).$$

See also the last part of the following subsection. $\square$

1.4. Let $V \in {}_J^J\mathcal{YD}$. Since $J^{\mathrm{cop}} = J$ as an algebra, V may be regarded as a left J^{cop}-module, which we denote by V^t. Then, V^t turns into an object in $_{J^{\mathrm{cop}}}^{J^{\mathrm{cop}}}\mathcal{YD}$ with respect to the modified comodule structure $v \mapsto \overline{\mathcal{S}}(v_J) \otimes v_V$. Moreover, $V \mapsto V^t$ gives a category isomorphism $_J^J\mathcal{YD} \overset{\simeq}{\longrightarrow} {}_{J^{\mathrm{cop}}}^{J^{\mathrm{cop}}}\mathcal{YD}$, which is involutive in the sense $V^{tt} = V$. This is a variation of (and is in fact essentially the same as) the isomorphism $_J^J\mathcal{YD} \overset{\simeq}{\longrightarrow} {}_{J^{\mathrm{op}}}^{J^{\mathrm{op}}}\mathcal{YD}$ given by Radford and Schneider [RS, Section 2].

Suppose that $R = (R, m, \Delta)$ is a braided bialgebra in $_J^J\mathcal{YD}$ with product m and coproduct Δ. Give on R^t the opposite product m^t defined by

$$m^t(r \otimes s) := m(s \otimes r) \quad (r, s \in R),$$

and the coproduct Δ^t defined by

$$\Delta^t := \mathrm{tw} \circ c_{R,R}^{-1} \circ \Delta,$$

where $c_{R,R}^{-1}$ is such as given by (1.4), and tw denotes the twist map $r \otimes s \mapsto s \otimes r$. Explicitly,

$$(1.12) \qquad \Delta^t(r) = (\overline{\mathcal{S}}(r_{(2)J}) \rightharpoonup r_{(1)}) \otimes r_{(2)R} \quad (r \in R).$$

Proposition 1.3. (1) *Given the same unit and counit as the original ones,* $R^t = (R^t, \Delta^t, m^t)$ *is a braided bialgebra in* $_{J^{\mathrm{cop}}}^{J^{\mathrm{cop}}}\mathcal{YD}$.
(2) *We have* $R^{tt} = R$, $P(R) = P(R^t)$.

Proof. (1) As is explained in [RS, Section 2], the category isomorphism $F : {}_J^J\mathcal{YD} \xrightarrow{\simeq} {}_{J^{\mathrm{cop}}}^{J^{\mathrm{cop}}}\mathcal{YD}$, $F(V) = V^t$ is tensorial with respect to id $: k \to k = F(k)$, $c_{W,V} \circ \mathrm{tw} : F(V) \otimes F(W) \xrightarrow{\simeq} F(V \otimes W)$. Therefore, $F(R)$ is a braided bialgebra in $_{J^{\mathrm{cop}}}^{J^{\mathrm{cop}}}\mathcal{YD}$. Our R^t is the braided opposite to $F(R)$.
(2) This is directly verified. $\qquad \square$

Obviously, this result is generalized to braided graded bialgebras.

Proposition 1.4. *Suppose that we are in the same situation as in Propositions 1.1, 1.2.*

(1) *R is a pre-Nichols algebra (resp., the Nichols algebra) of V if and only if R^t is a pre-Nichols algebra (resp., the Nichols algebra) of V^t.*
(2) *The pode $\overline{\mathcal{S}}$ of $R \rtimes J$ gives isomorphisms*

$$R \xrightarrow{\simeq} \overline{R}^t, \quad R^t \xrightarrow{\simeq} \overline{R}$$

of braided graded Hopf algebras in ${}_J^J\mathcal{YD}$, and in ${}_{J^{\mathrm{cop}}}^{J^{\mathrm{cop}}}\mathcal{YD}$, respectively.

Proof. (1) This follows from Proposition 1.3 (2).
(2) This follows from Proposition 1.1 (2), (3) and (1.12). $\qquad \square$

Part(2) above refines the isomorphism given in (1.11), more conceptually.

1.5. For a vector space X, we let $X^* = \mathrm{Hom}(X, k)$ denote the linear dual, and $\langle\ ,\ \rangle : X^* \times X \to k$ the evaluation map.

Let J° denote the dual Hopf algebra [Sw, Section 6.2] of J, which also has a bijective antipode. If $V \in {}_J^J\mathcal{YD}$ is finite-dimensional, V^* is in ${}_{J^\circ}^{J^\circ}\mathcal{YD}$ with respect to the structure $p \rightharpoonup v^*$, $v^* \mapsto v^*_{J^\circ} \otimes v^*_{V^*}$ determined by

$$\langle p \rightharpoonup v^*, v \rangle = \langle p, v_J \rangle \langle v^*, v_V \rangle,$$
$$\langle v^*_{J^\circ}, a \rangle \langle v^*_{V^*}, v \rangle = \langle v^*, a \rightharpoonup v \rangle,$$

where $v \in V$, $v^* \in V^*$, $a \in J$, $p \in J^\circ$, Moreover, $V \mapsto V^*$ gives a braided tensor contravariant-functor from the finite-dimensional objects in ${}_J^J\mathcal{YD}$ to those objects in ${}_{J^\circ}^{J^\circ}\mathcal{YD}$, where the tensorial structure is given by the canonical isomorphisms $k \xrightarrow{\simeq} k^*$, $V^* \otimes W^* \xrightarrow{\simeq} (V \otimes W)^*$.

Let $R = \bigoplus_{n=0}^\infty R(n)$ be a braided graded Hopf algebra in ${}_J^J\mathcal{YD}$ such that $R(0) = k$. Following [Sw, Section 11.2], we define the *graded dual* R^g of R by

$$R^g = \bigoplus_{n=0}^\infty R(n)^*.$$

This is a subalgebra of the dual algebra R^* of the coalgebra R.

Assume that R is *locally finite* [Sw, ibidem] in the sense that each component $R(n)$ is finite-dimensional. Since each $R(n)^*$ is then an object in ${}_{J^\circ}^{J^\circ}\mathcal{YD}$, R^g is, too. We see now easily the following.

Lemma 1.5. *Under the assumption above, R^g is a braided graded Hopf algebra in* $^{J^\circ}_{J^\circ}\mathcal{YD}$ *such that $R^g(0) = k$.*

1.6. Given distinct Hopf algebras J, K with bijective antipode, suppose $V \in {}^J_J\mathcal{YD}$, $W \in {}^K_K\mathcal{YD}$. A linear map $\psi : V \to W$ is called a *map of braided vector spaces* if there is a Hopf algebra map $\phi : J \to K$ with which ψ is ϕ-linear and colinear [RS] in the sense that

$$(1.13) \qquad\qquad \psi(a \rightharpoonup v) \;=\; \phi(a) \rightharpoonup \psi(v),$$

$$(1.14) \qquad\qquad \phi(v_J) \otimes \psi(v_V) \;=\; \psi(v)_K \otimes \psi(v)_W,$$

where $a \in J$, $v \in V$. In this case, to specify ϕ, we will say that ψ is *attended by* ϕ. Notice that ψ then preserves the braiding in the obvious sense. The definition above extends in the obvious way to braided (graded) bi- or Hopf algebras. For example in the situation of Lemma 1.5, the canonical isomorphism $R \xrightarrow{\simeq} (R^g)^g$ is a map of braided graded Hopf algebras, attended by the canonical Hopf algebra map $J \to (J^\circ)^\circ$.

2. Skew pairings on pre-Nichols algebras

2.1. Let A, X be vector spaces. Given a linear form $\tau : A \otimes X \to k$, we denote the adjoint linear maps by

$$\begin{aligned}
\tau^l &: & A &\to X^*, & \tau^l(a)(x) &= \tau(a, x), \\
\tau^r &: & X &\to A^*, & \tau^r(x)(a) &= \tau(a, x),
\end{aligned}$$

where $a \in A$, $x \in X$. For subspaces $B \subset A$, $Y \subset X$, we let

$$\tau_{BY} := \tau\,|_{B \otimes Y} \colon B \otimes Y \to k$$

denote the restriction.

Suppose that A, X are Hopf algebras with bijective antipode. A linear form $\tau : A \otimes X \to k$ is called a *skew pairing* [DT, Definition 1.3], if

$$\begin{aligned}
\tau(ab, x) &= \tau(a, x_1)\tau(b, x_2), \\
\tau(a, xy) &= \tau(a_1, y)\tau(a_2, x), \\
\tau(1, x) &= \varepsilon(x), \quad \tau(a, 1) = \varepsilon(a)
\end{aligned}$$

for all $a, b \in A, x, y \in X$, or equivalently if τ^l gives a Hopf algebra map $A^{\mathrm{cop}} \to X^\circ$, or equivalently if τ^r gives a Hopf algebra map $X \to (A^{\mathrm{cop}})^\circ$. A skew pairing τ necessarily has a convolution-inverse τ^{-1}, such that

$$\tau^{-1}(a, x) = \tau(\overline{\mathcal{S}}(a), x) = \tau(a, \mathcal{S}(x)) \quad (a \in A, x \in X).$$

2.2. In what follows until the end of Section 3, We will work in the following situation. First, let J, K be Hopf algebras with bijective antipode, and suppose that a skew pairing $\tau_0 : J \otimes K \to k$ is given. Next, let $V \in {}^J_J\mathcal{YD}$, $W \in {}^K_K\mathcal{YD}$, and suppose that $\tau_1 : V \otimes W \to k$ is a linear form such that

$$(2.1) \qquad\qquad \tau_1(a \rightharpoonup v, w) \;=\; \tau_0(a, w_K)\tau_1(v, w_W),$$

$$(2.2) \qquad\qquad \tau_1(v, x \rightharpoonup w) \;=\; \tau_0(v_J, \mathcal{S}(x))\tau_1(v_V, w)$$

for all $a \in J$, $x \in K$, $v \in V$, $w \in W$. If W is finite-dimensional, these last conditions are equivalent to that τ_1^l gives a map $V^t \to W^*$ of braided vector spaces, attended

by $\tau_0^l : J^{\mathrm{cop}} \to K^\circ$. Similarly we have another equivalent condition if V is finite-dimensional; see [RS, Proposition 4.2]. Finally, let R, S be pre-Nichols algebras of V, W, respectively.

Proposition 2.1. τ_0, τ_1 *extend uniquely to a skew pairing* $\tau : (R \rtimes J) \otimes (S \rtimes K) \to k$ *such that*

$$(2.3) \qquad \tau(a, w) = 0 = \tau(v, x)$$

for all $a \in J$, $x \in K$, $v \in V$, $w \in W$.

Proof. This follows by [M2, Theorem 5.3] if we take our $\tau_1(v, w)$ as $-\lambda(w, v)$ in [M2]. See also [RS, Theorem 8.3]. $\qquad\square$

Conversely, if τ is a skew pairing which extends τ_0, and satisfies (2.3), it restricts to the linear form τ_{VW} which satisfies (2.1) (2.2), as is easily seen. In what follows we keep to denote by τ the skew pairing as above.

Lemma 2.2. *If* $a \in K$, $x \in K$, $r, r' \in R$, $s, s' \in S$, *then we have*

$$(2.4) \qquad \tau(ra, sx) = \tau(r_J, x_1)\tau(r_R, s)\tau(a, x_2),$$

$$(2.5) \qquad \tau(r, ss') = \tau(r_{(1)}, s')\tau(r_{(2)}, s),$$

$$(2.6) \qquad \tau(rr', s) = \tau(r, \overline{\mathcal{S}}(s_{(2)K}) \rightharpoonup s_{(1)})\tau(r', s_{(2)S}),$$

Proof. We reach these formulae by proving the following, step by step.

$$\tau(a, s) = \varepsilon(a)\varepsilon(s), \quad \tau(r, x) = \varepsilon(r)\varepsilon(x),$$

$$(2.7) \qquad \tau(ra, s) = \varepsilon(a)\tau(r, s),$$

$$(2.8) \qquad \tau(r, sx) = \tau(r_J, x)\tau(r_R, s),$$

To prove (2.6), notice from (2.8) that

$$\tau(r, x \rightharpoonup s) = \tau(r_J, \mathcal{S}(x))\tau(r_R, s).$$

$\qquad\square$

Lemma 2.3. *Let* $\overline{\mathcal{S}}$ *denote the pode of* $R \rtimes J$. *If* $a \in J$, $x \in K$, $r, r' \in R$, $s, s' \in S$, *then we have*

$$(2.9) \qquad \tau(\overline{\mathcal{S}}(r)a, sx) = \tau(\overline{\mathcal{S}}(r), s)\tau(a, x),$$

$$(2.10) \qquad \tau(\overline{\mathcal{S}}(r)\overline{\mathcal{S}}(r'), s) = \tau(\overline{\mathcal{S}}(r), s_{(1)})\tau(\overline{\mathcal{S}}(r'), s_{(2)}),$$

$$(2.11) \qquad \tau(\overline{\mathcal{S}}(r), ss') = \tau(\overline{\mathcal{S}}(\overline{\mathcal{S}}(r_{(2)J}) \rightharpoonup r_{(1)}), s)\tau(\overline{\mathcal{S}}(r_{(2)R}), s').$$

Proof. Suppose $v \in V$. Then, $\overline{\mathcal{S}}(v) = -v_V\overline{\mathcal{S}}(v_J)$. By (2.8), we see

$$(2.12) \qquad \tau(\overline{\mathcal{S}}(v), sx) = -\varepsilon(x)\tau(v, x),$$

It follows by induction on n that if $v^1, \cdots, v^n \in V$, then

$$(2.13) \qquad \tau(\overline{\mathcal{S}}(v^1) \cdots \overline{\mathcal{S}}(v^n), sx) = (-1)^n \varepsilon(x)\tau(v^1, s_{(1)}) \cdots \tau(v^n, s_{(n)}).$$

This implies (2.9), (2.10); recall here (1.7). We see that (2.11) follows from (1.10), (2.9). $\qquad\square$

Recall from 1.3 that $\overline{R} = \overline{\mathcal{S}}(R)$ in $R \rtimes J$.

Corollary 2.4. (1) τ *is non-degenerate if and only if* $\tau_{\overline{R}S}$ *and* $\tau_0 (= \tau_{JK})$ *are both non-degenerate.*
   (2) $\tau(R(n) \otimes J, S(m) \otimes K) = 0$ *if* $n \neq m$.

Proof. (1) This follows by (2.9).

(2) One sees from (2.13) that

$$(2.14) \qquad \tau(\overline{R}(n), S(m)) = 0 \quad \text{if} \quad n \neq m.$$

Since $\overline{R}(n) \otimes J = R(n) \otimes J$ for each n, Part (2) follows by (2.9), again. $\qquad \square$

By Part (2) above, graded linear (at least) maps

$$R \to S^g, \quad S \to R^g, \quad \overline{R} \to S^g, \quad S \to \overline{R}^g$$

are given by τ_{RS}^l, τ_{RS}^r, $\tau_{\overline{RS}}^l$, $\tau_{\overline{RS}}^r$, respectively.

2.3. Set $\overline{V} = \overline{R}(1)$. Notice from Proposition 1.2 that $\overline{R}$ is a pre-Nichols algebra of $\overline{V}$ in $_{J^{\mathrm{cop}}}^{J^{\mathrm{cop}}}\mathcal{YD}$, such that $\overline{R} \rtimes J^{\mathrm{cop}} = (R \rtimes J)^{\mathrm{cop}}$. Recall from 1.4 that S^t is a pre-Nichols algebra of W^t in $_{K^{\mathrm{cop}}}^{K^{\mathrm{cop}}}\mathcal{YD}$.

Lemma 2.5. *Assume that V, W are both finite-dimensional so that R, S are locally finite.*

(1) τ_{RS}^l, τ_{RS}^r *give maps*

$$(2.15) \qquad R \to (S^t)^g, \quad S^t \to R^g$$

of braided graded Hopf algebras, which are graded dual to each other.

(2) $\tau_{\overline{RS}}^l$, $\tau_{\overline{RS}}^r$ *give maps*

$$(2.16) \qquad \overline{R} \to S^g, \quad S \to \overline{R}^g$$

of braided graded Hopf algebras, which are graded dual to each other.

Proof. (1) This follows from (2.5), (2.6). Notice that the map $R \to (S^t)^g$ above is attended by $\mathcal{S}_{K^\circ} \circ \tau_0^l : J \to (K^{\mathrm{cop}})^\circ$, since it is so, restricted to V, and R is generated by V.

(2) Similarly, this follows from (2.10), (2.11); see also Proposition 1.1 (3). $\qquad \square$

Proposition 2.6.   (1) *The following are equivalent:*

(a) τ_{RS} *is non-degenerate;*

(b) $\tau_1(= \tau_{VW})$ *is non-degenerate, and R, S are both Nichols algebras;*

(c) $\tau_{\overline{RS}}$ *is non-degenerate;*

(d) $\tau_{\overline{VW}}$ *is non-degenerate, and $\overline{R}$, S are both Nichols algebras.*

These conditions hold true if τ is non-degenerate.

(2) *Assume that V, W are both finite-dimensional. Then the equivalent conditions given above are further equivalent to each of the following:*

(e) *The maps given in (2.15) are isomorphisms;*

(f) *The maps given in (2.16) are isomorphisms.*

Proof. (1) Proposition 1.2 (2) and (2.12) prove (b) $\Leftrightarrow$ (d). To prove (a) $\Leftrightarrow$ (b), let $T = \mathrm{Im}\,\tau_{RS}^l$ denote the image of τ_{RS}^l. We see from (2.5) that T is a subcoalgebra of the dual coalgebra S° of the algebra S; see [Sw, Section 6.0]. Moreover, T is a graded coalgebra, and $\tau_{RS}^l : R \to T$ is a graded anti-coalgebra map. Since S is generated by $W = S(1)$, T is strictly graded, or $P(T) = T(1)$; see [Sw, Section 11.2]. It follows that τ_{RS}^l is injective if and only if τ_1^l is injective, and R is Nichols. Similarly, since by (2.6), $\tau_{RS}^r : S^t \to \mathrm{Im}\,\tau_{RS}^r$ is a graded coalgebra map onto a strictly graded coalgebra, it follows by using Proposition 1.4 (1) that τ_{RS}^r is injective if and only if τ_1^r is injective, and S is Nichols. These prove the desired

equivalence. Similarly, (c) $\Leftrightarrow$ (d) follows. The last statement follows by Corollary 2.4 (1).

(2) This follows, since obviously, (a) $\Leftrightarrow$ (e), (c) $\Leftrightarrow$ (f). $\qquad\qquad\square$

Remark 2.7. *As was seen above, the graded linear map $R \to S^g$ given by τ_{RS}^l is injective if and only if τ_1^l is injective, and R is Nichols.*

3. GENERALIZED q-BOSON ALGEBRAS AND THEIR INTEGRABLE MODULES

Throughout this section we let

$$(3.1) \qquad\qquad J, \ K, \ \tau_0, \ V, \ W, \ \tau_1, \ R, \ S$$

be as given at the beginning of 2.2

3.1. ¿From the proof of [M2, Theorem 5.3], we see that the tensor-product algebra $R \otimes S$ is uniquely deformed to such an algebra that includes $R (= R \otimes k)$, $S (= k \otimes S)$ as subalgebras, and obeys the rules of product

$$(3.2) \qquad\qquad rs = r \otimes s \quad (r \in R, s \in S),$$
$$(3.3) \qquad wv = \tau_0(v_J, w_J) v_V \otimes w_W + \tau_1(v, w) 1 \otimes 1 \quad (v \in V, w \in W).$$

Recall here that we take our $\tau_1(v, w)$ as $-\lambda(w, v)$ in [M2]; see also Remark 3.3 below.

Definition 3.1. *We denote this deformed algebra by $\mathcal{B}$, and call it the* generalized q-boson algebra *associated to the data (3.1).*

Notice that $\mathcal{B}$ has $1 \otimes 1$ as unit. Also, it is a $\mathbb{Z}$-graded algebra, by counting degrees so that

$$\deg R(n) = n, \quad \deg S(n) = -n \quad (n \geq 0)$$

3.2. To obtain a useful description of $\mathcal{B}$, let τ be the skew pairing given by Proposition 2.1. We set

$$\mathcal{A} = R \rtimes J, \quad \mathcal{H} = S \rtimes K,$$

so that τ is defined on $\mathcal{A} \otimes \mathcal{H}$.

Lemma 3.2 (cf. [Lu]). (1) $\mathcal{A}$ *is a left $\mathcal{H}$-module algebra under the action*

$$\xi \triangleright \alpha = \tau(\alpha_1, \xi) \alpha_2 \quad (\alpha \in \mathcal{A}, \ \xi \in \mathcal{H}).$$

(2) *The associated smash product $\mathcal{A} \# \mathcal{H}$ is the algebra defined on $\mathcal{A} \otimes \mathcal{H}$ with respect to the unit $1 \# 1$ and the product*

$$(\alpha \# \xi)(\beta \# \eta) = \tau(\beta_1, \xi_1) \alpha \beta_2 \# \xi_2 \eta,$$

where $\alpha, \beta \in \mathcal{A}$, $\xi, \eta \in \mathcal{H}$. Here, $\alpha \# \xi$ stands for $\alpha \otimes \xi$.

(3) *A linear representation $\varpi : \mathcal{A} \# \mathcal{H} \to \mathrm{End}(\mathcal{A})$ of $\mathcal{A} \# \mathcal{H}$ on $\mathcal{A}$ is given by*

$$\varpi(\alpha \# \xi)(\beta) = \alpha(\xi \triangleright \beta) \quad (\alpha, \beta \in \mathcal{A}, \ \xi \in \mathcal{H}).$$

This is easy to see. Lu [Lu] essentially proved the result above, when $\mathcal{H}$ is a finite-dimensional Hopf algebra, $\mathcal{A} = (\mathcal{H}^{\mathrm{cop}})^*$, and τ is the evaluation map; in this case, ϖ is necessarily an isomorphism.

Remark 3.3. *With the notation as above, $\sigma(\alpha \otimes \xi, \beta \otimes \eta) := \varepsilon(\alpha)\tau(\beta,\xi)\varepsilon(\eta)$ defines a 2-cocyle on $(\mathcal{A} \otimes \mathcal{H})^{\otimes 2}$; see [M2, Proposition 2.6], for example. We see that the algebra $\mathcal{A}\#\mathcal{H}$ coincides with the crossed product $_\sigma(\mathcal{A} \otimes \mathcal{H})$ associated to σ, and with the bicleft (especially, right $\mathcal{A} \otimes \mathcal{H}$-cleft) extension over k as given in the last paragraph of the proof of [M2, Theorem 5.3]. To prove the existance of such a skew pairing τ as given by Proposition 2.1, this last article first constructed that cleft extension, and then proved that the associated 2-cocyle arises precisely from the desired τ.*

Since R, S are left coideal subalgebras of $\mathcal{A} = R \rtimes J$, $\mathcal{H} = S \rtimes K$, respectively, we see that $R \otimes S(= (R \otimes k) \otimes (S \otimes k))$ is a subalgebra of $\mathcal{A}\#\mathcal{H}$. We denote the resulting algebra by $R\#S$.

Proposition 3.4.   (1) *The inclusions $R, S \hookrightarrow R\#S$ uniquely extend to an algebra isomorphism $\mathcal{B} \xrightarrow{\simeq} R\#S$.*
  (2) *A linear representation $\rho : R\#S \to \mathrm{End}(R)$ of $R\#S$ on R is induced from ϖ so that*
$$\rho(r\#s)(r') = r(s \triangleright r') \quad (r, r' \in R, \ s \in S).$$

Proof. (1) We see from (1.8) that the same relation as (3.3) holds true in $R\#S$. Therefore the inclusions above uniquely extend to an algebra map $\mathcal{B} \to R\#S$, which is an isomorphism since it is the identity map of $R \otimes S$, regarded as a linear map.
   (2) This follows since $r(s \triangleright r') \in R$. $\square$

Remark 3.5. *By (2.7), one sees*
$$(3.4) \qquad\qquad s \triangleright r = \tau(r_{(1)}, s)r_{(2)} \quad (r \in R, \ s \in S).$$
Let $w \in W(= S(1))$. One then sees
$$(3.5) \qquad w \triangleright rr' = (w \triangleright r)r' + \tau(r_J, w_K)r_R(w_S \triangleright r') \quad (r, r' \in R).$$
Thus, w acts on R as a sort of (braided) derivation.

3.3. We will identify so as $\mathcal{B} = R\#S$ via the isomorphism above. We regard R as a left $\mathcal{B}$-module by ρ. Notice that this $\mathcal{B}$-module is $\mathbb{Z}$-graded, or more explicitly
$$(3.6) \qquad R(l)(S(m) \triangleright R(n)) \subset R(l - m + n) \quad (l, m, n \geq 0),$$
where we suppose $R(n) = 0$ if $n < 0$. Set $\mathcal{I} = R \otimes (\bigoplus_{n>0} S(n))$ in $\mathcal{B}$; this is the left ideal generated by $S(1)$. We see that $\beta \mapsto \rho(\beta)(1)$ $(\beta \in \mathcal{B})$ induces the $\mathcal{B}$-linear isomorphism
$$(3.7) \qquad\qquad \mathcal{B}/\mathcal{I} \xrightarrow{\simeq} R.$$

Theorem 3.6. *Assume that $\tau_1^l : V \to W^*$ is injective, and R is the Nichols algebra of V. Then, R is simple as a left $\mathcal{B}$-module.*

To prove the theorem, let first
$$\nu : R \otimes R^g \rightarrowtail \mathrm{End}(R)$$
denote the canonical linear injection given by
$$\nu(r \otimes f)(r') = r\langle f, r'\rangle \quad (r, r' \in R, \ f \in R^g).$$
Since R is a braided Hopf algebra, we have the right R-linear isomorphism
$$\chi : R \otimes R \xrightarrow{\simeq} R \otimes R, \quad \chi(r \otimes r') = r_{(1)} \otimes r_{(2)}r',$$

just as in the case of ordinary Hopf algebras. The right R-linear dual induces a linear isomorphism,

$$\chi^\vee : \mathrm{End}(R) \xrightarrow{\simeq} \mathrm{End}(R).$$

Lemma 3.7. *Let μ denote the composite $\chi^\vee \circ \nu$.*

(1) *$\mu : R \otimes R^g \to \mathrm{End}(R)$ is the linear injection given by*

$$\mu(r \otimes f)(r') = r\langle f, r'_{(1)}\rangle r'_{(2)} \quad (r, r' \in R, \ f \in R^g).$$

(2) *The composite*

$$\mathcal{B} = R \otimes S \xrightarrow{\mathrm{id} \otimes \tau_{RS}^r} R \otimes R^g \xrightarrow{\mu} \mathrm{End}(R)$$

coincides with ρ.

This is easy to prove.

Here, recall that a vector space X given a topology is called a *topological vector space* [T], if the translation $y \mapsto x + y$ by each $x \in X$ is continuous and if X has a basis of neighborhoods of 0 consisting of (vector) subspaces. The topological vector spaces which we will encounter in this paper are all Hausdorff. An algebra is called a *topological algebra*, if it is a topological vector space and if the product is continuous.

Given a discrete vector space X, present it as a directed union $X = \bigcup_\lambda X_\lambda$ of finite-dimensional subspaces $X_\lambda \subset X$. Then we have the canonical isomorphism

$$\mathrm{End}(X) \simeq \varprojlim_\lambda \mathrm{Hom}(X_\lambda, X).$$

Through this, regard $\mathrm{End}(X)$ as the projective limit of discrete $\mathrm{Hom}(X_\lambda, X)$. Then, $\mathrm{End}(X)$ turns into a complete topological algebra; the thus introduced topology is independent of choice of presentations $X = \bigcup_\lambda X_\lambda$.

Proof of Theorem 3.6. Regarding $\mathrm{End}(R)$ as a topological vector space, as above, we claim that the image of ρ is dense in $\mathrm{End}(R)$. Notice that $\chi^\vee$ is an isomorphism of topological vector spaces since $\chi(C \otimes R) = C \otimes R$ for every finite-dimensional subcoalgebra $C \subset R$. It then suffices to prove that the image of the composite $\nu_\mathcal{B} := \nu \circ (\mathrm{id} \otimes \tau_{RS}^r)$ is dense. By Remark 2.7, the assumptions imply that τ_{RS}^l is injective. It follows that given a finite-dimensional subspace $X \subset R$, the composite

$$S \xrightarrow{\tau_{RS}^r} R^g \longrightarrow X^*$$

is surjective, whence so is the composite

$$\mathcal{B} = R \otimes S \xrightarrow{\nu_\mathcal{B}} \mathrm{End}(R) \longrightarrow \mathrm{Hom}(X, R),$$

where the second arrows are the restriction maps. This proves the claim. Therefore, given a finite number of linearly independent elements $r_1, \cdots, r_n$ in R and the same number of arbitrary elements $r'_1, \cdots, r'_n$ in R, there exists β in $\mathcal{B}$ such that $\beta r_i = r'_i$ for all $1 \le i \le n$. This fact even in $n = 1$ proves that R cannot include any non-trivial $\mathcal{B}$-submodule. $\qquad \square$

Definition 3.8. *Let M be a left $\mathcal{B}$-module. We define a subspace M_0 of M by*

$$M_0 = \{m \in M \mid S(1)m = 0\}.$$

We also define a linear map κ_M by

(3.8) $$\kappa_M : R \otimes M_0 \to M, \quad \kappa_M(r \otimes m) = rm.$$

Lemma 3.9. *Regard $R \otimes M_0$ as a left $\mathcal{B}$-module by letting $\mathcal{B}$ act on the factor R in $R \otimes M_0$. Then, κ_M is $\mathcal{B}$-linear.*

Proof. Let $\mathrm{Hom}_{\mathcal{B}}(R, M)$ denote the vector space of all $\mathcal{B}$-linear maps $\varphi : R \to M$. By (3.7), $\varphi \mapsto \varphi(1)$ gives the isomorphism

$$(3.9) \qquad \mathrm{Hom}_{\mathcal{B}}(R, M) \xrightarrow{\simeq} M_0.$$

Moreover, κ_M is identified with the evaluation map $R \otimes \mathrm{Hom}_{\mathcal{B}}(R, M) \to M$, which is obviously $\mathcal{B}$-linear. $\qquad\square$

Theorem 3.10. *Under the same assumptions as in Theorem 3.6, we have the following.*

(1) $R_0 = k$.
(2) κ_M *is injective for every left $\mathcal{B}$-module M.*

Proof. (1) By Theorem 3.6, R is simple. By Schur's Lemma, $D := \mathrm{End}_{\mathcal{B}}(R)$ is a division algebra, and the image $\mathrm{Im}\,\rho$ of ρ is included in $\mathrm{End}_D(R)$, which implies that $\mathrm{End}_D(R)$ is dense in $\mathrm{End}(R)$. But, D then must equal k, which proves (1) by (3.9).

(2) Since R is simple and $k = \mathrm{End}_{\mathcal{B}}(R)$, it follows by [AM, Proposition 3.1] (or [AMT, Proposition 12.5]) that the evaluation map cited in the last proof above is injective, which proves (2). $\qquad\square$

3.4. In what follows until the end of this section we assume that V, W are both finite-dimensional so that R, S are locally finite.

Definition 3.11. *A left $\mathcal{B}$-module M is said to be* integrable, *if each element $m \in M$ is annihilated by $S(n)$, where $n(> 0)$ is a sufficiently large integer that may depend on m. In the category of all left $\mathcal{B}$-modules, let $\mathcal{O}(\mathcal{B})$ denote the full subcategory consisting of the integrable modules; this is closed under sub, quotient and direct sum, whence it is abelian.*

The next lemma follows by (3.6).

Lemma 3.12. *The left $\mathcal{B}$-module R is integrable. Moreover, if X is a vector space, the left $\mathcal{B}$-module $R \otimes X$, in which $\mathcal{B}$ acts on the factor R, is integrable.*

Let $\mathcal{V}ec$ denote the category of the vector spaces (over k). We have functors,

$$(3.10) \qquad (\ )_0 \ : \ M \mapsto M_0, \quad \mathcal{O}(\mathcal{B}) \to \mathcal{V}ec,$$
$$(3.11) \qquad R \otimes \ : \ X \mapsto R \otimes X, \quad \mathcal{V}ec \to \mathcal{O}(\mathcal{B}).$$

Theorem 3.13. *Assume that τ_1 is non-degenerate, and R, S are both Nichols algebras. Then the functors $(\ )_0$, $R \otimes$ are quasi-inverses of each other, so that we have a category equivalence $\mathcal{O}(\mathcal{B}) \approx \mathcal{V}ec$.*

We have the natural maps

$$\iota_X \ : \ X \to (R \otimes X)_0, \quad \iota_X(x) = 1 \otimes x,$$
$$\kappa_M \ : \ R \otimes M_0 \to M, \text{ as given by (3.8).}$$

in $\mathcal{V}ec$, $\mathcal{O}(\mathcal{B})$, respectively. To prove the theorem above, notice that under (half of) the assumptions, ι_X is an isomorphism, and κ_M is injective, by Theorem 3.10. It then remains to prove that κ_M is surjective.

3.5. In this subsection we do not assume the assumptions above, while we keep to assume that V, W are finite-dimensional.

We will use the obvious identification $\mathcal{B} = \bigoplus_{n=0}^{\infty} R \otimes S(n)$. Set

$$\mathcal{I}_n = \bigoplus_{l \geq n} R \otimes S(l) \quad (n = 0, 1, \cdots)$$

in $\mathcal{B}$; this is the left ideal generated by $S(n)$. Regard $\mathcal{B}$ as a topological vector space which has $\mathcal{I}_n (n = 0, 1, \cdots)$ as a basis of neighborhoods of 0. Set

$$\hat{\mathcal{B}} = \prod_{n=0}^{\infty} R \otimes S(n), \quad \hat{\mathcal{I}}_n = \prod_{l \geq n} R \otimes S(l) \quad (n = 0, 1, \cdots).$$

These are the completions of $\mathcal{B}$, $\mathcal{I}_n$, respectively, and their topologies coincide with the direct product of the discrete topologies on $R \otimes S(n)$.

Proposition 3.14. (1) *$\mathcal{B}$ is a topological algebra, whence as its completion, $\hat{\mathcal{B}}$ turns into a complete topological algebra.*

(2) *Let $M \in \mathcal{O}(\mathcal{B})$. Then the $\mathcal{B}$-module structure $\mathcal{B} \times M \to M$ uniquely extends to $\hat{\mathcal{B}} \times M \to M$ so that each $m \in M$ is annihilated by $\hat{\mathcal{I}}_n$, where $n(> 0)$ is some integer that may depend on m. By this extended structure, M is a left $\hat{\mathcal{B}}$-module.*

Proof. (1) Let $\beta = \sum_i r_i \# s_i \in \mathcal{B}$, with $r_i \in R$, $s_i \in S$ homogeneous. One then sees that for each $n \geq 0$, $\beta \mathcal{I}_n \subset \mathcal{I}_n$, $\mathcal{I}_n \beta \subset \mathcal{I}_{n-l}$, where $l = \text{Max}\{\deg r_i\}$. It follows that the product on $\mathcal{B}$ is continuous, which proves (1).

(2) Suppose that M is discrete. The integrability is equivalent to that for each $m \in M$, the map $\mathcal{B} \to M$ given by $\beta \mapsto \beta m$ is continuous. The map has a completion $\hat{\mathcal{B}} \to M$; let $\hat{\beta}m$ denote the image of $\hat{\beta} \in \hat{\mathcal{B}}$. We then see that $(\hat{\beta}, m) \mapsto \hat{\beta}m$ gives the desired structure. $\square$

By Part (2) above, $\mathcal{O}(\mathcal{B})$ is now naturally identified with the category of those left $\hat{\mathcal{B}}$-modules M in which every element $m \in M$ is annihilated by some $\hat{\mathcal{I}}_n$.

3.6. Here we assume that the assumptions of Theorem 3.13 are satisfied, or equivalently that $\tau_{RS} : R \otimes S \to k$ is non-degenerate; see Proposition 2.6. By Corollary 2.4 (2), this last is equivalent to that $\tau_{R(n)S(n)} : R(n) \otimes S(n) \to k$ is non-degenerate for each $n \geq 0$. Choose bases (r_{i_n}) of $R(n)$, and (s_{i_n}) of $S(n)$ which are mutually dual with respect to $\tau_{R(n)S(n)}$.

Definition 3.15. *Let $\mathcal{S}$ denote the antipode of the braided graded Hopf algebra R, and define*

$$\gamma = \sum_{n=0}^{\infty} \sum_{i_n} \mathcal{S}(r_{i_n}) \# s_{i_n} \text{ in } \hat{\mathcal{B}}.$$

We call this γ the extremal projector.

The definition above is independent of choice of dual bases, as will be seen from the proof of Part (1) of the proposition below.

We remark that the antipode $\mathcal{S}$ of R is bijective. This follows from the formula given in [R, Proposition 2 (b)], since the antipode of J, and hence that of $\mathcal{A}$ are both bijective.

Proposition 3.16. (1) γ *is an idempotent in $\hat{\mathcal{B}}$ such that*

$$S(n)\gamma = 0 = \gamma R(n) \text{ for all } n > 0.$$

(2) *The infinite sum $\sum_{n=0}^{\infty} \sum_{i_n} r_{i_n} \gamma s_{i_n}$ converges in $\hat{\mathcal{B}}$ to the unit 1.*

(3) *Let $M \in \mathcal{O}(\mathcal{B})$. Then,*

$$\gamma \cdot : M \to M_0, \quad m \mapsto \gamma m$$

gives a projection from M onto the subspace M_0.

Proof. (1) Since $\tau_{RS}^r : S \to R^g$ is now an isomorphism, we see from the proof of Theorem 3.6 that $\rho : \mathcal{B} \to \mathrm{End}(R)$ is injective. Moreover, this is continuous, and its completion gives an isomorphism

$$\hat{\rho} : \hat{\mathcal{B}} \xrightarrow{\simeq} \mathrm{End}(R)$$

of topological algebras. We see

$$\begin{aligned}
\hat{\rho}(\gamma)(r) &= \sum_{n=0}^{\infty} \sum_{i_n} \mathcal{S}(r_{i_n})\tau(r_{(1)}, s_{i_n})r_{(2)} \\
&= \mathcal{S}(r_{(1)})r_{(2)} = \varepsilon(r)1,
\end{aligned}$$

so that $\hat{\rho}(\gamma)$ equals the projection $R \to R$ given by $r \mapsto \varepsilon(r)1$. This proves (1).

(2) This follows, since $r_{i_n} \gamma s_{i_n} \in \hat{\mathcal{I}}_n$, and we see

$$\sum_{n=0}^{\infty} \sum_{i_n} \hat{\rho}(r_{i_n} \gamma s_{i_n})(r) = \sum_{n=0}^{\infty} \sum_{i_n} r_{i_n} \tau(r_{(1)}, s_{i_n})\varepsilon(r_{(2)}) = r.$$

(3) If $m \in M$, $s \in S(1)$, then $s(\gamma m) = (s\gamma)m = 0$ by (1), whence $\gamma m \in M_0$. If $m \in M_0$, then it is annihilated by all s_{i_n}, where $n > 0$, and hence $\gamma m = m$. These two prove (3). $\square$

Proof of Theorem 3.13. To prove the remaining surjectivity of κ_M, let $m \in M$ ($\in \mathcal{O}(\mathcal{B})$). Then by Part (2) above,

$$(3.12) \qquad m = \sum_{n=0}^{\infty} \sum_{i_n} r_{i_n} \gamma s_{i_n} m.$$

Notice that by the integrability, this infinite sum is essentially finite. Since $\gamma s_{i_n} m \in M_0$ by Part (3) above, we see $m \in RM_0$, which proves the desired surjectivity. $\square$

Remark 3.17. *In the situation of Theorem 3.13, assume that one of R, S is finite-dimensional. Then both of them, and so $\mathcal{B}$ are all finite-dimensional. Moreover one sees from the proof of Theorem 3.6 that ρ gives an isomorphism $\mathcal{B} \xrightarrow{\simeq} \mathrm{End}(R)$, whence $\mathcal{B}$ is a matrix algebra, up to isomorphism. The category $\mathcal{O}(\mathcal{B})$ consists of all left $\mathcal{B}$-modules, and the category equivalence $\mathcal{O}(\mathcal{B}) \approx \mathcal{V}ec$ given by Theorem 3.13 coincides with the familiar one.*

4. Generalized q-boson algebras associated to generalized Kac-Moody algebras

Throughout this section we assume that the characteristic of k is zero, and let $q \in k \setminus 0$.

4.1. We assume that q is transcendental over $\mathbb{Q}$. Let I be a countable index set, $\mathbb{A} = (a_{ij})_{i,j \in I}$ a symmetrizable Borcherds-Cartan matrix, and $\mathbf{m} = (m_i \mid i \in I)$ a sequence of positive integers. Kang [Kan] defined the quantized enveloping algebra $U = U_q$ associated to $\mathbb{A}, \mathbf{m}$. We refer to [KT] for the definition, and let the symbols

$$s_i, \ \alpha_i, \ (h \mid h'), \ \xi_i, \ q^h, \ e_{ik}, \ f_{ik}, \ K_i, \ U^0, \ U^\pm, \ U^{\geq 0}, \ U^{\leq 0}$$

be the same as given in [KT, pp. 348-350]. In particular, the zero part U^0 of U is the group algebra $kP^\vee$ of the group $P^\vee$ of the dual weight lattice as given in [KT, (1.1)]. To apply the results obtained in the preceding sections, we will work in the special situation that $J = K = U^0$. Set

$$f'_{ik} = -\xi_i f_{ik} K_i \quad (\in U^{\leq 0}),$$

and let V denote the subspace of $U^{\leq 0}$ spanned by all f'_{ik}. Let W denote the subspace of U^+ spanned by all e_{ik}. Then, V, W turn into objects in ${}^{U^0}_{U^0}\mathcal{YD}$ with respect to the structures

$$
\begin{aligned}
q^h \rightharpoonup e_{ik} &= q^{\alpha_i(h)} e_{ik}, & e_{ik} &\mapsto K_i \otimes e_{ik}, \\
q^h \rightharpoonup f'_{ik} &= q^{-\alpha_i(h)} f'_{ik}, & f'_{ik} &\mapsto K_i \otimes f'_{ik},
\end{aligned}
$$

where $h \in P^\vee$, $i \in I$, $1 \leq k \leq m_i$. Let R denote the subalgebra of $U^{\leq 0}$ generated by V. Let S stand for U^+, which is by definition the subalgebra of $U^{\geq 0}$ generated by W. We then see that R, S are naturally pre-Nichols algebras of V, W, respectively, so that

$$R \rtimes U^0 = U^{\leq 0}, \quad S \rtimes U^0 = U^{\geq 0}.$$

Notice that if $\overline{\mathcal{S}}$ denotes the pode of $U^{\leq 0}$,

$$\overline{\mathcal{S}}(f'_{ik}) = \xi_i f_{ik}, \quad (\overline{R} =) \overline{\mathcal{S}}(R) = U^-.$$

By [KT, Proposition 2.1] (or our Proposition 2.1), the pairing $\tau_0 : U^0 \otimes U^0 \to k$ given by

$$\tau_0(q^h, q^{h'}) = q^{-(h|h')} \quad (h, h' \in P^\vee)$$

uniquely extends to a skew pairing $\tau : U^{\leq 0} \otimes U^{\geq 0} \to k$ so that

$$
\begin{aligned}
\tau(q^h, e_{ik}) &= 0 = \tau(f'_{ik}, q^h), \\
\tau(f'_{jl}, e_{ik}) &= \delta_{ij} \delta_{kl}
\end{aligned}
$$

where $h \in P^\vee$, $i, j \in I$, $1 \leq k \leq m_i$, $1 \leq l \leq m_j$. As is seen from the remark following Proposition 2.1 (or seen directly), the restriction $\tau_1 : V \otimes W \to k$ of τ satisfies (2.1), (2.2), and is obviously non-degenerate. Let $\mathcal{B}$ denote the associated generalized q-boson algebra. Then, $\mathcal{B}$ is generated by all f'_{ik}, e_{ik}, and is defined by the relations (R5) - (R8) in [KT, p.349], in which f_{ik} should read f'_{ik}, and by

$$e_{ik} f'_{jl} = q_{ij} f'_{jl} e_{ik} + \delta_{ij} \delta_{kl},$$

where $i, j \in I$, $1 \leq k \leq m_i$, $1 \leq l \leq m_j$, and we have set

$$(4.1) \qquad\qquad q_{ij} = q^{-s_i a_{ij}} \left(= q^{-s_j a_{ji}} \right).$$

By [KT, Theorem 2.5], the skew pairing τ above is non-degenerate, whence R, S are Nichols algebras, by Proposition 2.6 (1).

As was seen in 3.2, R has a natural left $\mathcal{B}$-module structure, in which each f'_{ik} in $\mathcal{B}$ acts by the left multiplication. One sees from (3.5) that e_{ik} acts so as

$$(4.2) \qquad e_{ik} \triangleright f'_{j_1 l_1} \cdots f'_{j_t l_t} = \sum_{p=1}^{t} \delta_{ij_p} \delta_{kl_p} q_{ij_1} \cdots q_{ij_{p-1}} f'_{j_1 l_1} \cdots \widehat{f'_{j_p l_p}} \cdots f'_{j_t l_t},$$

where q_{ij} is such as given in (4.1), and $\widehat{}$ means an omission. By Theorems 3.6, 3.10, we have the following.

Theorem 4.1. (1) *R is simple as a left $\mathcal{B}$-module.*
(2) *Given a left $\mathcal{B}$-module M, set*

$$M_0 = \{m \in M \mid e_{ik}m = 0 \text{ for all } i \in I, 1 \le k \le m_i\}.$$

Then,

$$\kappa_M : R \otimes M_0 \to M, \quad \kappa_M(r \otimes m) = rm$$

is a $\mathcal{B}$-linear injection.
(3) *In particular, $R_0 = k$.*

4.2. Keep $\mathcal{B}$ as above. Since V, W can be infinite-dimensional, we need to modify the definition of integrable $\mathcal{B}$-modules.

Definition 4.2. *A left $\mathcal{B}$-module M is said to be* integrable, *if for each $m \in M$, there exist an integer $n > 0$ and a finite subset $F \subset I$ such that m is annihilated by the product $e_{i_1 k_1} \cdots e_{i_t k_t}$ of length $t > 0$, if $t \ge n$, or if $i_p \notin F$ for some $1 \le p \le t$. In the category of all left $\mathcal{B}$-modules, let $\mathcal{O}(\mathcal{B})$ denote the full subcategory consisting of the integrable modules; this is abelian, being closed under sub, quotient and direct sum.*

If the index set I is finite, the definition above coincides with Definition 3.11.

Lemma 4.3. *The left $\mathcal{B}$-module R is integrable.*

Proof. This is seen from (4.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 4.4. *We have a category equivalence*

$$\mathcal{O}(\mathcal{B}) \approx \mathcal{V}ec,$$

which is given by the mutually quasi-inverse functors $R\otimes$, $(\ \ )_0$ defined by (3.10), (3.11).

Remark 4.5. *To prove this theorem in the same way of the proof of Theorem 3.13, we remark that the argument in 3.5, 3.6 can be modified, as follows, so as to fit in with the present situation. We may suppose that the index set I is (countably) infinite since if it is finite, we do not need any modification.*
(1) Given a subset $F \subset I$, we define

$$\mathbb{A}_F = (a_{ij})_{i,j\in F}, \quad \mathbf{m}_F = (m_i \mid i \in F)$$

with the restricted index set. Since $\mathbb{A}_F$ is still a symmetrizable Borcherds-Cartan matrix, we have the quantized enveloping algebra U_F, say, associated to $\mathbb{A}_F$, $\mathbf{m}_F$. Let $R_F, S_F, \mathcal{B}_F$ denote the associated objects corresponding to $R, S, \mathcal{B}$, respectively. Notice that $R_F \subset R$, $S_F \subset S$, and so $\mathcal{B}_F \subset \mathcal{B}$, since R_F, S_F are Nichols. If F is finite, the completion $\hat{\mathcal{B}}_F$ of $\mathcal{B}_F$ is defined so as in 3.5.

(2) Recall that $S(1)$ $(= W)$ has a basis, $(e_{ik} \mid i \in I,\ 1 \le k \le m_i)$. For a subset $F \subset I$, the subobject $S_F(1)$ of $S(1)$ in ${}^{U^0}_{U^0}\mathcal{YD}$ has a basis, $(e_{ik} \mid i \in F,\ 1 \le k \le m_i)$. We have a projection $\pi_F : S(1) \to S_F(1)$ given by

$$\pi_F(e_{ik}) = \begin{cases} e_{ik} & \text{if } i \in F, \\ 0 & \text{otherwise.} \end{cases}$$

This uniquely extends to a projection $S \to S_F$ onto the braided graded Hopf subalgebra $S_F \subset S$, which we call $\tilde{\pi}_F$. One sees that the kernel $\operatorname{Ker} \tilde{\pi}_F$ is the ideal of S generated by all e_{ik} with $i \notin F$.

(3) In what follows we restrict the subsets $F \subset I$ in our consideration, only to finite ones. The pairs (n, F) of an integer $n \ge 0$ and a finite subset $F \subset I$ form a directed set with respect to the natural order which is defined so that $(n, F) \le (l, G)$ if $n \le l$ and $F \subset G$. Given such a pair (n, F), define a left ideal of $\mathcal{B}$ by

$$\mathcal{I}_{n,F} := \bigoplus_{l \ge n} R \otimes S(l) + R \otimes \operatorname{Ker} \tilde{\pi}_F.$$

Notice that if $(n, F) \le (l, G)$, then $\mathcal{I}_{n,F} \supset \mathcal{I}_{l,G}$, whence $(\mathcal{I}_{n,F})$ form a projective system. We can regard $\mathcal{B}$ as a topological vector space which has $(\mathcal{I}_{n,F})$ as its basis of neighborhoods of 0. By modifying the proof of Proposition 3.14, we see the following.

- (i) *$\mathcal{B}$ is a topological algebra, whence its completion $\hat{\mathcal{B}}$ is a complete topological algebra.*
- (ii) *An object in $\mathcal{O}(\mathcal{B})$ is precisely a left $\mathcal{B}$-module M such that each element $m \in M$ is annihilated by some $\mathcal{I}_{n,F}$. Such a module in turn is identified with a left $\hat{\mathcal{B}}$-module M such that each element $m \in M$ is annihilated by the completion $\hat{\mathcal{I}}_{n,F}$ of some $\mathcal{I}_{n,F}$.*

(4) We claim that the linear representation $\rho : \mathcal{B} \to \operatorname{End}(R)$ is continuous, and its completion gives an isomorphism $\hat{\rho} : \hat{\mathcal{B}} \xrightarrow{\cong} \operatorname{End}(R)$ of complete topological algebras. To see this, notice that $\hat{\mathcal{B}}$ is the projective limit of $\prod_{n=0}^{\infty} R \otimes S_F(n)$ along the projections

$$\prod_{n=0}^{\infty} R \otimes S_G(n) \to \prod_{n=0}^{\infty} R \otimes S_F(n) \quad (F \subset G \subset I)$$

analogously defined as the $\tilde{\pi}_F$ above, while $\operatorname{End}(R)$ is the projective limit of $\operatorname{Hom}(R_F, R)$ along the restriction maps $\operatorname{Hom}(R_G, R) \to \operatorname{Hom}(R_F, R)$. The claim then follows since we see as in the proof of Theorem 3.6 that the linear representations $\mathcal{B}_F \to \operatorname{End}(R_F)$, with $S \otimes_{S_F}$ applied, give rise to isomorphisms

$$\prod_{n=0}^{\infty} R \otimes S_F(n) \xrightarrow{\cong} \operatorname{Hom}(R_F, R)$$

of complete topological vector spaces, which are compatible with the projective systems on both sides and whose projective limit coincides with $\hat{\rho}$. We define the extremal projector γ in $\hat{\mathcal{B}}$ to be a unique element such that $\hat{\rho}(\gamma)(r) = \varepsilon(r)1$ for every $r \in R$. By this definition, γ has the same property as described in Part (1) of Proposition 3.16. We see that this γ is characterized as the element which for every finite $F \subset I$, is mapped by the projection $\hat{\mathcal{B}} \to \prod_{n=0}^{\infty} R \otimes S_F(n)$ to the extremal projector in $\hat{\mathcal{B}}_F$ as defined by Definition 3.15.

(5) For each integer $n \geq 0$, we can choose a basis $(s_p \mid p \in \mathfrak{X}(n))$ of $S(n)$ so that each s_p is a monomial in e_{ik} of length n. For a finite subset $F \subset I$, let $\mathfrak{X}_F(n)$ denote the set of those indicies $p \in \mathfrak{X}(n)$ for which s_p is a monomial only in e_{ik} with $i \in F$. Then, $(s_p \mid p \in \mathfrak{X}_F(n))$ is a basis of $S_F(n)$. Set

$$\mathfrak{X} = \bigsqcup_{n=0}^{\infty} \mathfrak{X}(n), \qquad \mathfrak{X}_{n,F} = \bigsqcup_{l<n} \mathfrak{X}_F(l).$$

Then we have

$$\mathcal{B} = \bigoplus_{p \in \mathfrak{X}} R \otimes s_p, \quad \mathcal{I}_{n,F} = \bigoplus_{p \notin \mathfrak{X}_{n,F}} R \otimes s_p, \quad \hat{\mathcal{B}} = \prod_{p \in \mathfrak{X}} R \otimes s_p, \quad \hat{\mathcal{I}}_{n,F} = \prod_{p \notin \mathfrak{X}_{n,F}} R \otimes s_p.$$

Let $\mathcal{S}$ denote the antipode of R. Since it is bijective (see the remark just above Proposition 3.16), there uniquely exists a family $(r_p \mid p \in \mathfrak{X})$ of elements in R such that

$$\gamma = \sum_{p \in \mathfrak{X}} \mathcal{S}(r_p) \otimes s_p.$$

The characterization of γ noted last in (4) above proves that for each pair (n, F), $(r_p \mid p \in \mathfrak{X}_F(n))$ is the basis of $R_F(n)$ dual to $(s_p \mid p \in \mathfrak{X}_F(n))$, and so that $(r_p \mid p \in \mathfrak{X})$ is a (unique) basis of R such that $\tau(r_p, s_q) = \delta_{pq}$. Obviously, γ has the same property as described in Part (3) of Proposition 3.16. The statement of the remaining Part (2) is modified so that the sequence

$$\sum_{p \in \mathfrak{X}_{n,F}} r_p \gamma s_p \quad (\in \mathcal{B})$$

directed by the pairs (n, F) converges in $\hat{\mathcal{B}}$ to the unit 1.

Proof of Theorem 4.4. We may suppose as above that I is infinite. To prove the remaining surjectivity of κ_M, let $m \in M$ ($\in \mathcal{O}(\mathcal{B})$). Then the last statement of Remark 4.5 proves

$$m = (\lim_{n,F} \sum_{p \in \mathfrak{X}_{n,F}} r_p \gamma s_p)m = \lim_{n,F}(\sum_{p \in \mathfrak{X}_{n,F}} r_p \gamma s_p m).$$

By the integrability this last limit equals the finite sum $\sum_{p \in \mathfrak{X}_{n,F}} r_p \gamma s_p m$, where (n, F) is large enough. The desired surjectivity follows since $\gamma s_p m \in M_0$ by the property $S(n)\gamma = 0$ $(n > 0)$; see Remark 4.5 (4). $\square$

Remark 4.6. *Tan* [Tan, Proposition 3.2(3)] *asserts essentially the same result as above, in the slightly specialized situation when all m_i equal 1. But, the argument of his proof, asserting that $Xg(u_2) = X$ on Page 4346, line-3, is wrong. His definition of integrability given in* [Tan, p.4343, lines 3-5] *is incomplete, I think.*

4.3. Suppose in particular that the index set I is finite, $\mathbb{A}$ is a symmetrizable generalized Cartan matrix, and all m_i equal 1. Let e_i, f_i denote the present generators e_{i1}, f_{i1} of U^+, U^-, respectively. In this specialized situation the result essentially the same as our last theorem was first announced by Kashiwara [K, Remark 3.4.10], and then proved by Nakashima [N, Theorem 6.1], who introduced the extremal projector in his situation; cf. Definition 3.15. More precisely, Nakashima worked on the algebra $B = B_q(\mathfrak{g})$, which is bigger than our $\mathcal{B}$, including U^0, too. We remark that his B coincides with the (U, U')-bicleft extension over k which was given by [M1, Proposition 4.9], where U' denote the (graded) Hopf algebra defined just as

U, but the relation $[e_i, f_j] = \delta_{ij}\xi_i^{-1}(K_i - K_i^{-1})$ is replaced by $[e_i, f_j] = 0$. We also remark that Nakashima's [N, Theorem 6.1] can be reformulated as a category equivalence between $\mathcal{O}(B)$ and the category of all P-graded vector spaces, where P denotes the group of the weight lattice; see [N, p.286].

4.4. Suppose further that the generalized Cartan matrix $\mathbb{A}$ is of finite type. We choose a diagonal matrix $\mathbb{D} = \mathrm{diag}(s_i \mid i \in I)$ which makes $\mathbb{DA}$ symmetric, in the standard manner so that $1 \leq s_i \leq 3$, in particular. Suppose that q is a root of 1 whose order N, say, is odd, and is not divided by 3 if the Dynkin diagram of $\mathbb{A}$ includes a connected component of type G_2. The Frobenius-Lusztig kernel $u = u_q$ is the finite-dimensional quotient Hopf algebra of $U = U_q$ subject to those relations which say that the N-th powers of $q^h (h \in P^\vee)$ and of the positive root vectors in $U^\pm$ are 1 and 0, respectively; see [L]. Let u^0, $u^{\geq 0}$, $u^{\leq 0}$, R', S' denote the natural images of U, $U^{\geq 0}$, $U^{\leq 0}$, R, S, respectively, in u. It is known that R', S' are Nichols algebras in $^{u^0}_{u^0}\mathcal{YD}$ so that $R' \rtimes u^0 = u^{\leq 0}$, $S' \rtimes u^0 = u^{\geq 0}$. We see that the skew pairing τ above factors through $u^{\leq 0} \otimes u^{\geq 0} \to k$, which is a non-degenerate skew pairing. Therefore, to the associated generalized q-boson algebra now of finite dimension, Remark 3.17 can apply.

Acknowledgments

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Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan
    E-mail address: `akira@math.tsukuba.ac.jp`