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Vector and axial currents in Wilson chiral perturbation theory

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We reconsider the construction of the vector and axial-vector currents in Wilson Chiral Perturbation Theory, the low-energy effective theory for lattice QCD with Wilson fermions. We discuss in detail the finite renormalization of the currents that has to be taken into account in order to properly match the currents. We explicitly show that imposing the chiral Ward identities on the currents does, in general, affect the axial-vector current at $O(a)$. As an application of our results we compute the pion decay constant to one loop in the two-flavor theory. Our result differs from previously published ones.

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I. INTRODUCTION

Chiral perturbation theory (ChPT) is widely regarded to be an important tool for lattice QCD. It provides analytic guidance for the chiral extrapolation of the lattice data obtained at quark masses heavier than in nature. Standard ChPT, as formulated in Refs. [1,2], is based on the symmetries and the particular symmetry breaking of continuum QCD. The generalization to lattice QCD with Wilson fermions, taking into account the explicit chiral symmetry breaking of Wilson's fermion discretization [3], was given in Ref. [4]. The resulting low-energy effective theory, often called Wilson chiral perturbation theory (WChPT), is a double expansion in the quark mass and the lattice spacing, the two parameters of explicit chiral symmetry breaking.

Massless continuum QCD is invariant under various nonsinglet chiral transformations. This invariance implies the existence of conserved currents (which are obtained by the Noether theorem) and various chiral Ward identities. ChPT is constructed in such a way that these Ward identities are correctly reproduced, order by order in the chiral expansion. And since conserved currents do not renormalize it is straightforward to maintain the normalization of the currents.

The construction of WChPT is slightly more complicated compared to continuum ChPT. Because of the explicit breaking of chiral symmetry by the Wilson term there does not exist a conserved axial-vector current for vanishing bare quark mass. And even though a conserved vector current exists for degenerate quark masses, it is often not used in practice. The local, nonconserved vector current is employed instead, even though it requires the computation of a renormalization constant Z_V. The renormalization constant Z_A is also needed for the axial-vector current.

The explicit breaking of chiral symmetry and the need for renormalizing the currents raises the question how to construct the effective currents in Wilson ChPT. The "Noether link" does not hold anymore. Also the renormalization of the currents has to be taken into account for a proper matching of the effective theory to the fundamental lattice theory.

Some results concerning the currents can be found in the literature [5,6], but they are in conflict. Reference [5] calculates the pion decay constant using the current obtained by the naive Noether procedure as the axial-vector current [7]. Reference [6] introduces source terms for the currents as in continuum ChPT, and constructs the generating functional for correlation functions of the currents. The resulting axial-vector current contains an additional $O(a)$ contribution that is not present in the Noether current. Consequently, the resultant f_π contains an extra term and differs from that of Ref. [5].

Besides this discrepancy the issue of renormalization has not been properly taken into account in either work. No particular renormalization condition for the axial-vector current has been imposed as is necessary for a proper matching of the currents. It has been argued in Ref. [6] that the results for the currents derived there should hold for any choice of lattice operators which are correctly normalized in the continuum limit. However, the validity of this expectation has not been shown so far.

In this paper we reconsider the construction and mapping of the vector and axial-vector currents in WChPT to $O(a)$.[1] We proceed in two steps. First, we write down the most general expressions for the currents which are compatible with locality and the symmetries of the underlying lattice theory. With this procedure we reproduce the results of Ref. [6] (which are more fully justified in Ref. [9]). In the second step we impose the chiral Ward identities as particular renormalization conditions for the currents. This choice, suggested in Refs. [10–12], is widely used in practice. We find that this renormalization condition does have an impact at $O(a)$ on the axial-vector current. Consequently, our current differs from the ones in

[1]Preliminary results have already been presented in Ref. [8]. Details have changed, but the overall conclusions are unaltered.

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Refs. [5,6]. As an application of our results we finally compute the pion decay constant to one loop, including the $O(a)$ correction to the chiral logarithm, which also differs from the results in [5,6].

This paper is organized as follows. In Sec. II we first summarize some definitions of the lattice theory with two flavors of Wilson quarks, in particular, the various vector and axial-vector currents used in numerical simulations. This is followed by the Symanzik expansion of the currents close to the continuum limit. The currents in the Symanzik effective theory are then mapped to their counterparts in the chiral effective theory. Section III discusses the renormalization of the vector and axial-vector currents in the lattice theory and how this is carried over to the effective theory. The results for the decay constant are given in Sec. IV, followed by some concluding remarks in Sec. V. Appendix A is devoted to an alternative but equivalent derivation of the currents based on the generating functional, while Appendix B consists of details concerning the calculation of Z_A in the effective theory.

II. CURRENTS IN WCHPT

A. Definitions in the lattice theory

We consider lattice QCD with Wilson fermions on a hypercubic lattice with lattice spacing a. For simplicity we study $N_f = 2$ quarks with equal quark mass. The fermion action is of the form

$$S_f = S_W + c_{\rm SW} S_{\rm clover}, \tag{1}$$

where the first part denotes the standard Wilson action [3] with bare quark mass m_0. We also allow for a clover-leaf term with coefficient $c_{\rm SW}$. The details of the gauge action are not important in the following so we leave it unspecified.

It is common to use the local expressions for the vector and axial-vector currents in numerical simulations,

$$V^a_{\mu,\rm Loc}(x) = \bar\psi(x)\gamma_\mu T^a \psi(x), \tag{2}$$

$$A^a_{\mu,\rm Loc}(x) = \bar\psi(x)\gamma_\mu\gamma_5 T^a \psi(x). \tag{3}$$

The T^a are the Hermitian $SU(N_f)$ generators, normalized according to $\mathrm{tr}(T^aT^b) = \delta^{ab}/2$. In the case of $N_f = 2$, the one we are considering, this normalization corresponds to $T^a = \sigma^a/2$, where σ^a are the usual Pauli matrices.

For degenerate quark masses the fermion action (1) is invariant under $SU(N_f)$ flavor transformations. The associated conserved vector current differs from (2) and reads [10]

$$\begin{aligned} V^a_{\mu,\rm Con}(x) = \frac{1}{2}\{&\bar\psi(x+a\hat\mu)\gamma_\mu T^a U_\mu(x)\psi(x) \\ &+ \bar\psi(x)\gamma_\mu T^a U^\dagger_\mu(x)\psi(x+a\hat\mu) \\ &+ \bar\psi(x+a\hat\mu)T^a U_\mu(x)\psi(x) \\ &- \bar\psi(x)T^a U^\dagger_\mu(x)\psi(x+a\hat\mu)\}. \end{aligned} \tag{4}$$

No conserved axial-vector current exists due to the explicit chiral symmetry breaking by the Wilson term in S_W.

In on-shell $O(a)$-improved lattice theories with degenerate quarks one defines improved currents by adding terms involving lattice derivatives to the local currents [13,14],

$$V^a_{\mu,\rm Imp}(x) = (1 + b_V am) \times \left[V^a_{\mu,\rm Loc}(x) + c_V \frac{1}{2}(\nabla^+_\nu + \nabla_\nu) T^a_{\mu\nu}(x)\right], \tag{5}$$

$$A^a_{\mu,\rm Imp}(x) = (1 + b_A am) \times \left[A^a_{\mu,\rm Loc}(x) + c_A \frac{1}{2}(\nabla^+_\mu + \nabla_\mu) P^a(x)\right], \tag{6}$$

where

$$\begin{aligned} T^a_{\mu\nu,\rm Loc}(x) &= \bar\psi(x) i\sigma_{\mu\nu} T^a \psi(x) \quad \text{and} \\ P^a_{\rm Loc}(x) &= \bar\psi(x)\gamma_5 T^a \psi(x). \end{aligned} \tag{7}$$

The coefficients $b_{V,A}$ and $c_{V,A}$—together with $c_{\rm SW}$—can be nonperturbatively tuned such that the cutoff effects are of $O(a^2)$ instead of linear in a.[2]

In order to correctly approach the continuum limit the nonconserved currents need to be renormalized. Thus one introduces

$$V^a_{\mu,\rm ren,Loc} = Z_{V,\rm Loc} V^a_{\mu,\rm Loc}, \qquad V^a_{\mu,\rm ren,Imp} = Z_{V,\rm Imp} V^a_{\mu,\rm Imp}, \tag{8}$$

$$A^a_{\mu,\rm ren,Loc} = Z_{A,\rm Loc} A^a_{\mu,\rm Loc}, \qquad A^a_{\mu,\rm ren,Imp} = Z_{A,\rm Imp} A^a_{\mu,\rm Imp}. \tag{9}$$

In the following, we will often use $Z_{V,A}$ generically, without specifying the underlying current.

The Z-factors (which depend not only on the choice of currents but also on the action) can be fixed by imposing chiral Ward identities [10–12]. We will come back to this important issue in Sec. III. The Z-factor for the conserved vector current is 1, of course.

[2] Note that the mass m in Eqs. (5) and (6) denotes the renormalized mass containing the additive mass renormalization proportional to $1/a$ and the renormalization factor: $m = Z_m(m_0 - m_{\rm cr})/a$. The factors of $(1 + b_{V,A}am)$ can also be considered to be part of the renormalization factor, but it is notationally convenient here to include them in the bare currents.

B. The Symanzik effective theory

According to Symanzik the lattice theory can be described by an effective continuum theory provided one is close to the continuum limit [15,16]. This effective theory is defined by an effective action and effective operators, and both are strongly restricted by the locality and the symmetries of the underlying lattice theory. The leading terms are, by construction, the familiar expressions of continuum QCD. Lattice artifacts appear as higher dimensional operators multiplied by appropriate powers of the lattice spacing. The effective action, for example, reads [17]

$$S_{\rm Sym} = S_{\rm ct} + a\bar{c}_{\rm SW} \int d^4x \bar{\psi}(x) i\sigma_{\mu\nu} F_{\mu\nu}(x)\psi(x) + O(a^2). \tag{10}$$

The first term is just the continuum QCD action for two flavors with degenerate quark mass. The leading cutoff effects are described by a single correction term, a Pauli term containing the field strength tensor $F_{\mu\nu}(x)$ multiplied by an unknown coefficient (“low-energy constant”) $\bar{c}_{\rm SW}$. Many more terms appear at $O(a^2)$ [17].

The mapping of the bare local currents of Eqs. (2) and (3) into the Symanzik theory is [13,14]:

$$V^a_{\mu,\rm Loc} \simeq V^a_{\mu,\rm Sym,Loc} = \frac{1}{Z^0_V}(1 + \bar{b}_V am)(V^a_{\mu,\rm ct} + a\bar{c}_V \partial_\nu T^a_{\mu\nu,\rm ct}) + O(a^2), \tag{11}$$

$$A^a_{\mu,\rm Loc} \simeq A^a_{\mu,\rm Sym,Loc} = \frac{1}{Z^0_A}(1 + \bar{b}_A am)(A^a_{\mu,\rm ct} + a\bar{c}_A \partial_\mu P^a_{\rm ct}) + O(a^2) \tag{12}$$

$$P^a_{\rm Loc} \simeq P^a_{\rm Sym,Loc} = \frac{1}{Z^0_P}(1 + \bar{b}_P am)P^a_{\rm ct} + O(a) \tag{13}$$

$$T^a_{\mu\nu,\rm Loc} \simeq T^a_{\mu\nu,\rm Sym,Loc} = \frac{1}{Z^0_T}(1 + \bar{b}_T am)T^a_{\mu\nu,\rm ct} + O(a) \tag{14}$$

where the continuum bilinears take their usual forms

$$V^a_{\mu,\rm ct} = \bar{\psi}\gamma_\mu T^a \psi, \qquad T^a_{\mu\nu,\rm ct} = \bar{\psi}\sigma_{\mu\nu}T^a\psi, \qquad A^a_{\mu,\rm ct} = \bar{\psi}\gamma_\mu\gamma_5 T^a\psi, \qquad P^a_{\rm ct} = \bar{\psi}\gamma_5 T^a\psi. \tag{15}$$

The mapping of operators between effective theories,

$$O_{\rm Lat} \simeq O_{\rm Sym}, \tag{16}$$

is defined so that

$$\langle O_{\rm Lat}(x) O(\psi^{\rm Lat}, \bar{\psi}^{\rm Lat}, A^{\rm Lat}_\mu, \mathbf{y})\rangle_{S_{\rm Lat}} = \langle O_{\rm Sym}(x) O(\psi^{\rm Sym}, \bar{\psi}^{\rm Sym}, A^{\rm Sym}_\mu, \mathbf{y})\rangle_{S_{\rm Sym}}. \tag{17}$$

Here $O(\psi^{\rm Lat}, \bar{\psi}^{\rm Lat}, A^{\rm Lat}_\mu, \mathbf{y})$ and $O(\psi^{\rm Sym}, \bar{\psi}^{\rm Sym}, A^{\rm Sym}_\mu, \mathbf{y})$ are arbitrary multilocal operators consisting of quark and gluon fields at positions $\mathbf{y} = y_1, y_2, \cdots$ which all differ from x. The above vacuum expectation values are calculated with $S_{\rm Lat}$ on the left-hand-side and $S_{\rm Sym}$ on the right-hand-side. We note that the equations of motion have been used in order to reduce the number of $O(a)$ terms on the right-hand sides of (10)–(12). This is legitimate as long as we consider the Symanzik theory as an on-shell effective theory, set up to reproduce physical quantities like masses, decay constants, etc..

We stress that the currents $V^a_{\mu,\rm Sym,Loc}$ and $A^a_{\mu,\rm Sym,Loc}$ are the result of matching the *bare* lattice currents into the Symanzik theory, and thus must include the renormalization constants $Z^0_{V,A}$. These have perturbative expansions of the form $Z^0_{V,A} = 1 + O[g(a)^2]$, but do not contain any contributions of $O(a)$, since all $O(a)$ terms are explicitly accounted for. They are completely determined in principle once one has specified the action and $g(a)$, but are only known approximately in practice. This will not present a problem, since we ultimately will normalize the currents nonperturbatively and all dependence on $Z^0_{V,A}$ will cancel. The superscript “0,” which indicates that this quantity is of 0’th order in an expansion in a, distinguishes these Z-factors from the nonperturbatively determined renormalization constants $Z_{V,A}$ introduced before. The latter, as we will see, depend linearly on a.

We can also map the renormalized currents of Eqs. (8) and (9) into the Symanzik theory. For $V_{\mu,\rm ren,Loc}$ and $A_{\mu,\rm ren,Loc}$, the result will be the same as in (11) and (12) except for multiplication by overall factors of (the to-be-determined quantities) $Z_{V,\rm Loc}$ and $Z_{A,\rm Loc}$, respectively.

The results (11) and (12) also hold for improved currents of the form of Eqs. (5) and (6), although the coefficients $Z^0_{V,A}$ and $\bar{c}_{V,A}$ will differ. It is important to keep in mind that these coefficients, as well as $\bar{c}_{\rm SW}$, are *a priori* unknown, their values depending on the details of the lattice theory. If all these parameters vanish we say that the lattice theory and current matrix elements are $O(a)$–improved. For this to happen the parameters c_X and b_X of the lattice theory need to be tuned to appropriate nonzero values.[3]

The vector current in (11) is the effective current into which the local lattice current is mapped. The conserved lattice vector current is mapped onto the most general conserved effective vector current. This current $V^a_{\mu,\rm Sym,Con}$ can be obtained by starting from (11) (with an *a priori* different coefficient $\bar{c}_V$ in front of the tensor term, and with $Z^0_V = 1$, $\bar{b}_V = 0$) and then imposing current

[3]Explicitly, $c_V = -(Z^0_T/Z^0_V)\bar{c}_V$, $c_A = -(Z^0_P/Z^0_A)\bar{c}_A$, $b_V = -\bar{b}_V$, and $b_A = -\bar{b}_A$.

conservation. The current in (11) is, however, already conserved,

$$\partial_\mu V^a_{\mu,\mathrm{Sym,Loc}} = 0, \tag{18}$$

which is a consequence of the particular structure of the $O(a)$ correction (total derivative of an antisymmetric tensor). Hence, both lattice currents are mapped onto the same form of effective current at this order in the Symanzik expansion. Violations of current conservation which imply a difference between the local and the conserved current are expected to show up at $O(a^2)$.

We mention two special cases of the Symanzik effective theory at $O(a)$ that we will need in the next section for the mapping to the chiral effective theory. Let us first consider the Symanzik effective theory in which $\bar{c}_{\mathrm{SW}}$ is nonzero but $\bar{c}_{V,A}$ and $\bar{b}_{V,A}$ vanish, and $Z^0_{V,A} = 1$. In this case the Symanzik currents coincide with the continuum QCD currents. These transform into each other under infinitesimal $SU(2)$ axial rotations $\delta\psi = \omega^a T^a \gamma_5 \psi$, $\delta\bar{\psi} = \omega^a \bar{\psi}\gamma_5 T^a$:

$$\delta V^a_\mu = i\epsilon^{abc}\omega^b A^c_\mu, \qquad \delta A^a_\mu = i\epsilon^{abc}\omega^b V^c_\mu. \tag{19}$$

This leads to various chiral Ward identities, schematically written as $\langle \delta S_{\mathrm{Sym}} \mathcal{O}_{\mathrm{Sym}}\rangle = \langle \delta \mathcal{O}_{\mathrm{Sym}}\rangle$, where $\mathcal{O}_{\mathrm{Sym}}$ denotes some product of vector and axial-vector currents. The form of these Ward identities is as in continuum QCD, since the right-hand side reads $\langle \delta \mathcal{O}_{\mathrm{Sym}}\rangle = \langle \delta \mathcal{O}_{\mathrm{ct}}\rangle$. The only difference is the appearance of an extra term proportional to $a\bar{c}_{\mathrm{SW}}$ in the variation of the effective action, caused by the Pauli term in S_{Sym}. Note, however, that these simple QCD-like Ward identities only hold at $O(a)$, and are violated at $O(a^2)$ by the terms of this order in the effective action and effective currents.

The second special case is obtained by setting $\bar{c}_{\mathrm{SW}} = 0$, $\bar{b}_{V,A} = 0$ and $Z^0_{V,A} = 1$, in which case $O(a)$ corrections stem entirely from the $\bar{c}_{V,A}$ terms in the effective currents. This implies, for example, that the correlation function of two axial-vector currents reads

$$\begin{aligned}\langle A^a_{\mu,\mathrm{Sym}}(x) A^a_{\nu,\mathrm{Sym}}(y)\rangle = {}& \langle A^a_{\mu,\mathrm{ct}}(x) A^a_{\nu,\mathrm{ct}}(y)\rangle \\ & + a\bar{c}_A \langle A^a_{\mu,\mathrm{ct}}(x)\partial_\nu P^a_{\mathrm{ct}}(y) \\ & + \partial_\mu P^a_{\mathrm{ct}}(x) A^a_{\nu,\mathrm{ct}}(y)\rangle.\end{aligned} \tag{20}$$

Here the expectation values are defined with Boltzmann factor $\exp(-S_{\mathrm{ct}})$ only. Hence, the $O(a)$ correction is given by the correlation function between the axial-vector current and the derivative of the pseudoscalar density. Again, this result will be violated as soon as one includes the corrections of $O(a^2)$.

C. Matching to ChPT

The appropriate chiral effective theory is obtained by writing down the most general chiral effective Lagrangian and effective currents which are compatible with the symmetries of the underlying Symanzik theory. A standard spurion analysis is employed in order to properly incorporate explicit symmetry breaking terms. For example, the Symanzik effective action is invariant under the chiral symmetry group G, parity P and charge conjugation C, provided both the mass and the coefficient $a\bar{c}_{\mathrm{SW}}$ are promoted to space-time dependent external fields M and A, which are postulated to transform according to [4,5,18]

$$\begin{aligned} &M \xrightarrow{G} LMR^\dagger, \qquad M \xrightarrow{P} M^\dagger, \qquad M \xrightarrow{C} M^T, \\ &A \xrightarrow{G} LAR^\dagger, \qquad A \xrightarrow{P} A^\dagger, \qquad A \xrightarrow{C} A^T. \end{aligned} \tag{21}$$

The "physical" values are obtained by setting $M \to m$ and $A \to a\bar{c}_{\mathrm{SW}}$. In an intermediate step, however, the spurion fields M and A are used together with the standard field

$$\Sigma(x) = \exp\left(\frac{2i}{f}\pi^a(x) T^a\right) \tag{22}$$

in order to write down the most general scalar that is invariant under G, P and C. This has been done in Refs. [5,18], and part of the result reads (in Euclidean space-time)

$$\begin{aligned}\mathcal{L}_{\mathrm{chiral}} = {}& \frac{f^2}{4}\langle \partial_\mu \Sigma(\partial_\mu \Sigma)^\dagger\rangle - \frac{f^2}{4} 2Bm\langle \Sigma^\dagger + \Sigma\rangle \\ & + L_{45} 2Bm \langle \partial_\mu \Sigma(\partial_\mu\Sigma)^\dagger\rangle\langle\Sigma^\dagger + \Sigma\rangle \\ & + W_{45}\hat{a}\bar{c}_{\mathrm{SW}}\langle \partial_\mu \Sigma(\partial_\mu\Sigma)^\dagger\rangle\langle\Sigma^\dagger + \Sigma\rangle \\ & - W_{68} 2Bm\hat{a}\bar{c}_{\mathrm{SW}}\langle\Sigma + \Sigma^\dagger\rangle^2 + \ldots .\end{aligned} \tag{23}$$

Here the angled brackets denote traces in flavor space.[4] The lattice spacing appears in the combination

$$\hat{a} = 2W_0 a, \tag{24}$$

which is of dimension two.[5] We have dropped a number of terms of $O(p^4)$ and $O(a^2)$, which we will not need in the following. Note that we have absorbed the term $f^2 W_0 a\bar{c}_{\mathrm{SW}}\langle\Sigma^\dagger + \Sigma\rangle/2$ in the definition of the mass, so m in (23) corresponds already to the so-called shifted mass [4,6].

Taking the naive continuum limit $a \to 0$ we recover the correct terms of the continuum chiral Lagrangian with the familiar low-energy coefficients f, B and $L_{45} = L_4 + L_5/2$ of continuum ChPT [1,2]. W_0 and $W_{45} = W_4 + W_5/2$ are low-energy constants associated with breaking terms due to the nonzero lattice spacing [5].[6]

[4] In the last section we used the same notation for correlation functions. The context usually tells unambiguously what is meant by $\langle \ldots \rangle$.

[5] W_0, a LEC that enters the chiral Lagrangian at $O(a)$, is of dimension three [5].

[6] Our notation for the low-energy coefficients differs slightly compared to the notation in other references, since our m is already the shifted mass: our $W_{45}\bar{c}_{\mathrm{SW}}$ and $W_{68}\bar{c}_{\mathrm{SW}}$ correspond to the combinations $W_{45} - L_{45}$ and $W_{68} - 2L_{68}$ in Ref. [5]. These combinations are abbreviated to $\tilde{W}$ and W in Ref. [6].

We now apply the same procedure to derive expressions for the effective operators. The currents in the Symanzik effective theory are given in Eqs. (11) and (12)—forms which, as noted above, hold for all the lattice currents of interest. To simplify the following discussion, we will map the Symanzik currents *without* overall Z-factors and b_X corrections into the chiral effective theory. These factors can be added at the end. Thus we consider the mappings

$$V^a_{\mu,\mathrm{ct}} + a\bar{c}_V \partial_\nu T^a_{\mu\nu,\mathrm{ct}} \longrightarrow V^a_{\mu,\mathrm{eff}}, \tag{25}$$

$$A^a_{\mu,\mathrm{ct}} + a\bar{c}_A \partial_\mu P^a_{\mathrm{ct}} \longrightarrow A^a_{\mu,\mathrm{eff}}. \tag{26}$$

The leading contributions are just the continuum expressions for the currents,

$$V^a_{\mu,\mathrm{ct}} = \bar{\psi}_R \gamma_\mu T^a \psi_R + \bar{\psi}_L \gamma_\mu T^a \psi_L, \tag{27}$$

$$A^a_{\mu,\mathrm{ct}} = \bar{\psi}_R \gamma_\mu T^a \psi_R - \bar{\psi}_L \gamma_\mu T^a \psi_L, \tag{28}$$

where we decomposed the currents into chiral fields. The vector current is just the sum of the right- and left-handed current, the axial-vector current is given by the difference.

The $O(a)$ corrections couple fields with opposite chirality, and the currents in the Symanzik theory do not transform under chiral rotations as continuum vector and axial-vector currents. However, the continuum transformation behavior can be enforced by promoting the coefficients $\bar{c}_V$, $\bar{c}_A$ to spurion fields C_V and C_A with nontrivial transformation behavior under G, P and C:

$$C_X \xrightarrow{G} L C_X R^\dagger, \qquad C_X \xrightarrow{P} C_X^\dagger, \qquad C_X \xrightarrow{C} C_X^T, \qquad X = V, A. \tag{29}$$

Note that these transformation laws are identical to the ones of the other $O(a)$ spurion A, cf. (21).

It is now easily checked that the $O(a)$ corrections written in the form

$$\begin{aligned} V^a_{\mu,a\text{-corr}} = {}& (\partial_\nu \bar{\psi}_L) i\sigma_{\mu\nu} C_V T^a \psi_R + \bar{\psi}_L i\sigma_{\mu\nu} T^a C_V \partial_\nu \psi_R \\ & + (\partial_\nu \bar{\psi}_R) i\sigma_{\mu\nu} C_V^\dagger T^a \psi_L + \bar{\psi}_R i\sigma_{\mu\nu} T^a C_V^\dagger \partial_\nu \psi_L, \end{aligned} \tag{30}$$

$$\begin{aligned} A^a_{\mu,a\text{-corr}} = {}& (\partial_\mu \bar{\psi}_L) \gamma_5 C_A T^a \psi_R + \bar{\psi}_L \gamma_5 T^a C_A \partial_\mu \psi_R \\ & + (\partial_\mu \bar{\psi}_R) \gamma_5 C_A^\dagger T^a \psi_L + \bar{\psi}_R \gamma_5 T^a C_A^\dagger \partial_\mu \psi_L, \end{aligned} \tag{31}$$

transform as the continuum currents. Setting the spurion field to its physical value, $C_V \to a\bar{c}_V$ and $C_A \to a\bar{c}_A$, one recovers the desired Symanzik currents.

The currents in the chiral effective theory are now obtained by writing down the most general vector and axial-vector current built of the chiral field Σ, its derivatives and the spurion fields. The spurions necessary for the construction of the vector current are M and A, the ones we already encountered in the construction of the effective action, and C_V. For the axial-vector C_A must be used instead.

In order to write down the currents we first recall that our effective theory has to reproduce continuum ChPT if we send $a \to 0$. This requirement implies that the continuum part of the currents are just the expressions given by Gasser and Leutwyler. At leading order these read

$$V^a_{\mu,\mathrm{LO}} = \frac{f^2}{2} \langle T^a (\Sigma^\dagger \partial_\mu \Sigma + \Sigma \partial_\mu \Sigma^\dagger) \rangle, \tag{32}$$

$$A^a_{\mu,\mathrm{LO}} = \frac{f^2}{2} \langle T^a (\Sigma^\dagger \partial_\mu \Sigma - \Sigma \partial_\mu \Sigma^\dagger) \rangle. \tag{33}$$

Obviously these expressions transform as vector and axial-vector currents under G, P and C. Moreover, these currents are properly normalized in order to satisfy the current algebra.

In order to construct the leading $O(a)$ corrections we need at least one power of either A or C_X and one partial derivative of Σ. It is easily checked that the following terms transform as desired:

$$\begin{aligned} &V1) \quad V^a_{\mu,\mathrm{LO}} \langle \Sigma^\dagger A + \Sigma A^\dagger \rangle, \\ &V2) \quad V^a_{\mu,\mathrm{LO}} \langle \Sigma^\dagger C_V + \Sigma C_V^\dagger \rangle, \\ &V3) \quad \langle T^a (\partial_\mu \Sigma^\dagger A - A \partial_\mu \Sigma^\dagger + \partial_\mu \Sigma A^\dagger - A^\dagger \partial_\mu \Sigma) \rangle, \\ &V4) \quad \langle T^a (\partial_\mu \Sigma^\dagger C_V - C_V \partial_\mu \Sigma^\dagger + \partial_\mu \Sigma C_V^\dagger - C_V^\dagger \partial_\mu \Sigma) \rangle, \end{aligned} \tag{34}$$

for the vector current, and

$$\begin{aligned} &A1) \quad A^a_{\mu,\mathrm{LO}} \langle \Sigma^\dagger A + \Sigma A^\dagger \rangle, \\ &A2) \quad A^a_{\mu,\mathrm{LO}} \langle \Sigma^\dagger C_A + \Sigma C_A^\dagger \rangle, \\ &A3) \quad \langle T^a (\partial_\mu \Sigma^\dagger A + A \partial_\mu \Sigma^\dagger - \partial_\mu \Sigma A^\dagger + A^\dagger \partial_\mu \Sigma) \rangle, \\ &A4) \quad \langle T^a (\partial_\mu \Sigma^\dagger C_A + C_A \partial_\mu \Sigma^\dagger - \partial_\mu \Sigma C_A^\dagger + C_A^\dagger \partial_\mu \Sigma) \rangle, \end{aligned} \tag{35}$$

for the axial-vector current. Setting the external fields to their final value, $C_X \to a\bar{c}_X$, we obtain the following expressions for the currents in the effective theory:

$$V^a_\mu = V^a_{\mu,\mathrm{LO}} \left(1 + \frac{4}{f^2} \hat{a} (W_{V1} \bar{c}_{\mathrm{SW}} + W_{V2} \bar{c}_V) \langle \Sigma + \Sigma^\dagger \rangle \right), \tag{36}$$

$$\begin{aligned} A^a_\mu = {}& A^a_{\mu,\mathrm{LO}} \left(1 + \frac{4}{f^2} \hat{a} (W_{A1} \bar{c}_{\mathrm{SW}} + W_{A2} \bar{c}_A) \langle \Sigma + \Sigma^\dagger \rangle \right) \\ & + 4\hat{a} (W_{A3} \bar{c}_{\mathrm{SW}} + W_{A4} \bar{c}_A) \partial_\mu \langle T^a (\Sigma - \Sigma^\dagger) \rangle. \end{aligned} \tag{37}$$

The coefficients W_X are unknown low-energy constants (LECs). In order to make W_{X1}, W_{X2} dimensionless we included the factor $4/f^2$. Note that the second line in (37) is proportional to the continuum pseudoscalar density, $P^a_{\mathrm{ct}} = f^2 B \langle T^a (\Sigma - \Sigma^\dagger) \rangle / 2$.

The number of unknown LECs in the currents can be reduced using the freedom of field redefinition [6,19]. Explicitly, performing $\Sigma \to \Sigma + \delta\Sigma$ with

$$\delta\Sigma = \frac{4\hat{a}}{f^2}\bar{c}_{\rm SW}\Delta W(\Sigma^2 - 1), \tag{38}$$

we obtain the same effective Lagrangian and currents with the transformed coefficients

$$\begin{aligned} W_{45} &\to W_{45} + \Delta W, \qquad W_{68} \to W_{68} + \Delta W/2, \\ W_{V1} &\to W_{V1} + \Delta W, \qquad W_{A1} \to W_{A1} + \Delta W, \\ & W_{A3} \to W_{A3} - \Delta W. \end{aligned} \tag{39}$$

Therefore, depending on the particular choice for ΔW we may make one LEC vanish. In the following we choose this to be W_{A3} in the expression for the axial-vector current.

So far the expressions in (36) and (37) are the most general currents compatible with the symmetries that have the correct continuum limit. In order to match the currents properly we have to impose the constraints that these currents have to obey at $O(a)$, for instance current conservation for the vector current. This will relate some of the LECs W_X to those in the effective action.

In order to discuss this we first derive the EOM corresponding to the Lagrangian in Eq. (23) without the continuum NLO term proportional to L_{45} and without the $O(am)$ correction proportional to W_{68}.[7] Using the shorthand notation $\chi = 2Bm$, $\rho = 2W_{45}W_0 a\bar{c}_{\rm SW}$ and

$$R = \frac{4}{f^2}\langle\Sigma + \Sigma^\dagger\rangle, \qquad T = \frac{4}{f^2}\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\rangle, \tag{40}$$

the leading order EOM reads

$$\begin{aligned} &[\Sigma(\partial_\mu\partial_\mu\Sigma^\dagger) - (\partial_\mu\partial_\mu\Sigma)\Sigma^\dagger](1 + \rho R) \\ &\quad - [\Sigma - \Sigma^\dagger](\chi + \rho T) \\ &= 2\lambda + [\partial_\mu\Sigma\Sigma^\dagger - \Sigma\partial_\mu\Sigma^\dagger]\rho\partial_\mu R. \end{aligned} \tag{41}$$

The parameter λ is the Lagrange multiplier associated with the constraint $\det\Sigma = 1$. Setting the lattice spacing to zero (i.e. $\rho = 0$) Eq. (41) reproduces the EOM in continuum ChPT [2].

Using (41) we find the condition

$$\partial_\mu V_\mu^a = 0 \Leftrightarrow W_{V1} = W_{45} \quad \text{and} \quad W_{V2} = 0. \tag{42}$$

Therefore, the conserved vector current in the chiral effective theory is given by

$$V^a_{\mu,\rm eff} = V^a_{\mu,\rm LO}\left(1 + \frac{4}{f^2}\hat{a}W_{45}\bar{c}_{\rm SW}\langle\Sigma + \Sigma^\dagger\rangle\right). \tag{43}$$

[7] These can and need to be included if the NLO terms of $O(p^2 m)$ and $O(p^2 ma)$ are included in the current.

This expression agrees with the one in Ref. [6] obtained from the generating functional. Note that at this order this current coincides with the Noether current associated with vector transformations. This will, however, no longer be true at higher order in the chiral expansion.[8]

Note that there is no term proportional to $\bar{c}_V$ in (43), which is a consequence of the $\partial_\nu T^a_{\mu\nu}$ structure of the $O(a)$ correction. We need three partial derivatives in order to construct such a term in the chiral effective theory. Hence, this correction is of $O(ap^3)$, which is of higher order than we consider here.

In order to obtain the proper result for the axial-vector current we have to make sure that the properties (19) and (20) of the underlying theory are correctly reproduced. The result (43) for the vector current together with (19) (for $\bar{c}_V = \bar{c}_A = 0$) immediately implies $W_{A1} = W_{45}$. On the other hand, setting $\bar{c}_{\rm SW} = 0$ and demanding (20) leads to $W_{A2} = 0$. So the final result for the axial-vector current reads

$$\begin{aligned} A^a_{\mu,\rm eff} = A^a_{\mu,\rm LO}&\left(1 + \frac{4}{f^2}\hat{a}W_{45}\bar{c}_{\rm SW}\langle\Sigma + \Sigma^\dagger\rangle\right) \\ &+ 4\hat{a}W_A\bar{c}_A\partial_\mu\langle T^a(\Sigma - \Sigma^\dagger)\rangle, \end{aligned} \tag{44}$$

where we abbreviated $W_{A4} = W_A$. This expression also agrees with the result in Ref. [6].[9]

So far we have discussed the currents at leading order in the chiral expansion including the first correction of $O(a)$. Higher-order contributions to the currents can be derived in the same fashion. The terms without factors of the lattice spacing are the familiar contributions of continuum ChPT,

$$\begin{aligned} \delta V^a_{\mu,\rm eff} = \frac{f^2}{2}&\langle T^a(\Sigma^\dagger\partial_\mu\Sigma + \Sigma\partial_\mu\Sigma^\dagger)\rangle \\ &\times\left(\frac{4}{f^2}2BmL_{45}\langle\Sigma + \Sigma^\dagger\rangle\right), \end{aligned} \tag{45}$$

$$\begin{aligned} \delta A^a_{\mu,\rm eff} = \frac{f^2}{2}&\langle T^a(\Sigma^\dagger\partial_\mu\Sigma - \Sigma\partial_\mu\Sigma^\dagger)\rangle \\ &\times\left(\frac{4}{f^2}2BmL_{45}\langle\Sigma + \Sigma^\dagger\rangle\right). \end{aligned} \tag{46}$$

In order to construct the first subleading $O(a)$ corrections we have to form vector and axial-vector currents with one power of either A or C_X, and either three derivatives (corresponding to the O $(ap^2$ contributions), or one derivative and one power of the mass spurion field M (the $O(am)$ contributions). Simple examples for such terms are

[8] At higher order in the chiral expansion the currents receive contributions which are not present in the Noether current. The vector current contribution proportional to L_9 [2], which captures the dominant contribution of the pion form factor, is an example for this. We thank J. Bijnens for pointing this out to us.

[9] The result in Ref. [6] is obtained by setting $W_{10} = 2W_A\bar{c}_A$ and $\tilde{W} = 2W_{45}\bar{c}_{\rm SW}$.

products of the leading $O(a)$ terms in (34) and (35) with either $\langle \Sigma M^\dagger + M\Sigma^\dagger \rangle$ or $\langle \partial_\mu \Sigma (\partial_\mu \Sigma)^\dagger \rangle$. An example is the correction

$$\delta A^a_{\mu,\mathrm{eff}} = \frac{f^2}{2} \langle T^a(\Sigma^\dagger \partial_\mu \Sigma - \Sigma \partial_\mu \Sigma^\dagger)\rangle (X \bar{c}_A \hat{a} m \langle \Sigma + \Sigma^\dagger \rangle^2), \tag{47}$$

with X being the low-energy constant associated with this correction.

There are more terms possible and it is a straightforward but tedious exercise to find all possible terms contributing to the currents. For the rest of this paper, however, we do not need these higher-order terms.

We conclude this section by discussing the pseudoscalar density, which we need later on for the computation of the PCAC mass. In the Symanzik effective theory the pseudoscalar density is given by P^a_{ct}. The corresponding expression in the chiral effective theory is constructed in the same way as the currents. Since the density in the Symanzik theory has no $O(a)$ term, we only have the spurion A for the construction of the density, and we find[10]

$$P^a_{\mathrm{eff}} = \frac{f^2 B}{2} \langle T^a(\Sigma^\dagger - \Sigma)\rangle \left(1 + \frac{4}{f^2} \hat{a} W_{68} \bar{c}_{\mathrm{SW}} \langle \Sigma + \Sigma^\dagger \rangle\right). \tag{48}$$

As already mentioned, our expressions for the effective currents agree with the results in Ref. [6], although the two calculations use different methods. In the approach of Ref. [6], which follows the procedure used by Gasser and Leutwyler in continuum ChPT [1], sources for the currents are introduced and the generating functional is constructed. The currents are then obtained from the generating functional by taking derivatives with respect to the sources. The analysis is complicated, however, by the presence in the Symanzik theory of $O(a)$ violations of the local gauge invariance used to restrict the mapping to the chiral effective theory. This complication, missed in Ref. [6], was noted in Ref. [9], and an appropriate extension outlined. The outcome of this extension was that the form of the results of Ref. [6] still holds after a redefinition of the LECs. Since a systematic and complete treatment of the impact of the $O(a)$ effects in the generating functional method has not been presented, however, we provide such a treatment in Appendix A. In particular, we show that the same results (43) and (44) are obtained using this method.

III. RENORMALIZATION

We now return to the issue of the normalization of the currents. The results of the previous section allow us to determine the form that a given lattice current will have when mapped into WChPT. For example, putting back the overall factors, the renormalized local vector and axial-vector currents of Eqs. (8) and (9) map as

$$V_{\mu,\mathrm{ren,Loc}} \simeq \frac{Z_{V,\mathrm{Loc}}}{Z^0_V}(1 + \bar{b}_V am) V^a_{\mu,\mathrm{eff}}, \tag{49}$$

$$A_{\mu,\mathrm{ren,Loc}} \simeq \frac{Z_{A,\mathrm{Loc}}}{Z^0_A}(1 + \bar{b}_A am) A^a_{\mu,\mathrm{eff}}, \tag{50}$$

with $V^a_{\mu,\mathrm{eff}}$ and $A^a_{\mu,\mathrm{eff}}$ given by Eqs. (43) and (44), respectively. In the lattice theory, the renormalization factors are determined by imposing particular conditions at nonzero lattice spacing. Hence, in order to properly match the effective currents we should impose the same conditions in the effective theory.

A. Renormalization of the lattice currents

Since the conserved vector current does not need to be renormalized it can be used to normalize the local current and to define $Z_{V,\mathrm{Loc}}$ (at the lattice level) according to [20,21]

$$Z_{V,\mathrm{Loc}} = \frac{\langle f|V^a_{\mu,\mathrm{Con}}|i\rangle}{\langle f|V^a_{\mu,\mathrm{Loc}}|i\rangle}. \tag{51}$$

Here i and f denote arbitrary initial and final states, though it is convenient to choose zero-momentum pseudoscalar states. Note that the dependence of $Z_{V,\mathrm{Loc}}$ on the particular states can be sizable, in particular, if the theory is not $O(a)$-improved [20].

An alternative definition for $Z_{V,\mathrm{Loc}}$, which does not use the conserved current, is given by [20–22]

$$Z_{V,\mathrm{Loc}} \langle \pi^a(\vec{p})|V^b_{0,\mathrm{Loc}}|\pi^c(\vec{p})\rangle = \epsilon^{abc} 2E. \tag{52}$$

The matrix element on the left-hand side can be obtained in the usual way by calculating the ratio of two correlation functions, where pseudoscalar sources are used to project onto the pion states. Although we have specified initial and final pion states, the renormalization factor still depends on the momenta of the pions [20]. For massive pions one usually chooses zero spatial momentum, $\vec{p} = 0$.

The two renormalization conditions (51) and (52) do not specify the renormalization condition completely, since the matrix elements still depend on the quark mass. Theoretically preferable are mass independent renormalization schemes, in which the renormalization condition is imposed at zero quark mass. This implies some technical difficulties, because numerical simulations cannot be performed directly for vanishing quark masses. One way to circumvent this technical limitation is to calculate the Z-factors for various small quark masses and extrapolate

[10] This agrees with the result of Ref. [6], with W_{68} related to W of that reference by $W = W_{68}\bar{c}_{\mathrm{SW}}$.

the results to the massless limit. This may introduce some extrapolation error, but in principle is a viable procedure.

A different strategy to define and compute the renormalization factors makes use of Schrödinger functional boundary conditions [23]. With this setup it is possible to compute $Z_{V,\mathrm{Loc}}$ and other renormalization factors for a vanishing partially conserved axial-vector current (PCAC) quark mass. Even though the renormalization conditions are imposed at vanishing quark mass, they now depend on details of the Schrödinger functional setup, e.g. the size and geometry of the finite volume. This dependence can, in principle, be removed by extrapolating to infinite volume.

Being able to work at vanishing quark mass has another advantage: chiral Ward identities involving the axial-vector current simplify significantly. In Ref. [24], for example, the Ward identity

$$\int_{\partial R} d\sigma_\mu(x)\epsilon^{abc}\langle f|A^a_{\mu,\mathrm{ren}}(x)A^b_{\nu,\mathrm{ren}}(y)|i\rangle = 2i\langle f|V^c_{\nu,\mathrm{ren}}(y)|i\rangle \tag{53}$$

is imposed. This identity is the Euclidean analogue of the current algebra relation stating that the commutator of two axial-vector currents is equal to the vector current [25]. The region R is chosen to be the space-time volume between two hyper-planes at $x_0 = y_0 \pm t$. The equation for $\nu = 0$ between pseudoscalar states has been used to determine the renormalization factor Z_A. For more details see Ref. [24].

B. Renormalization of the effective currents

Having chosen particular renormalization conditions for the lattice currents, we have to impose the same conditions in the chiral effective theory.

Some conditions are harder to implement than others. For example, matrix elements between the vacuum and a vector meson state in Eq. (51) are not easily accessible in standard mesonic ChPT, since the vector meson is not a degree of freedom in the chiral effective theory. Conditions involving quark states (the so-called "RIMOM" scheme [26]) are also out of reach. In practice, only conditions involving pseudoscalar states can be treated in the chiral effective theory.

In the following we carry out the matching for one particular class of renormalization conditions. For the vector current we assume that either the condition (51) or (52) is imposed with single pion states at zero spatial momentum and at vanishing bare PCAC mass. For the axial-vector current we assume that condition (53) is employed to fix Z_A. We impose these conditions in infinite volume. Finite volume can also be considered, but it does not make a difference at the order in the chiral expansion to which we work.

As a first step we have to calculate the PCAC mass and set it to zero. The PCAC mass is defined by

$$m_{\mathrm{PCAC}} = \frac{\langle 0|\partial_\mu A^a_{\mu,\mathrm{eff}}|\pi^a\rangle}{2\langle 0|P^a_{\mathrm{eff}}|\pi^a\rangle}. \tag{54}$$

Expanding the Σ fields in (44) and (48), keeping only one power of π^a, the ratio of expectation values on the right-hand side is easily obtained[11]

$$m_{\mathrm{PCAC}} = \frac{M_\pi^2}{2B}\left(1 + \frac{8}{f^2}\hat{a}[2(W_{45} - W_{68})\bar{c}_{\mathrm{SW}} + W_A\bar{c}_A]\right), \tag{55}$$

in agreement with Ref. [6]. The PCAC quark mass is proportional to the pion mass, which is given by [6]

$$M_\pi^2 = 2Bm\left(1 + \frac{16}{f^2}\hat{a}(2W_{68} - W_{45})\bar{c}_{\mathrm{SW}}\right). \tag{56}$$

Recall that m denotes the shifted mass including the leading $O(a)$ shift.

At higher order in the chiral expansion the right-hand side of (56) contains an additional correction of $O(a^2)$ [4]. A vanishing pion mass therefore corresponds to $m = O(a^2)$. Since we ignore $O(a^2)$ corrections, we conclude that a vanishing PCAC quark mass is equivalent to $m = 0$, and in the following we assume this condition for the shifted mass.[12]

The next step is the determination of $Z_{V,\mathrm{Loc}}$ using either (51) or (52). Both conditions are easily calculated using the chiral effective theory "image" of the local lattice current, Eq. (49) with $m = 0$, and the expression (43). In both cases we find[13]

$$Z_{V,\mathrm{Loc}} = Z_V^0. \tag{57}$$

Even though the result is the same for both renormalization conditions, the way it arises differs in the two cases. The result for the first condition is obviously trivial since both the local and the conserved effective vector currents have the same form in the effective theory [cf. (43)], differing only by overall factors. In (52), on the other hand, the two-

[11]In practice, to calculate m_{PCAC} requires knowledge of the renormalization constants of the lattice axial-vector current and pseudoscalar density, which can, as the present work shows, introduce additional $O(a)$ corrections. These do not, however, change the key result being derived here, namely, that $m_{\mathrm{PCAC}} \propto m$ up to $O(a^2)$ corrections.

[12]The $O(a^2)$ correction implies a nontrivial phase structure of the theory with two qualitatively different scenarios [4]. One of these is characterized by a first-order phase transition and the pion mass is nonzero for all values of m. For $m = 0$, however, the pion mass assumes its minimal value $M^2_{\pi,\mathrm{min}} = O(a^2)$. Since here we ignore the $O(a^2)$ corrections we also find for this scenario that a vanishing PCAC mass is given by a vanishing shifted mass, at least to the order we are working.

[13]It is a simple matter to restore the mass dependence, in which case (57) would read $Z_{V,\mathrm{Loc}} = Z_V^0(1 + \bar{b}_V am)$.

pion states contribute a wave function renormalization factor Z_π, which reads[14]

$$Z_\pi = 1 - \frac{16}{f^2}\hat{a}W_{45}\bar{c}_{SW}, \tag{58}$$

and which exactly cancels the $O(a)$ correction coming from the current.

Let us consider higher-order corrections to the result (57). Expanding the Σ fields in the vector current (43) the first correction terms contain four pion fields. Two of these need to be contracted and form a loop. Hence the result will be proportional to $M_\pi^2 \ln M_\pi^2$, which vanishes for $m_{PCAC} = 0$. Higher-order analytic parts, on the other hand, will contribute corrections of $O(a^2)$, which we do not consider here.

The implication of Eq. (57) is that, after one or other of the renormalization conditions has been enforced, the renormalized local vector current maps simply into $V^a_{\mu,\text{eff}}$, since the prefactors on the right-hand side of Eq. (49) cancel. The same holds true for any lattice vector current which is renormalized in this way.

We now proceed to the local axial-vector current, whose normalization $Z_{A,\text{Loc}}$ is to be fixed by imposing the Ward identity (53), using external one-pion states having definite (nonzero) momenta. Following Ref. [24] we choose the region R to be the space-time volume between two hyperplanes at $x_0 = y_0 \pm t$ with some finite time separation t. The equation for $\nu = 0$ can be brought into the form [24]

$$\int d\vec{x}\epsilon^{abc}\epsilon^{cde}\langle\pi^d(\vec{p})|[A^a_{0,\text{ren}}(y_0+t,\vec{x}) - A^a_{0,\text{ren}}(y_0-t,\vec{x})] \times A^b_{0,\text{ren}}(y)|\pi^e(\vec{q})\rangle = 2i\epsilon^{cde}\langle\pi^d(\vec{p})|V^c_{0,\text{ren}}(y)|\pi^e(\vec{q})\rangle. \tag{59}$$

The matrix element on the right-hand side of this equation is essentially the one in (52) that we used to fix $Z_{V,\text{Loc}}$. Imposing (59) thus provides a condition for $Z_{A,\text{Loc}}$: simply compute the two sides in the effective theory and set them equal. The calculation is straightforward but rather technical. For this reason we defer the details of the calculation to Appendix B. The final result is (recall that $m = 0$ so there is no $\bar{b}_A$ term)

$$\frac{Z_{A,\text{Loc}}}{Z_A^0} = 1 - \frac{4\hat{a}}{f^2}(W_{45}\bar{c}_{SW} + W_A\bar{c}_A)z_A(t), \tag{60}$$

$$z_A(t) = 1 - \cosh[t(|\vec{p}| - |\vec{q}|)]\exp[-|t||\vec{p}-\vec{q}|]. \tag{61}$$

Since this result is determined by a ratio of physical correlation functions in which the axial currents are separated, it must depend on the physical combination of LECs $W_{45}\bar{c}_{SW} + W_A\bar{c}_A$ [6]. This point is also explained at the end of Appendix A, and provides a check of our result. We emphasize that (60) is the complete result to $O(a)$, there are no corrections of $O(am)$ since we work at zero quark mass. The next correction to (60) is of $O(a^2)$, which we do not consider here.

In contrast to the vector current we do find a nonvanishing correction of $O(a)$. That this correction depends on the separation between the axial currents, t, and upon the external state, is as expected. The t dependence is proportional to an integral (a sum on the lattice) of the divergence of the axial current, which does not vanish at $O(a)$, even when $m = 0$. The dependence of the external state is a generic feature of $O(a)$ corrections in an unimproved theory.

Using Eqs. (50) and (60) we find that the renormalized local axial-vector current maps into WChPT as

$$A^a_{\mu,\text{ren,Loc}} \simeq A^a_{\mu,\text{ren,Loc,eff}} = \left[1 - \frac{4\hat{a}}{f^2}(W_{45}\bar{c}_{SW} + W_A\bar{c}_A)z_A(t)\right]A^a_{\mu,\text{eff}}. \tag{62}$$

This is the main new result of this paper. We see that the a dependence of $A^a_{\mu,\text{eff}}$, derived using symmetries and given in Eq. (44), is supplemented by an additional discretization error resulting from the application of the normalization condition at nonzero a.

Note that the quantity Z_A^0 appears only in the combination $Z_{A,\text{Loc}}/Z_A^0$ in Eqs. (50) and (60), and the individual value of Z_A^0 is not necessary. This factor is needed only when expressing the bare lattice current in the intermediate Symanzik theory. In fact, the combination $Z_{A,\text{Loc}}/Z_A^0$, and also the analogue for the vector current, $Z_{V,\text{Loc}}/Z_V^0$, may be interpreted as renormalization constants $Z_{A,\text{eff}}$ and $Z_{V,\text{eff}}$ in the chiral effective theory [8].

We also remark that the *form* of $A^a_{\mu,\text{ren,Loc,eff}}$ applies to any lattice current—local or improved. These cases only differ in the values of the LECs. Thus in the following we will drop the subscript ''Loc'' on the renormalized current in the chiral effective theory.

We close this section by investigating the dependence of $z_A(t)$ on t, $\vec{p}$ and $\vec{q}$. There turn out to be three distinct cases (recalling that $\vec{p}, \vec{q} \neq 0$):

(i) $\vec{p} = \vec{q}$. This is the simplest case to implement practically, and leads to $z_A(t) = 0$. Thus it turns out that, in this case, there are no additional $O(a)$ terms introduced by the current normalization.

(ii) $\vec{p}$ parallel to $\vec{q}$. Then, for $|t| \gg 1/|\vec{p}-\vec{q}|$, the product of cosh and exponential becomes $1/2$, and so $z_A \to 1/2$.

(iii) All other nonvanishing $\vec{p}$ and $\vec{q}$. Here, for $|t| \gg 1/|\vec{p}-\vec{q}|$, the exponential overwhelms the cosh and $z_A \to 1$.

We stress, however, that in both the second and third cases z_A depends on t for nonasymptotic values of t.

[14] Note that there are no terms proportional to the quark mass in the chiral Lagrangian (23), since we have set m to zero.

IV. APPLICATION: PION DECAY CONSTANT

At this point we have completed the construction and matching of the effective currents. Now we can proceed and compute correlation functions involving these currents. As a simple but important example we calculate the pion decay constant. Given the presence of the $z_A(t)$ contribution, our result differs, in general, from previously published ones.

The literature usually distinguishes two quark mass regimes: (i) the GSM regime with $m \sim a\Lambda_{\rm QCD}^2$ and (ii) the regime with $m \sim a^2\Lambda_{\rm QCD}^3$, sometimes called LCE regime.[15] Each of these regimes has its associated power-counting. In this section we do not, however, work within either regime, but rather calculate to one-loop order keeping terms of $O(a)$, but not $O(m)$, in the tree-level result. This choice is made for simplicity, and because we aim only to illustrate the impact of using the correctly normalized currents, and not to provide a compendium of results. In particular, we do not include correction of $O(a^2)$, which are required for an NLO result in the LCE regime. The NLO result for the GSM regime can, however, be obtained from our formulae by dropping some higher-order terms.

A. Decay constant at tree level

Expanding the renormalized axial-vector current in powers of the pion fields we obtain to leading order the expression

$$A^a_{\mu,\rm ren,eff} = if\frac{\partial_\mu \pi^a}{\sqrt{Z_\pi}} \times \left(1 + \frac{4}{f^2}\hat{a}(W_{45}\bar{c}_{\rm SW} + W_A\bar{c}_A)[2 - z_A(t)]\right). \tag{63}$$

This depends, as required, only on the physical combination $W_{45}\bar{c}_{\rm SW} + W_A\bar{c}_A$ of LECs. We can immediately read off the following expression for the tree-level decay constant:

$$f_{\pi,\rm tree} = f\left(1 + \frac{4}{f^2}\hat{a}(W_{45}\bar{c}_{\rm SW} + W_A\bar{c}_A)[2 - z_A(t)]\right). \tag{64}$$

We see that the $O(a)$ corrections depend not only on the form of the underlying lattice action and currents (through the LECs) but also on the choice of renormalization condition [through $z_A(t)$]. The $z_A(t)$ term always reduces the size of these corrections, although by no more than a factor of 2 (which happens in the third case discussed at the end of the previous section). The decay constant is free of $O(a)$ corrections only if both the action and the axial-vector current are improved, i.e. for $\bar{c}_{\rm SW} = \bar{c}_A = 0$, in accordance with what we know from the Symanzik effective theory.

[15] GSM stands for *generically small quark masses* [6] and LCE for *large cutoff effects* [27].

We want to comment on the origin of discrepancies between (64) and previously published results for the decay constant. Reference [5] finds[16]

$$f^{\rm RS}_{\pi,\rm tree} = f\left(1 + \frac{8}{f^2}\hat{a}W_{45}\bar{c}_{\rm SW}\right). \tag{65}$$

There is no correction proportional to $W_A\bar{c}_A$ since their calculation used the Noether current as the axial-vector current [7]. This misses the $O(a)$ correction in (12), and cannot be correct since the result (65) is $O(a)$ improved if only the action (and not the current) is improved.

Reference [6] finds the same form as (64) except without the $z_A(t)$ contribution. These authors assumed that a nonperturbative renormalization condition had been applied, but did not include the impact of applying the condition at nonvanishing lattice spacing.

B. Decay constant to one loop

It is straightforward to compute the leading correction to the tree-level result (64). Expanding the axial-vector current in Eq. (37) to higher powers of the pion fields one obtains the one-loop contributions to f_π. The integrals that appear can be regularized using dimensional regularization. The counterterms for the divergences are provided by the tree-level contribution of the NLO terms in the axial-vector current, c.f. (46) and (47). Even though we have not explicitly given all possible NLO corrections of $O(am, ap^2)$, it is easy to convince oneself that all contributing tree-level terms with one partial derivative are of the form $if\partial_\mu\pi^a \cdot am$. Expressing m by the tree-level pion mass according to (56), we obtain the counterterm

$$A^a_\mu[aM_\pi^2]_{\rm CT} = if\partial_\mu\pi^a\tilde{W}_{A3}\hat{a}M_\pi^2/f^4. \tag{66}$$

For simplicity we have absorbed the coefficients $\bar{c}_{\rm SW}$ and $\bar{c}_A$, and the contribution proportional to $z_A(t)$, in the LEC $\tilde{W}_{A3}$, since in practice these and the W coefficients are difficult to disentangle. The additional factor $1/f^4$ is introduced for convenience, since it leads to a dimensionless coefficient $\tilde{W}_{A3}$.

The final one-loop result for f_π is then given by

$$f_{\pi,1\text{-loop}} = f\left(1 + \frac{\hat{a}}{f^2}\tilde{W}_{A1} - \frac{1}{16\pi^2 f^2}\left[1 + \frac{\hat{a}}{f^2}\tilde{W}_{A2}\right] \times M_\pi^2 \ln\frac{M_\pi^2}{\mu^2} + \frac{8}{f^2}M_\pi^2\left[L_{45} + \frac{\hat{a}}{f^2}\tilde{W}_{A3}\right]\right). \tag{67}$$

The coefficients are $\tilde{W}_{A1} = 4(W_{45}\bar{c}_{\rm SW} + W_A\bar{c}_A)\times[2 - z_A(t)]$ (as above) and $\tilde{W}_{A2} = 4(W_{45}\bar{c}_{\rm SW} + W_A\bar{c}_A)\times[1 - z_A(t)]$. Note that both coefficients depend on the physical combination of LECs, as expected.

[16] Ref. [5] quotes explicitly the three-flavor result, which we changed to the corresponding two-flavor result to make the comparison. We also dropped the one-loop correction.

A couple of comments concerning (67) are in order. In the continuum limit we reproduce the familiar result for f_π from continuum ChPT for $N_f = 2$ [1]. Away from the continuum limit the result is modified by terms of $O(a)$, $O(aM_\pi^2)$ and $O(aM_\pi^2 \ln M_\pi^2)$. Dropping the latter two, i.e. setting $\tilde{W}_{A2} = \tilde{W}_{A3} = 0$, we obtain the NLO result for the GSM regime. Taking in addition $z_A(t) = 0$ we reproduce the NLO result in Ref. [6].

The coefficients $\tilde{W}_{A1}$ and $\tilde{W}_{A2}$ are in general not independent. For example, the first case discussed at the end of the last section has $\tilde{W}_{A1} = 2\tilde{W}_{A2}$ (for asymptotically large t values). For the third case we even find $\tilde{W}_{A2} = 0$, so the coefficient of the chiral logarithm is free of $O(a)$ artifacts. Since in this case also $\tilde{W}_{A1}$ assumes its minimal value this is the theoretically preferred renormalization condition.

Except for the special case with $\tilde{W}_{A2} = 0$ the coefficient of the chiral logarithm receives an $O(a)$ correction in the form of the factor $[1 + \hat{a}\tilde{W}_{A2}/f^2]$. Consequently, the coefficient of the chiral logarithm is, in contrast to continuum ChPT, not a universal coefficient depending on f (and N_f) only, but on the (nonuniversal) lattice artifacts too. This fact has previously been stressed in Ref. [28].

Note that the combination L_{45} appears effectively in the lattice spacing dependent combination $L_{45}^{\rm eff}(a) = L_{45} + \hat{a}\tilde{W}_{A3}/f^2$. Determinations of L_{45} based on simulations at one lattice spacing only are potentially dangerous because the size of the contribution $\hat{a}\tilde{W}_{A3}/f^2$ is *a priori* unknown.

V. CONCLUDING REMARKS

In this paper we have reconsidered the construction of the vector and axial-vector currents in WChPT. Because of the explicit chiral symmetry breaking in lattice QCD with Wilson fermions two aspects need to be taken into account which are not present in continuum ChPT.

First, the local lattice currents are not conserved, and in general they do not map onto the conserved currents in WChPT. In particular, the WChPT currents are not the Noether currents associated with the chiral symmetries, not even at leading order in the chiral expansion. The reason is that the currents in the Symanzik theory have $O(a)$ corrections which are not related to the effective action.

Second, the matching of the currents needs to take into account the finite renormalization of the local lattice currents. In order to properly match the currents the same renormalization conditions that have been employed for the lattice currents must be imposed on the WChPT currents.

Depending on the particular choice for the renormalization conditions the expressions for the renormalized currents differ by terms of $O(a)$. As a result, the WChPT predictions for matrix elements of the currents are different as well. Consequently, a result for an observable like f_π should make reference to the renormalization condition one has adopted. This has to be so at some level, since the lattice data differs too depending on the condition one has chosen. What we find is that the dependence enters at $O(a)$.

At a technical level, our result does not change the number of low-energy coefficients that enter into WChPT predictions at $O(a)$. In particular, as stressed in Ref. [6], there are only two combinations of the LECs that can enter into physical quantities, allowing their $O(a)$ corrections to be related. What changes is the nature of these relations, which now depend on the choice of normalization condition.

The considerations of this paper apply more generally. Another example from WChPT is the ratio of the renormalization factors for pseudoscalar and scalar densities. This must be determined nonperturbatively, e.g. by enforcing Ward identities, and will presumably receive $O(a)$ corrections similar to those for the axial current, although we have not worked out the details. The combination mS is, however, protected by exact lattice WTIs.

More generally still, our results emphasize the (perhaps rather obvious) point that nonperturbative renormalization conditions generically introduce additional discretization errors. These must be (and in practice usually are being) accounted for when extrapolating to the continuum limit.

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APPENDIX A: THE GENERATING FUNCTIONAL METHOD

In Sec. II C we constructed the effective currents in a rather direct way. We first wrote down the most general currents that transform as vector and axial-vector currents. In a second step we imposed appropriate Ward identities that these currents must obey, and this led to various constraints on the LECs in these currents.

We already mentioned at the end of Sec. II C that this is not the way one proceeds in continuum ChPT [1,2]. Instead, there one sets up the generating functional for correlation functions involving the currents (and the scalar and pseudoscalar density), and matches this to the analogous generating functional in the chiral effective theory. In this way the currents are obtained by functional derivatives with respect to the sources.

A crucial link in the matching is the invariance under local chiral transformations, which plays the role of a gauge symmetry. The local invariance implies that the sources enter the chiral Lagrangian in a very restricted way, namely, in form of a gauge-covariant derivative.

One may ask if the same procedure is also possible once we include the lattice spacing corrections. In order to answer this question we derive in detail the generating functional for the Symanzik effective theory and match it to the one in WChPT. We will see that we indeed obtain the same currents as the ones in (43) and (44). However, we will also see that it is more complicated to maintain the invariance under local chiral transformations.

The "generating functional method" has been used before in WChPT in Ref. [6]. However, as discussed in Ref. [9], the generating functional of Ref. [6] violates the invariance under local chiral transformations at nonzero lattice spacing; only the leading (continuum) part respects this symmetry. Consequently, the construction of the axial-vector current requires some care: part of the $O(a)$ corrections have to be mapped separately into the effective theory [9]. The final result found in [9] for the currents turns out, however, to have the form given in Ref. [6], as we derive here in a systematic way.

1. Generating functional in continuum QCD and ChPT

In order to prepare our discussion we first give a brief review of the construction in continuum ChPT, mainly to introduce our notation. We start by defining a (Euclidean) QCD Lagrangian including a part that includes sources for the currents and densities,

$$\mathcal{L} = \mathcal{L}_{\rm QCD} - i\mathcal{L}_{\rm Source}, \tag{A1}$$

where $\mathcal{L}_{\rm QCD}$ is the usual massless QCD Lagrangian, while the second part

$$\mathcal{L}_{\rm Source} = \bar\psi\gamma_\mu[v_\mu(x) + \gamma_5 a_\mu(x)]\psi + \bar\psi[s(x) + \gamma_5 p(x)]\psi \tag{A2}$$

contains sources for the vector and axial-vector currents (v_μ, a_μ) and for the scalar and pseudoscalar densities (s, p). These sources are matrix valued fields, given by

$$v_\mu(x) = v_\mu^a(x)T^a, \qquad a_\mu(x) = a_\mu^a(x)T^a, \qquad s(x) = s^a(x)T^a, \qquad p(x) = p^a(x)T^a, \tag{A3}$$

where T^a are the Hermitian $SU(N_f)$ generators, normalized according to $\mathrm{tr}(T^aT^b) = \delta^{ab}/2$. We are interested in $N_f = 2$, for which $T^a = \sigma^a/2$, with σ^a the usual Pauli matrices.

After integrating over space-time we obtain the action in the presence of the sources, $S = \int d^4x\mathcal{L}(x)$. By taking functional derivatives with respect to the sources we obtain the vector and axial-vector current and the scalar and pseudoscalar densities:

$$i\frac{\delta S}{\delta v_\mu^a(x)} = \bar\psi(x)\gamma_\mu T^a\psi(x) = V_\mu^a(x), \tag{A4}$$

$$i\frac{\delta S}{\delta a_\mu^a(x)} = \bar\psi(x)\gamma_\mu\gamma_5 T^a\psi(x) = A_\mu^a(x), \tag{A5}$$

$$i\frac{\delta S}{\delta s^a(x)} = \bar\psi(x)T^a\psi(x) = S^a(x), \tag{A6}$$

$$i\frac{\delta S}{\delta p^a(x)} = \bar\psi(x)\gamma_5 T^a\psi(x) = P^a(x). \tag{A7}$$

The key observation is that the Lagrangian (A1) is invariant under *local* $SU(N_f)_R \times SU(N_f)_L$ transformations, which act on the fermion fields according to

$$\psi(x) \longrightarrow \psi'(x) = R(x)P_+\psi(x) + L(x)P_-\psi(x), \qquad \bar\psi(x) \longrightarrow \bar\psi'(x) = \bar\psi(x)P_+L^\dagger(x) + \bar\psi(x)P_-R^\dagger(x). \tag{A8}$$

The projectors are defined in the usual way, $P_\pm = (1 \pm \gamma_5)/2$, and project onto fields with definite chirality,

$$\psi_R = P_+\psi, \qquad \psi_L = P_-\psi, \qquad \bar\psi_R = \bar\psi P_-, \qquad \bar\psi_L = \bar\psi P_+. \tag{A9}$$

Crucial for the local invariance is the nontrivial transformation of the source fields

$$v_\mu + a_\mu \longrightarrow v'_\mu + a'_\mu = R(v_\mu + a_\mu)R^\dagger + iR\partial_\mu R^\dagger, \tag{A10}$$

$$v_\mu - a_\mu \longrightarrow v'_\mu - a'_\mu = L(v_\mu - a_\mu)L^\dagger + iL\partial_\mu L^\dagger, \tag{A11}$$

$$s + p \longrightarrow s' + p' = L(s + p)R^\dagger, \tag{A12}$$

$$s - p \longrightarrow s' - p' = R(s - p)L^\dagger. \tag{A13}$$

(For notational simplicity we drop from now on the argument x). The invariance is more easily seen if we express the Lagrangian in terms of left- and right-handed fields,

$$\mathcal{L}_{\rm Source} = \bar\psi_R\gamma_\mu[v_\mu + a_\mu]\psi_R + \bar\psi_L\gamma_\mu[v_\mu - a_\mu]\psi_L + \bar\psi_L[s + p]\psi_R + \bar\psi_R[s - p]\psi_L, \tag{A14}$$

which obviously is invariant under global transformations. Note that global transformations leave $\mathcal{L}_{\rm QCD}$ and $\mathcal{L}_{\rm Source}$ independently invariant. Under local transformations the derivative part of $\mathcal{L}_{\rm QCD}$ produces extra terms which are cancelled by the derivative terms in (A10) and (A11). The reason why this cancellation works is related to the fact that the currents are the conserved Noether currents associated with chiral transformations and hence stem from local variations of the derivative term in the Lagrangian.

The Lagrangian $\mathcal{L}$ can be conveniently rewritten in a form that makes the local invariance more transparent.

First, the source term in (A14) suggests to introduce sources for right- and left-handed currents,

$$r_\mu = v_\mu + a_\mu, \qquad l_\mu = v_\mu - a_\mu, \tag{A15}$$

which transform according to

$$\begin{aligned} r_\mu \to r'_\mu &= R r_\mu R^\dagger + iR\partial_\mu R^\dagger, \\ l_\mu \to l'_\mu &= L l_\mu L^\dagger + iL\partial_\mu L^\dagger. \end{aligned} \tag{A16}$$

Vector and axial-vector currents are then obtained by

$$V^a_\mu(x) = i\left[\frac{\delta S}{\delta r^a_\mu(x)} + \frac{\delta S}{\delta l^a_\mu(x)}\right], \tag{A17}$$

$$A^a_\mu(x) = i\left[\frac{\delta S}{\delta r^a_\mu(x)} - \frac{\delta S}{\delta l^a_\mu(x)}\right]. \tag{A18}$$

The right- and left-handed sources allow the definition of covariant derivatives

$$D^R_\mu = D^{\mathrm{color}}_\mu - ir_\mu, \qquad D^L_\mu = D^{\mathrm{color}}_\mu - il_\mu, \tag{A19}$$

where D^{color}_μ is the covariant derivative with respect to local $SU(3)$ color transformations. By construction, $D^R_\mu \psi_R$, $D^L_\mu \psi_L$ transform as the chiral fields themselves, and $\mathcal{L}$ is simply given by

$$\mathcal{L} = \bar\psi_R \gamma_\mu D^R_\mu \psi_R + \bar\psi_L \gamma_\mu D^L_\mu \psi_L + \mathcal{L}_{\mathrm{Gauge}}, \tag{A20}$$

where $\mathcal{L}_{\mathrm{Gauge}}$ is the gauge field part of the QCD Lagrangian containing the gluon fields $G^a_\mu(x)$. Obviously, (A20) is invariant under local chiral transformations.

Having $\mathcal{L}$ in hand we can now define a generating functional for correlation functions involving the currents and densities,

$$Z_{\mathrm{QCD}}[r_\mu, l_\mu, s, p] = \frac{1}{Z_0}\int \mathcal{D}[G_\mu, \bar\psi, \psi] e^{-\int d^4x \mathcal{L}[r_\mu, l_\mu, s, p]}, \tag{A21}$$

where Z_0 denotes the partition function, i.e. Z_{QCD} for vanishing sources.[17] Taking functional derivatives with respect to the sources one generates all possible correlation functions involving the currents and densities; the generating functional for connected correlation functions can be defined as usual by the logarithm of Z_{QCD}.

The matching to the chiral effective theory is now done by requiring that the generating functional in the chiral effective theory is the same as the one in the fundamental theory,

$$Z_{\mathrm{QCD}}[r_\mu, l_\mu, s, p] = Z_{\mathrm{chiral}}[r_\mu, l_\mu, s, p], \tag{A22}$$

[17]Strictly speaking, the scalar source should not vanish but be set to the physical quark mass.

$$Z_{\mathrm{chiral}}[r_\mu, l_\mu, s, p] = \frac{1}{Z_0}\int \mathcal{D}[\pi] e^{-\int d^4x \mathcal{L}_{\mathrm{chiral}}[r_\mu, l_\mu, s, p]}. \tag{A23}$$

The equality in (A22) is meant in the sense that the two sides coincide order by order in a low-energy expansion, where the long-distance correlation functions are dominated by the pion pole [29]. Correlation functions in the effective theory are obtained by the same functional derivatives as in the underlying theory. Provided the generating functionals, as functions of the sources, are the same, arbitrary correlation functions also agree. Moreover, the QCD symmetries are carried over to the effective theory, i.e. the Ward identities of QCD are correctly reproduced by the effective theory. Consequently, the right-hand side must also be invariant under local chiral transformations, and this invariance provides one constraint in the construction of $\mathcal{L}_{\mathrm{chiral}}$. Other constraints are provided by parity (P) and charge conjugation (C).

The effective Lagrangian $\mathcal{L}_{\mathrm{chiral}}[r_\mu, l_\mu, s, p]$ is obtained in a systematic low-energy expansion, and it has been derived by Gasser and Leutwyler through next-to-leading order [1,2]. We do not repeat their result here but emphasize that the local chiral invariance implies that the sources for the right- and left-handed currents can only enter the effective Lagrangian through the covariant derivative

$$D_\mu \Sigma = \partial_\mu \Sigma + i\Sigma r_\mu - il_\mu \Sigma \tag{A24}$$

on the usual chiral field Σ, and through the field strength tensors

$$\begin{aligned} r_{\mu\nu} &= \partial_\mu r_\nu - \partial_\nu r_\mu + i[r_\mu, r_\nu], \\ l_{\mu\nu} &= \partial_\mu l_\nu - \partial_\nu l_\mu + i[l_\mu, l_\nu]. \end{aligned} \tag{A25}$$

The transformation behavior of these objects is as expected, taking into account $\Sigma \to L\Sigma R^\dagger$,

$$D_\mu \Sigma \to L(D_\mu \Sigma)R^\dagger, \qquad r_{\mu\nu} \to R r_{\mu\nu} R^\dagger, \qquad l_{\mu\nu} \to L l_{\mu\nu} L^\dagger. \tag{A26}$$

2. Generating functional in the Symanzik effective theory

Following the development in the continuum, we want to define a generating functional in the Symanzik effective theory, which may then be matched to the chiral effective theory. As before we want to obtain the currents and densities by taking derivatives with respect to the source fields. As we will see this is not as straightforward as in continuum QCD. For simplicity we will deal here only with the currents and ignore the densities. This reveals the main obstacles in the procedure.

As in the continuum case we want to define a source term in the Lagrangian, such that taking the derivatives with respect to sources produce the Symanzik currents, given in (11) and (12). In addition, we want to do this in a way that maintains invariance under local chiral transformations, since this symmetry provides one of the links in the matching to the chiral effective theory.

It is illustrative to first show that the naive generalization of the continuum case fails in this respect. Suppose we write down a source term that directly couples source fields to the vector and axial-vector current,

$$\mathcal{L}_{\text{Source,Sym}} = v^a(x)V^a_{\mu,\text{Sym,Loc}}(x) + a^a_\mu(x)A^a_{\mu,\text{Sym,Loc}}(x). \tag{A27}$$

Just as in the continuum—cf. Eqs. (A4) and (A5)—functional derivatives with respect to $v^a(x)$, $a^a(x)$ produce the desired currents. However, local invariance is lost: The $O(a)$ corrections in the currents and the derivative terms in (A10) and (A11), generate $O(a)$ terms under local chiral transformations that are not cancelled by the variation of the Symanzik effective action in (10). That is of course no surprise; it simply reflects the fact that chiral symmetry is explicitly broken in the Symanzik effective theory and that the currents in (11) and (12) are not Noether currents associated with exact chiral symmetries.

In order to maintain local chiral invariance we proceed differently and introduce separate sources fields for the $O(a)$ corrections in the currents. Our source Lagrangian has the form

$$\mathcal{L}_{\text{Source,Sym}} = \mathcal{L}_{\text{Source,ct}} + \mathcal{L}_{\text{Source},a}. \tag{A28}$$

The first term with subscript "ct" is the familiar one from continuum QCD, defined in (A2). As before, one can rewrite it in terms of right- and left-handed sources using (A15). The second part reads

$$\begin{aligned}\mathcal{L}_{\text{Source},a} = {}& (D^L_\nu \bar\psi_L) i\sigma_{\mu\nu} C_V r_{V,\mu}\psi_R + \bar\psi_L i\sigma_{\mu\nu} l_{V,\mu} C_V D^R_\nu \psi_R + (D^R_\nu\bar\psi_R) i\sigma_{\mu\nu} C^\dagger_V l_{V,\mu}\psi_L + \bar\psi_R i\sigma_{\mu\nu} r_{V,\mu} C^\dagger_V D^L_\nu \psi_L \\ &+ (D^L_\mu\bar\psi_L)\gamma_5 C_A r_{A,\mu}\psi_R + \bar\psi_L\gamma_5 l_{A,\mu} C_A D^R_\mu\psi_R + (D^R_\mu\bar\psi_R)\gamma_5 C^\dagger_A l_{A,\mu}\psi_L + \bar\psi_R\gamma_5 r_{A,\mu} C^\dagger_A D^L_\mu \psi_L.\end{aligned} \tag{A29}$$

This is essentially nothing but a source term for the $O(a)$ corrections to the currents as given in (30) and (31). We have introduced sources $r^a_{X,\mu}$, $l^a_{X,\mu}$, $X = V$ or A, which transform according to

$$r_{X,\mu} \to r'_{X,\mu} = R r_{X,\mu} R^\dagger \qquad l_{X,\mu} \to l'_{X,\mu} = L l_{X,\mu} L^\dagger. \tag{A30}$$

Note that this is the transformation law under *local* transformations even though there are no terms involving derivatives of L and R. Note also that $\mathcal{L}_{\text{Source},a}$ contains the covariant derivatives D^R_μ, D^L_μ, defined in (A19), which involve the sources l_μ, r_μ for the continuum parts in the Symanzik currents.

It is easily checked that the source term (A29) is invariant under local chiral transformations (taking into account the transformation laws for C_V, C_A given in (29)). Therefore, also the total Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{Sym}} - i\mathcal{L}_{\text{Source,Sym}} \tag{A31}$$

is locally invariant. The currents are then obtained from $S = \int d^4x \mathcal{L}(x)$ according to

$$V^a_{\mu,\text{Sym}}(x) = i\left[\frac{\delta S}{\delta r^a_\mu(x)} + \frac{\delta S}{\delta l^a_\mu(x)} + \frac{\delta S}{\delta r^a_{V,\mu}(x)} + \frac{\delta S}{\delta l^a_{V,\mu}(x)}\right], \tag{A32}$$

$$A^a_{\mu,\text{Sym}}(x) = i\left[\frac{\delta S}{\delta r^a_\mu(x)} - \frac{\delta S}{\delta l^a_\mu(x)} + \frac{\delta S}{\delta r^a_{A,\mu}(x)} - \frac{\delta S}{\delta l^a_{A,\mu}(x)}\right], \tag{A33}$$

which is a generalization of the prescription in the continuum, (A17) and (A18). Note that we implicitly assume here that the spurion fields A, C_V, C_A are set to their physical values $a\bar c_{\text{SW}}$, $a\bar c_V$, $a\bar c_A$ after the derivatives have been taken.

So far we focused on the invariance under local chiral transformations. For the matching to the chiral effective theory the discrete symmetries P and C are also needed. In addition, there is one property in the source term that we also have to preserve. The source term in (A29) depends on the sources and the spurion fields C_V and C_A. The dependence is such that the vector current sources r_V, l_V only couple to C_V and the axial-vector sources only to C_A. A mixed term, involving $C_A r_{V,\mu}$ for example, is not present. It is mandatory to preserve this feature, otherwise the vector current in the effective theory may end up with a contribution proportional to $\bar c_A$, which is certainly not the case. In order to achieve this we choose to generalize the source term without changing the results for the currents derived from it.

Instead of one spurion field C_V we introduce two of them, $C_{V,1}$ and $C_{V,2}$, and make the replacements

$$C_V r_{V,\mu} \to C_{V,1} r_{V,\mu} \qquad l_{V,\mu} C_V \to l_{V,\mu} C_{V,2}. \tag{A34}$$

The symmetry properties of $\mathcal{L}_{\text{Source},a}$ remain unchanged if both $C_{V,1}$ and $C_{V,2}$ transform as C_V, and if both have the same physical value $a\bar c_V$. However, the source term is now invariant under more general symmetry transformations, namely

$$\begin{aligned} r_{V,\mu} &\to H_V r_{V,\mu} R^\dagger, \qquad & C_{V,1} &\to L C_{V,1} H^\dagger_V, \\ l_{V,\mu} &\to L l_{V,\mu} G^\dagger_V, \qquad & C_{V,2} &\to G_V C_{V,2} R^\dagger, \end{aligned} \tag{A35}$$

where H_V and G_V are two independent local $SU(2)$ matrices. The origin of these two *hidden local symmetries* is the fact that the right- and left-handed sources and the spurions

$C_{V,i}$ only appear next to each other in the source term. Note that we recover the previously discussed transformation laws with $H_V = R$ and $G_V = L$.

The same generalization can be done with the axial-vector part in the source term after we have introduced $C_{A,1}$ and $C_{A,2}$ and postulated the transformation behavior

$$\begin{aligned} r_{A,\mu} &\to H_A r_{A,\mu} R^\dagger, \qquad C_{A,1} \to L C_{A,1} H_A^\dagger, \\ l_{A,\mu} &\to L l_{A,\mu} G_A^\dagger, \qquad C_{A,2} \to G_A C_{A,2} R^\dagger. \end{aligned} \tag{A36}$$

Since the symmetries H_A, G_A are independent of the ones in the vector current part we can no longer form invariants with vector current sources and axial-vector spurions $C_{A,i}$ and vice-versa. This will become important in the next section where we construct the currents in the chiral effective theory.

In analogy to (A21) we can now define a generating functional in the Symanzik effective theory. The number of external fields, however, has grown significantly,

$$\begin{aligned} Z_{\mathrm{Sym}} = Z_{\mathrm{Sym}}[&r_\mu, l_\mu, r_{V,\mu}, l_{V,\mu}, r_{A,\mu}, l_{A,\mu}, A, C_{V,1}, \\ &C_{V,2}, C_{A,1}, C_{A,2}], \end{aligned} \tag{A37}$$

and here we still have not included any sources associated with the scalar and pseudoscalar density. Apparently, the method of constructing correlation functions via a generating functional loses its simplicity away from the continuum limit. In the naive continuum limit (A37) reduces to Z_{QCD} with its dependence on r_μ, l_μ only.

3. Matching to WChPT

Similarly to the continuum case we now match the Symanzik effective theory to Wilson ChPT by demanding that the generating functionals in both theories agree

$$Z_{\mathrm{Sym}} = Z_{\mathrm{chiral}}, \tag{A38}$$

where the right-hand side is defined in analogy to (A23). For simplicity we have dropped the dependence on the large number of external fields, specified in (A37).

The effective Lagrangian entering the right-hand side in (A38) is built in terms of the pseudoscalar field Σ and its covariant derivative $D_\mu \Sigma$ as well as all external fields. Many terms have already been constructed and can be found in the literature. First, one finds the familiar Gasser-Leutwyler Lagrangian [1,2], which provides the continuum part. Terms involving the spurion field A are given in Ref. [6]. Hence we focus here on the new contributions stemming from the $O(a)$ terms in the currents, i.e. those involving the fields $r_{V,\mu}$, $l_{V,\mu}$, $r_{A,\mu}$, $l_{A,\mu}$, $C_{V,i}$, $C_{A,i}$.

It will be useful to introduce the following combinations:

$$\begin{aligned} S_{V,\mu} &= i(C_{V,1} r_{V,\mu} + l_{V,\mu} C_{V,2}), \\ S_{A,\mu} &= i(C_{A,1} r_{A,\mu} + l_{A,\mu} C_{A,2}), \\ A_{V,\mu} &= i(C_{V,1} r_{V,\mu} - l_{V,\mu} C_{V,2}), \\ A_{A,\mu} &= i(C_{A,1} r_{A,\mu} - l_{A,\mu} C_{A,2}). \end{aligned} \tag{A39}$$

These combinations are automatically invariant under the four hidden symmetries involving H_X, G_X with $X = V, A$. The factor i is just convention, inspired by the observation that the terms involving the continuum sources also always include an i. For convenience we have summarized the transformation behavior of these quantities in Table I. We also list the transformation rules under parity and charge conjugation, which one also needs for the construction of invariants under all symmetries.

We are now in the position to construct the terms of $O(a)$ that will contribute to the currents. For this we need invariants involving one power of the sources in (A39). Lorentz invariance then requires one covariant derivative D_μ. The simplest invariants that can be formed are

$$\langle D_\mu \Sigma A_{X,\mu}^\dagger + A_{X,\mu}(D_\mu \Sigma)^\dagger \rangle, \qquad X = V, A. \tag{A40}$$

Analogous invariants involving $S_{X,\mu}$ cannot be built since they violate charge conjugation.

We do not need to consider any invariants involving $D_\mu S_{X,\mu}$, $D_\mu A_{X,\mu}$ or terms quadratic in $S_{X,\mu}$ or $A_{X,\mu}$. Even though there exist nonvanishing invariants they will not give contributions to the vector or axial-vector current. The reason is that all these terms contain more than one source field and derivatives of C_X. Setting in the end the sources to zero and C_X to its constant final value all these terms vanish. We also do not consider the terms involving $\langle \Sigma^\dagger D_\mu \Sigma \pm \Sigma (D_\mu \Sigma)^\dagger \rangle$ since

$$\mathrm{tr}\,(\Sigma^\dagger D_\mu \Sigma) = \partial_\mu \ln\det\Sigma = 0. \tag{A41}$$

It turns out that the two terms given in (A40) are the only independent invariants at the order we are interested in.

TABLE I. Summary of transformation properties. The short-hand notation $(-1)^\mu$ in the vector quantities represents the sign flip in the spatial components under parity.

Field	Chiral	Charge Conj. C	Parity P
Σ	$L\Sigma L^\dagger$	Σ^T	$\Sigma^\dagger$
$D_\mu \Sigma$	$L D_\mu \Sigma R^\dagger$	$(D_\mu \Sigma)^T$	$(-1)^\mu (D_\mu \Sigma)^\dagger$
A	$LAR^\dagger$	A^T	$A^\dagger$
$r_{X,\mu}$	$H_X r_{X,\mu} R^\dagger$	$-l_{X,\mu}^T$	$(-1)^\mu l_{X,\mu}$
$l_{X,\mu}$	$L r_{X,\mu} G_X^\dagger$	$-r_{X,\mu}^T$	$(-1)^\mu r_{X,\mu}$
$C_{X,1}$	$L C_X H_X^\dagger$	$C_{X,1}^T$	$C_{X,1}^\dagger$
$C_{X,2}$	$G_X C_X R^\dagger$	$C_{X,2}^T$	$C_{X,2}^\dagger$
$S_{X,\mu}$	$L S_{X,\mu} R^\dagger$	$-S_{X,\mu}^T$	$-(-1)^\mu S_{X,\mu}^\dagger$
$A_{X,\mu}$	$L A_{X,\mu} R^\dagger$	$A_{X,\mu}^T$	$(-1)^\mu A_{X,\mu}^\dagger$

In the last section we stressed the importance of the local hidden symmetries for the correct chiral Lagrangian. If instead of (A35) and (A36) we assume the weaker transformation laws (A30) and (29), we can construct the additional invariants

$$\langle \Sigma^\dagger D_\mu \Sigma r_{X,\mu} + \Sigma (D_\mu \Sigma)^\dagger l_{X,\mu} \rangle \langle C_X \Sigma^\dagger + \Sigma C_X^\dagger \rangle. \quad \text{(A42)}$$

These invariants require to "split" the right- and left-handed source fields from the $O(a)$ spurions C_X and build products of two trace terms, something that is forbidden by the transformation rule (A35) and (A36).

It is straightforward to compute the $O(a)$ corrections to the currents following from (A40). In the case of the vector current the correction vanishes. For the axial-vector current we find the correction

$$2a\bar{c}_A \partial_\mu \langle T^a(\Sigma - \Sigma^\dagger) \rangle. \quad \text{(A43)}$$

In order to derive the complete currents including all $O(a)$ corrections we also need other terms which have already been given by Ref. [6]. Carrying over their notation, the relevant terms in the Lagrangian are

$$\begin{aligned}\mathcal{L} = {} & \frac{f^2}{4} \langle D_\mu \Sigma (D_\mu \Sigma)^\dagger \rangle + W_{45} \langle D_\mu \Sigma (D_\mu \Sigma)^\dagger \rangle \\ & \times \langle A\Sigma^\dagger + \Sigma A^\dagger \rangle + W_{10} \langle D_\mu \Sigma (D_\mu A)^\dagger \\ & + D_\mu A (D_\mu \Sigma)^\dagger \rangle + 2W_A \langle D_\mu \Sigma A_{A,\mu}^\dagger + A_{A,\mu} (D_\mu \Sigma)^\dagger \rangle.\end{aligned} \quad \text{(A44)}$$

The last term, coming with a new LEC W_A, is the one we found above. We do not include the term with $X = V$ since its contribution to the vector current vanishes anyway.

Before deriving the currents we are free to make a field redefinition in order to simplify $\mathcal{L}$. Following Ref. [6] we perform the change $\Sigma \to \Sigma + \delta\Sigma$ with

$$\delta\Sigma = \frac{4W_{10}}{f^2} (\Sigma A^\dagger \Sigma - A) \quad \text{(A45)}$$

and obtain $\mathcal{L}$ in (A44) with the modified coefficient $W_{45} \to W_{45} + W_{10}$. Therefore, without loss we can drop the W_{10} term in (A44), as long as we consider only physical quantities.

In fact, if we work only to linear order in the sources, it is easy to see that the W_A and W_{10} terms in (A44) are proportional, so that W_A can also be absorbed into W_{45} by a change of variables [9]. This holds as long as we consider only physical quantities *and* if all currents are placed at different space-time points. In order to avoid the latter restriction, and maintain generality, we do not make this change of variables. Nevertheless, for the quantities we calculate in this paper, which are both physical and involve separated currents, the possibility of this change of variables implies that the LECs must enter in the combination [6] $W_{45}\bar{c}_{SW} + W_A\bar{c}_A$. This provides a check on the results.

We can now compute the vector and axial-vector current according to (A32) and (A33), and find exactly the same expressions given before in Eqs. (43) and (44). We therefore conclude that we obtain identical results for the currents with the generating functional as with our "direct method."

APPENDIX B: COMPUTATION OF $Z_{A,\mathrm{Loc}}$.

In this appendix we derive the result (60) for the renormalization constant $Z_{A,\mathrm{Loc}}$, which follows from imposing the Ward-Takahashi identity (WTI) in (59). It will be useful to first show that (59) is indeed an identity in (massless) continuum ChPT. The generalization to WChPT is then straightforward.

We start by establishing

$$\begin{aligned}&\int d\vec{x} \epsilon^{abc} \langle \{A^a_{0,\mathrm{LO}}(y_0 + t, \vec{x}) - A^a_{0,\mathrm{LO}}(y - t, \vec{x})\} A^b_{0,\mathrm{LO}}(y) \mathcal{O}_{\mathrm{out}} \rangle \\ &= 2i \langle V^c_{0,\mathrm{LO}}(y) \mathcal{O}_{\mathrm{out}} \rangle,\end{aligned} \quad \text{(B1)}$$

at leading order in continuum ChPT. The operator $\mathcal{O}_{\mathrm{out}}$ is composed of fields with support outside the time interval $[y_0 - t, y_0 + t]$. We will not need to specify $\mathcal{O}_{\mathrm{out}}$ in detail until we consider the $O(a)$ corrections; for now, we only assume that it creates an even number of pion fields, that the two-pion component occurs at LO in a chiral expansion, and that the two-pion component contains no zero-momentum pions.

Expanding the vector current to leading order in the pion fields,

$$\begin{aligned}V^a_{\mu,\mathrm{LO}} &= \frac{f^2}{2} \langle T^a (\Sigma^\dagger \partial_\mu \Sigma + \Sigma \partial_\mu \Sigma^\dagger) \rangle \\ &= i\epsilon^{abc} \pi^b \pi^c_\mu + O(\pi^4),\end{aligned} \quad \text{(B2)}$$

(using abbreviation $\pi^a_\mu = \partial_\mu \pi^a$), the right-hand side of the identity we want to show is simply given by

$$\mathrm{rhs}_{\mathrm{LO}} = 2i \langle V^c_{0,\mathrm{LO}}(y) \mathcal{O}_{\mathrm{out}} \rangle_{\mathrm{LO}} \quad \text{(B3)}$$

$$= -2\epsilon^{cab} \langle \pi^a(y) \pi^b_0(y) \mathcal{O}_{\mathrm{out}} \rangle_{\mathrm{LO}} \quad \text{(B4)}$$

where $\langle \mathcal{O} \rangle_{\mathrm{LO}}$ denotes functional integrals with $\mathcal{L}^{\mathrm{LO}}_{\mathrm{chiral}} = \pi^2_\mu/2$ in the Boltzmann weight, and interactions, which stem from the expansion

$$\begin{aligned}\mathcal{L}_{\mathrm{chiral}} &= \frac{f^2}{4} \langle \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \rangle \\ &= \frac{1}{2} \left[\pi^2_\mu + \frac{1}{3f^2} \{ (\pi \cdot \pi_\mu)^2 - \pi^2 \pi^2_\mu \} \right] + O(\pi^6),\end{aligned} \quad \text{(B5)}$$

used at the lowest order giving a nonvanishing result. In fact, to evaluate $\mathrm{rhs}_{\mathrm{LO}}$ we need no interactions, given that $\mathcal{O}_{\mathrm{out}}$ has a LO two-pion component.

For the left-hand side we need the expansion of the axial-vector current,

$$A^a_{\mu,\mathrm{LO}} = \frac{f^2}{2}\langle T^a(\Sigma^\dagger\partial_\mu\Sigma - \Sigma\partial_\mu\Sigma^\dagger)\rangle = if\left[\pi^a_\mu + \frac{2}{3f^2}\{\pi^a(\pi\cdot\pi_\mu) - \pi^a_\mu\pi^2\}\right] + O(\pi^5). \tag{B6}$$

The contribution with the leading-order term in both axial-vector currents on the left-hand side vanishes,

$$-f^2\int d\vec{x}\langle\{\pi^a_0(y_0+t,\vec{x}) - \pi^a_0(y_0-t,\vec{x})\}\pi^b_0(y)\mathcal{O}_{\mathrm{out}}\rangle_{\mathrm{LO}} = 0, \tag{B7}$$

since $\mathcal{O}_{\mathrm{out}}$ does not contain zero-momentum pion fields.

The nonvanishing contribution to the left-hand side stems from the three-pion term in one of the axial-vector currents or the four-pion term in $\mathcal{L}_{\mathrm{chiral}}$.

We obtain the first contribution using the three-pion term in the current $A^b_0(y)$, finding

$$\mathrm{lhs}_{1,\mathrm{LO}} = -\frac{2}{3}\epsilon^{abc}\int d\vec{x}\langle\{\pi^a_0(y_0+t,\vec{x}) - \pi^a_0(y_0-t,\vec{x})\} \times\{\pi^b(\pi\cdot\pi_0)(y) - \pi^b_0\pi^2(y)\}\mathcal{O}_{\mathrm{out}}\rangle_{\mathrm{LO}}. \tag{B8}$$

Performing the Wick contractions this can be written as

$$\mathrm{lhs}_{1,\mathrm{LO}} = 2\epsilon^{cab}\langle\pi^a(y)\pi^b_0(y)\mathcal{O}_{\mathrm{out}}\rangle_{\mathrm{LO}} \times\int d\vec{x}\partial_0\{G(t,\vec{x}-\vec{y}) - G(-t,\vec{x}-\vec{y})\}, \tag{B9}$$

where G denotes the massless pion propagator,

$$\langle\pi^a(x)\pi^b(y)\rangle_{\mathrm{LO}} = \delta^{ab}G(x-y) = \delta^{ab}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}e^{ip(x-y)}. \tag{B10}$$

The integral over $\vec{x}$ is easily evaluated. Using

$$\int d\vec{x}\partial_0 G(x) = \begin{cases} -\frac{1}{2} & \text{for } x_0 > 0 \\ +\frac{1}{2} & \text{for } x_0 < 0 \end{cases} \tag{B11}$$

the integral reduces to -1 and we find that

$$\mathrm{lhs}_{1,\mathrm{LO}} = \mathrm{rhs}_{\mathrm{LO}}. \tag{B12}$$

We obtain the second contribution to the left-hand side using the three-pion term for $A^a_0(y_0\pm t,\vec{x})$, yielding

$$\mathrm{lhs}_{2,\mathrm{LO}} = 2\epsilon^{abc}\int d\vec{x}\langle\{\pi^a(y_+)\pi^b_0(y_+)\partial_0G(t,\vec{x}-\vec{y}) - \pi^a(y_-)\pi^b_0(y_-)\partial_0G(-t,\vec{x}-\vec{y})\}\mathcal{O}_{\mathrm{out}}\rangle_{\mathrm{LO}}, \tag{B13}$$

where we introduced the shorthand notation $y_\pm = (y_0 \pm t,\vec{x})$.

The third contribution stems from the four-pion terms in the chiral Lagrangian (B5), and we find

$$\mathrm{lhs}_{3,\mathrm{LO}} = \frac{1}{6}\epsilon^{abc}\int d\vec{x}\left\langle\{\pi^a_0(y_+) - \pi^a_0(y_-)\} \times\int d^4z\{(\pi\cdot\pi_\mu)^2 - \pi^2\pi^2_\mu\}(z)\pi_0(y)\mathcal{O}_{\mathrm{out}}\right\rangle_{\mathrm{LO}}. \tag{B14}$$

This reduces to

$$\mathrm{lhs}_{3,\mathrm{LO}} = \epsilon^{abc}\int d^4z d\vec{x}\langle\pi^a(z)\pi^b_\mu(z)\{\partial_\mu\partial_0G(y_+-z) - \partial_\mu\partial_0G(y_--z) + \partial_0G(y_+-z)\partial_\mu - \partial_0G(y_--z)\partial_\mu\}\partial_0G(z-y)\mathcal{O}_{\mathrm{out}}\rangle_{\mathrm{LO}}. \tag{B15}$$

Since

$$\int d\vec{x}\partial_\mu\partial_0G(y_\pm - z) = -\delta_{\mu0}\delta(y_0\pm t - z_0), \tag{B16}$$

the first two terms in (B15) become

$$-\epsilon^{abc}\int d\vec{x}\langle\{\pi^a(y_+)\pi^b_0(y_+)\partial_0G(t,\vec{x}-\vec{y}) - \pi^a(y_-)\pi^b_0(y_-)\partial_0G(-t,\vec{x}-\vec{y})\}\mathcal{O}_{\mathrm{out}}\rangle_{\mathrm{LO}} \tag{B17}$$

where we renamed $\vec{z}$ as $\vec{x}$. Using (B11) for the integral over $\partial_0G(y_\pm - z)$ the remaining two terms become

$$-\epsilon^{abc}\int_{y_0-t}^{y_0+t}dz_0\int d\vec{z}\langle\pi^a(z)\pi^b_\mu(z)\partial_\mu\partial_0G(z-y)\mathcal{O}_{\mathrm{out}}\rangle_{\mathrm{LO}},$$

and after partial integration this can be written as

$$-\epsilon^{abc}\int d\vec{x}\langle\{\pi^a(y_+)\pi^b_0(y_+)\partial_0G(t,\vec{x}-\vec{y}) - \pi^a(y_-)\pi^b_0(y_-)\partial_0G(-t,\vec{x}-\vec{y})\}\mathcal{O}_{\mathrm{out}}\rangle_{\mathrm{LO}} + \epsilon^{abc}\int_{y_0-t}^{y_0+t}dz_0\int d\vec{z}\langle\pi^a(z)\partial_\mu\partial_\mu\pi^b(z)\partial_0 \times G(z-y)\mathcal{O}_{\mathrm{out}}\rangle_{\mathrm{LO}}.$$

Since $\partial^2_\mu\pi^b(z)$ is contracted with the on-shell fields in $\mathcal{O}_{\mathrm{out}}$ (recall our assumption!), the second line vanishes. The remaining first line is the same as in (B17). Hence, in total the third contribution reduces to

$$\mathrm{lhs}_{3,\mathrm{LO}} = -\mathrm{lhs}_{2,\mathrm{LO}}, \tag{B18}$$

so the second and third contributions cancel. This, together with Eq. (B12), proves the Ward-Takahashi identity (B1) to first nontrivial order in the chiral expansion.

Repeating this calculation with the lattice spacing corrections included is now straightforward. What changes are the expansions of the currents and the effective Lagrangian in terms of the pion fields:

$$\mathcal{L}_{\rm chiral} = \frac{1}{2}(1+X_1)\pi_\mu^2 + \frac{1}{6f^2}[(1+X_1)(\pi\cdot\pi_\mu)^2 - (1+X_2)\pi^2\pi_\mu^2] + O(\pi^6), \tag{B19}$$

$$V^a_{\mu,\rm eff} = i\epsilon^{abc}\pi^b\pi^c_\mu(1+X_1) + O(\pi^4), \tag{B20}$$

$$A^a_{\mu,\rm eff} = if(1+Y_1)\pi^a_\mu + i\frac{2}{3f}[(1+Y_2)\pi^a(\pi\cdot\pi_\mu) - (1+Y_3)\pi^a_\mu\pi^2], \tag{B21}$$

where we introduced the coefficients

$$X_1 = \frac{16}{f^2}W_{45}\hat{a}\bar{c}_{\rm SW}, \qquad X_2 = \frac{40}{f^2}W_{45}\hat{a}\bar{c}_{\rm SW},$$
$$Y_1 = X_1 + \frac{8}{f^2}W_A\hat{a}\bar{c}_A, \qquad Y_2 = X_1 - \frac{4}{f^2}W_A\hat{a}\bar{c}_A, \tag{B22}$$
$$Y_3 = \frac{28}{f^2}W_{45}\hat{a}\bar{c}_{\rm SW} + \frac{2}{f^2}W_A\hat{a}\bar{c}_A.$$

The modification of the kinetic term by the factor $1+X_1$ yields a slightly different pion propagator,

$$\langle\pi^a(x)\pi^b(y)\rangle_{{\rm LO}(a)} = \frac{\delta^{ab}}{1+X_1}\langle\pi^a(x)\pi^b(y)\rangle_{\rm LO} = \frac{\delta^{ab}}{1+X_1}G(x-y), \tag{B23}$$

with G defined in (B10) and $\langle\mathcal{O}\rangle_{{\rm LO}(a)}$ denotes functional integrals with $\mathcal{L}^{\rm LO}_{\rm chiral} = (1+X_1)\pi_\mu^2/2$ in the Boltzmann weight. Notice that $1+X_1 = Z_\pi^{-1}$ [cf. (58)]. Hence, in terms of the renormalized pion fields

$$\tilde{\pi}^a(x) = Z_\pi^{-1/2}\pi^a(x) \tag{B24}$$

Eq. (B23) assumes its standard form

$$\langle\tilde{\pi}^a(x)\tilde{\pi}^b(y)\rangle_{{\rm LO}(a)} = \delta^{ab}G(x-y). \tag{B25}$$

The right-hand side of the WTI now reads

$$\text{rhs}_{{\rm LO}(a)} = -2\epsilon^{cab}(1+X_1)\langle\pi^a(y)\pi^b_\mu(y)\mathcal{O}_{\rm out}\rangle_{{\rm LO}(a)}, \tag{B26}$$

while the three contributions on the left-hand side are modified as follows:

$$\text{lhs}_{1,{\rm LO}(a)} = \frac{[1+(3Y_1+Y_2+2Y_3)/3]}{(1+X_1)}(-2\epsilon^{cab}) \times\langle\pi^a(y)\pi^b_\mu(y)\mathcal{O}_{\rm out}\rangle_{{\rm LO}(a)}, \tag{B27}$$

$$\text{lhs}_{2,{\rm LO}(a)} = \frac{[1+(3Y_1+Y_2+2Y_3)/3]}{(1+X_1)}(\text{lhs}_{2,+} - \text{lhs}_{2,-}), \tag{B28}$$

$$\text{lhs}_{2,\pm} = 2\epsilon^{abc}\int d\vec{x}\,\partial_0 G(\pm t, \vec{x}-\vec{y}) \times\langle\pi^a(y_\pm)\pi^b_0(y_\pm)\mathcal{O}_{\rm out}\rangle_{{\rm LO}(a)}, \tag{B29}$$

$$\text{lhs}_{3,{\rm LO}(a)} = -\frac{[1+2Y_1+(X_1+2X_2)/3]}{(1+X_1)^2} \times(\text{lhs}_{2,+} - \text{lhs}_{2,-}). \tag{B30}$$

Here the factors in the numerator come from (B22) while those in the denominator arise from the presence of (one or two) pion propagators in the manipulations given earlier in the appendix.

To determine the renormalization constant we need the ratio of the two sides of the original WTI. It follows from (B26) and (B27) that

$$\frac{\text{lhs}_{1,{\rm LO}(a)}}{\text{rhs}_{{\rm LO}(a)}} = \frac{[1+(3Y_1+Y_2+2Y_3)/3]}{(1+X_1)^2}, \tag{B31}$$

independent of the detailed form of $\mathcal{O}_{\rm out}$.

For the remaining ratios we need to specify the external fields. We take

$$\mathcal{O}^c_{\rm out} = \epsilon^{cde}\tilde{\pi}^d(T,\vec{p})\tilde{\pi}^e(-T,-\vec{q}) \tag{B32}$$

with $\vec{p}$, $\vec{q} \neq \vec{0}$. Here $\tilde{\pi}^d(T,\vec{p})$ is the Fourier transform of $\tilde{\pi}^d(T,\vec{x})$ with respect to the three spatial coordinates. For this choice, the correlators we need are

$$\text{rhs}_{{\rm LO}(a)} = -12\partial_0^T\{G(T-y_0,\vec{p})G(T+y_0,\vec{q})\}, \tag{B33}$$

$$\text{lhs}_{2,\pm} = 12\frac{\partial_0 G(\pm t,\vec{p}-\vec{q})}{(1+X_1)} \times\partial_0^T\{G(T-y_0\mp t,\vec{p})G(T+y_0\pm t,\vec{q})\}, \tag{B34}$$

where $\partial_0^T = \partial/\partial T$. For simplicity, and without loss of generality, we have set $\vec{y}=0$, which avoids an overall phase.

To evaluate these expressions we need the hybrid position-momentum pion propagator,

$$G(t,\vec{p}) = \int\frac{dp_0}{2\pi}\frac{e^{ip_0t}}{p_0^2+\vec{p}^2} = \frac{e^{-E_pt}}{2E_p}, \qquad t\geq 0,\ E_p = |\vec{p}|, \tag{B35}$$

using which we find

$$\text{rhs}_{{\rm LO}(a)} = \frac{3(E_p+E_q)}{E_pE_q}e^{[-(E_p+E_q)T+y_0(E_p-E_q)]}, \tag{B36}$$

$$\text{lhs}_{2,\pm} = \frac{\pm 1}{(1+X_1)} \times\frac{3(E_p+E_q)}{2E_pE_q}e^{[-|t|E_{p-q}-(E_p+E_q)T+(y_0\pm t)(E_p-E_q)]}, \tag{B37}$$

where $E_{p-q} = |\vec{p} - \vec{q}|$. Inserting these results into Eqs. (B26), (B28), and (B30) we obtain

$$\frac{\text{lhs}_{2,\text{LO}(a)}}{\text{rhs}_{\text{LO}(a)}} = \frac{[1 + (3Y_1 + Y_2 + 2Y_3)/3]}{(1 + X_1)^2} \times \cosh[t(|\vec{p}| - |\vec{q}|)] \exp[-|t||\vec{p} - \vec{q}|], \quad \text{(B38)}$$

$$\frac{\text{lhs}_{3,\text{LO}(a)}}{\text{rhs}_{\text{LO}(a)}} = -\frac{[1 + 2Y_1 + (X_1 + 2X_2)/3]}{(1 + X_1)^3} \times \cosh[t(|\vec{p}| - |\vec{q}|)] \exp[-|t||\vec{p} - \vec{q}|], \quad \text{(B39)}$$

and thus, finally,

$$\frac{\text{lhs}_{\text{LO}(a)}}{\text{rhs}_{\text{LO}(a)}} = \frac{\text{lhs}_{1,\text{LO}(a)} + \text{lhs}_{2,\text{LO}(a)} + \text{lhs}_{3,\text{LO}(a)}}{\text{rhs}_{\text{LO}(a)}} \quad \text{(B40)}$$

$$= 1 + \frac{8\hat{a}}{f^2}(W_{45}\bar{c}_{\text{SW}} + W_A\bar{c}_A)\{1 - \cosh[t(|\vec{p}| - |\vec{q}|)] \times \exp[-|t||\vec{p} - \vec{q}|]\}. \quad \text{(B41)}$$

This ratio should be unity when multiplied by the renormalization factor $(Z_{A,\text{Loc}}/Z_A^0)^2$, leading to the result (60) in the main text.

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