Technical Notes and Correspondence

Exact Discretization of a Matrix Differential Riccati Equation With Constant Coefficients

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Abstract—**An exact method is presented for discretizing a constant-coefficient, non-square, matrix differential Riccati equation, whose solution is assumed to exist. The resulting discrete-time equation gives the values that have no error at discrete-time instants for any discrete-time interval. The method is based on a matrix fractional transformation, which is more general than existing ones, for linearizing the differential Riccati equation. A numerical example is presented to compare the proposed method with that based on gage invariance and bilinearization, which has better performances than the conventional forward-difference method.**

Index Terms—**Differential Riccati equations, discrete time Riccati equations, exact discretization, exact linearization, nonlinear systems.**

I. Introduction

Finding solutions to differential Riccati equations has been a subject of extensive research in various areas of science and engineering [1], [2]. In view of continuing advances in digital devices, discretization of Riccati equations has also been an important subject. Bilinear transformation is used in [3] to solve differential Riccati equations and to derive their discrete-time versions such that gauge invariance is preserved. Unfortunately, this invariance does not guarantee the exactness of the discrete-time equations for finite discrete-time intervals. For linear systems, a large number of discretization methods have been proposed, including an exact discretization [4], [5] and an approximate, but always stable, discretization of closed-loop systems [6]. In contrast, a relatively small number of exact discretization methods have been presented for nonlinear systems and a simple but very approximate model is usually used [7]. Recently, a method was presented in [8] for discretizing exactly a scalar differential Riccati equation that has constant coefficients. Unlike the bilinear method, which yields an unstable higher-order linear system, this method transforms the original nonlinear equation into a stable linear system with no increase in order. However, to reduce the differential Riccati equation to a linear one, one must solve a related algebraic Riccati equation, which is not easy for matrix cases. In the present study, the matrix differential Riccati equation is linearized using a fractional variable transformation, which increases the order, but does not require such solutions, and is more general than that used in [1]–[3] in the sense that a free parameter can be incorporated without loosing the exactness of solutions. An exact discrete-time equation with the same order as the original equation is then obtained based on this linear equation. Although such exact discretization may not exist for all types nonlinear systems, trying to find such models and expand their classes are nevertheless important tasks.

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II. Exact Discretization of a Matrix Differential Riccati Equation

The main theorem is presented first, followed by two lemmas that will be used to prove this theorem.

Theorem: Consider a system given by the following non-square, matrix, differential Riccati equation with constant coefficients:

$$\frac{d\mathbf{X}}{dt} = \mathbf{XAX} + \mathbf{B}_1\mathbf{X} + \mathbf{XB}_2 + \mathbf{C} \tag{1}$$

where $\mathbf{X}(t) \in \mathbb{R}^{p\times q}, t \in \mathbb{R}, \mathbf{A} \in \mathbb{R}^{q\times p}, \mathbf{B}_1 \in \mathbb{R}^{p\times p}, \mathbf{B}_2 \in \mathbb{R}^{q\times q}$, and $\mathbf{C} \in \mathbb{R}^{p\times q}$. It is assumed that the continuous-time solution $\mathbf{X}(t)$ exists within an appropriate time interval. An exact discretization, whose state values match exactly those of the above at all discrete-time instants for any discretization interval, can be expressed in delta form as

$$\begin{aligned}\delta\mathbf{X}_k &= \{\mathbf{X}_k\mathbf{M}_{12}\mathbf{X}_k + \mathbf{M}_{22}\mathbf{X}_k + \mathbf{X}_k\mathbf{M}_{11} + \mathbf{M}_{21}\}\\ &\quad\cdot\{\mathbf{I} - T(\mathbf{M}_{11} + \mathbf{M}_{12}\mathbf{X}_k)\}^{-1},\\ \mathbf{X}_0 &= \mathbf{X}(0)\end{aligned} \tag{2}$$

where $\mathbf{X}_k$ is a discrete-time signal with $t = kT$, $k \in \mathbb{Z}$, and T being a uniform discretization interval. Moreover, δ is the delta operator defined [4] as

$$\delta\mathbf{X}_k \triangleq \frac{\mathbf{X}_{k+1} - \mathbf{X}_k}{T} \tag{3}$$

and matrices $\mathbf{M}_{ij}$ with appropriate dimensions satisfy

$$\begin{aligned}\mathbf{M} &\triangleq \begin{bmatrix} -\mathbf{M}_{11} & -\mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix}\\ &\triangleq \frac{e^{\mathbf{H}T} - \mathbf{I}}{T} \quad (\rightarrow \mathbf{H} \text{ as } T \rightarrow 0)\end{aligned} \tag{4}$$

$$\mathbf{H} \triangleq \begin{bmatrix} -\mathbf{B}_2 & -\mathbf{A} \\ \mathbf{C} & \mathbf{B}_1\end{bmatrix}. \tag{5}$$

In order for this discrete-time model to exist, the inverse matrix in the right-hand side of (2) must exist, which holds true at least as $T \rightarrow 0$.□

The following lemma is a generalization of that presented in [1]–[3] regarding a higher-order linear expression of the Riccati equation.

Lemma 1: Let $\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2]$ with $\mathbf{U}_1 \in \mathbb{R}^{q\times p}$, $\mathbf{U}_2 \in \mathbb{R}^{q\times q}$ and $\mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2]$ with $\mathbf{V}_1 \in \mathbb{R}^{p\times p}$, $\mathbf{V}_2 \in \mathbb{R}^{p\times q}$ be solutions on the interval $[0, t_1]$ of

$$\begin{bmatrix} \frac{d\mathbf{U}}{dt} \\ \frac{d\mathbf{V}}{dt}\end{bmatrix} = (\mathbf{H} + \mu\mathbf{I})\begin{bmatrix}\mathbf{U}\\ \mathbf{V}\end{bmatrix} \tag{6}$$

where μ is an arbitrary scalar. Assume that the variable $\mathbf{X}$ can be defined by the following matrix fractional transformation:

$$\begin{aligned}\mathbf{X} &= (\mathbf{V\Xi})(\mathbf{U\Xi})^{-1}\\ &= (\mathbf{V}_1\Xi_1 + \mathbf{V}_2\Xi_2)(\mathbf{U}_1\Xi_1 + \mathbf{U}_2\Xi_2)^{-1}\end{aligned} \tag{7}$$

where $\Xi = [\Xi_1^T, \Xi_2^T]^T$ with $\Xi_1 \in \mathbb{R}^{p\times q}, \Xi_2 \in \mathbb{R}^{q\times q}$ is a constant matrix such that $\mathbf{U\Xi}$ is nonsingular. Then, $\mathbf{X}$ satisfies the differential Riccati equation (1) on $[0, t_1]$. Conversely, if $\mathbf{X}$ is a solution of (1) on $[0, t_1]$ and $\mathbf{U}$ is a fundamental solution of

$$\frac{d\mathbf{U}}{dt} = (-\mathbf{AX} - \mathbf{B}_2 + \mu\mathbf{I})\mathbf{U} \tag{8}$$

then the pair $\mathbf{U}$ and $\mathbf{V} = \mathbf{XU}$ is a solution of (6) on $[0, t_1]$. □

Proof of lemma 1: Since $(d\mathbf{Y}^{-1}/dt) = -\mathbf{Y}^{-1}(d\mathbf{Y}/dt)\mathbf{Y}^{-1}$, the differentiation of (7), using (6), yields

$$\begin{aligned}\frac{d\mathbf{X}}{dt} &= \left(\frac{d\mathbf{V}}{dt}\Xi\right)(\mathbf{U}\Xi)^{-1} - (\mathbf{V}\Xi)(\mathbf{U}\Xi)^{-1}\left(\frac{d\mathbf{U}}{dt}\Xi\right)(\mathbf{U}\Xi)^{-1} \\ &= \{\mathbf{C}(\mathbf{U}\Xi) + (\mathbf{B}_1 + \mu\mathbf{I})(\mathbf{V}\Xi)\}(\mathbf{U}\Xi)^{-1} \\ &\quad - \mathbf{X}\{-(\mathbf{B}_2 - \mu\mathbf{I})(\mathbf{U}\Xi) - \mathbf{A}(\mathbf{V}\Xi)\}(\mathbf{U}\Xi)^{-1} \\ &= \mathbf{C} + \mathbf{B}_1\mathbf{X} + \mathbf{X}\mathbf{B}_2 + \mathbf{X}\mathbf{A}\mathbf{X} + \mu\mathbf{X} - \mathbf{X}\mu \end{aligned} \tag{9}$$

which is (1). As long as a solution of the differential Riccati equation exists, matrix Ξ can be chosen such that $(\mathbf{U}\Xi)^{-1}$ exists (Theorem 6.4, [2]). Conversely, if $\mathbf{X}$ is a solution of (1) on $[0, t_1]$ and $\mathbf{U}$ is a fundamental solution of (8), then the pair $\mathbf{U}$ and $\mathbf{V} = \mathbf{X}\mathbf{U}$ satisfies

$$\begin{aligned}\begin{bmatrix}\frac{d\mathbf{U}}{dt}\\ \frac{d\mathbf{V}}{dt}\end{bmatrix} &= \begin{bmatrix}(-\mathbf{A}\mathbf{X} - \mathbf{B}_2 + \mu\mathbf{I})\mathbf{U}\\ \frac{d\mathbf{X}}{dt}\mathbf{U} + \mathbf{X}\frac{d\mathbf{U}}{dt}\end{bmatrix} \\ &= \begin{bmatrix}-\mathbf{B}_2 + \mu\mathbf{I} & -\mathbf{A}\\ \mathbf{C} & \mathbf{B}_1 + \mu\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{U}\\ \mathbf{V}\end{bmatrix}\end{aligned} \tag{10}$$

which is (6). [End of proof]

In the digital control community, the following discrete-time system is often called the step-invariant model [4] of (6).

Lemma 2: The exact discretization of (6) is given by the following:

$$\begin{bmatrix}\delta\mathbf{U}_k\\ \delta\mathbf{V}_k\end{bmatrix} = \{(1 + \mu' T)\mathbf{M} + \mu'\mathbf{I}\}\begin{bmatrix}\mathbf{U}_k\\ \mathbf{V}_k\end{bmatrix} \tag{11}$$

where

$$\mu' = \frac{e^{\mu T} - 1}{T} \tag{12}$$

which approaches μ as $T \to 0$. □

Proof of lemma 2: It is known [4] that the discrete-time model whose state matches exactly that of a continuous-time original is possible for piece-wise constant inputs. In the present case, the linear system (6) has no input and this scheme is applicable directly, so that

$$\begin{bmatrix}\delta\mathbf{U}_k\\ \delta\mathbf{V}_k\end{bmatrix} = \frac{1}{T}\left\{e^{(\mathbf{H}+\mu\mathbf{I})T} - \mathbf{I}\right\}\begin{bmatrix}\mathbf{U}_k\\ \mathbf{V}_k\end{bmatrix}. \tag{13}$$

This can be rearranged as

$$\begin{aligned}\begin{bmatrix}\delta\mathbf{U}_k\\ \delta\mathbf{V}_k\end{bmatrix} &= \frac{1}{T}\left\{(e^{\mathbf{H}T} - \mathbf{I}) + e^{\mathbf{H}T}(e^{\mu T} - 1)\right\}\begin{bmatrix}\mathbf{U}_k\\ \mathbf{V}_k\end{bmatrix} \\ &= \{\mathbf{M} + (T\mathbf{M} + \mathbf{I})\mu'\}\begin{bmatrix}\mathbf{U}_k\\ \mathbf{V}_k\end{bmatrix} \\ &= \{(1 + \mu' T)\mathbf{M} + \mu'\mathbf{I}\}\begin{bmatrix}\mathbf{U}_k\\ \mathbf{V}_k\end{bmatrix}\end{aligned} \tag{14}$$

which is (11). [End of Proof]

The theorem is now proven below using Lemmas 1 and 2.

Proof of theorem: From the definition of delta operator (3), it can be shown that for the product of two nonsingular matrices $\mathbf{Y}_k$ and $\mathbf{Z}_k$, its difference quotient can be written as

$$\delta(\mathbf{Y}_k\mathbf{Z}_k) = (\delta\mathbf{Y}_k)\mathbf{Z}_k + \mathbf{Y}_k(\delta\mathbf{Z}_k) + T(\delta\mathbf{Y}_k)(\delta\mathbf{Z}_k). \tag{15}$$

Letting $\mathbf{Z}_k = \mathbf{Y}_k^{-1}$ in the above and using $\delta\mathbf{I} = \mathbf{0}$, one obtains

$$\delta\left(\mathbf{Z}_k^{-1}\right) = -(\mathbf{Z}_k + T\delta\mathbf{Z}_k)^{-1}\cdot\delta\mathbf{Z}_k\cdot\mathbf{Z}_k^{-1} \tag{16}$$

where the non-singularity of $\mathbf{Z}_k + T\delta\mathbf{Z}_k (= \mathbf{Z}_{k+1})$ trickles down to the existence of the discrete-time model later in (21). Equations (15) and (16) yield

$$\delta\left(\mathbf{Y}_k\mathbf{Z}_k^{-1}\right) = \left[\delta\mathbf{Y}_k - (\mathbf{Y}_k + T\delta\mathbf{Y}_k)(\mathbf{Z}_k + T\delta\mathbf{Z}_k)^{-1}\delta\mathbf{Z}_k\right]\mathbf{Z}_k^{-1}. \tag{17}$$

Therefore, the difference quotient of (7) can be written as

$$\begin{aligned}\delta\mathbf{X}_k &= \delta\left[(\mathbf{V}_k\Xi)(\mathbf{U}_k\Xi)^{-1}\right] \\ &= \left[(\delta\mathbf{V}_k)\Xi - \{(\mathbf{V}_k + T\delta\mathbf{V}_k)\Xi\}\right. \\ &\quad \left.\cdot\{(\mathbf{U}_k + T\delta\mathbf{U}_k)\Xi\}^{-1}(\delta\mathbf{U}_k)\Xi\right](\mathbf{U}_k\Xi)^{-1} \\ &= (\delta\mathbf{V}_k)\Xi(\mathbf{U}_k\Xi)^{-1} - \{\mathbf{V}_k\Xi + T\delta\mathbf{V}_k\Xi\}(\mathbf{U}_k\Xi)^{-1} \\ &\quad \cdot\left\{(\mathbf{U}_k\Xi + T\delta\mathbf{U}_k\Xi)(\mathbf{U}_k\Xi)^{-1}\right\}^{-1} \\ &\quad \cdot(\delta\mathbf{U}_k)\Xi(\mathbf{U}_k\Xi)^{-1}.\end{aligned} \tag{18}$$

On the other hand, the exact discretization (11) in Lemma 2 and (6) in Lemma 1 yield

$$\begin{aligned}(\delta\mathbf{U}_k)\Xi(\mathbf{U}_k\Xi)^{-1} &= -(1 + \mu' T)\{(\mathbf{M}_{11} - \eta\mathbf{I}) + \mathbf{M}_{12}\mathbf{X}_k\} \\ (\delta\mathbf{V}_k)\Xi(\mathbf{U}_k\Xi)^{-1} &= (1 + \mu' T)\{\mathbf{M}_{21} + (\mathbf{M}_{22} + \eta\mathbf{I})\mathbf{X}_k\}\end{aligned} \tag{19}$$

where

$$\eta = \frac{\mu'}{1 + \mu' T}. \tag{20}$$

Using equations in (19) and the fact that $(\mathbf{I} + T\mathbf{Z})^{-1}\mathbf{Z} = \mathbf{Z}(\mathbf{I} + T\mathbf{Z})^{-1}$, (18) can be arranged further as

$$\begin{aligned}\delta[\mathbf{X}_k] &= (1 + \mu' T)\{\mathbf{M}_{21} + (\mathbf{M}_{22} + \eta\mathbf{I})\mathbf{X}_k\} \\ &\quad + \left[\mathbf{X}_k + T(1 + \mu' T)\{\mathbf{M}_{21} + (\mathbf{M}_{22} + \eta\mathbf{I})\mathbf{X}_k\}\right] \\ &\quad \cdot(1 + \mu' T)\{(\mathbf{M}_{11} - \eta\mathbf{I}) + \mathbf{M}_{12}\mathbf{X}_k\} \\ &\quad \cdot\left[\mathbf{I} - T(1 + \mu' T)\{(\mathbf{M}_{11} - \eta\mathbf{I}) + \mathbf{M}_{12}\mathbf{X}_k\}\right]^{-1} \\ &= (1 + \mu' T)\left[\mathbf{M}_{21} + \mathbf{M}_{22}\mathbf{X}_k + \eta\mathbf{X}_k\right. \\ &\quad \left. + \mathbf{X}_k\{(\mathbf{M}_{11} - \eta\mathbf{I}) + \mathbf{M}_{12}\mathbf{X}_k\}\right] \\ &\quad \cdot\left[\mathbf{I} - T(1 + \mu' T)\{(\mathbf{M}_{11} - \eta\mathbf{I}) + \mathbf{M}_{12}\mathbf{X}_k\}\right]^{-1} \\ &= (1 + \mu' T)[\mathbf{M}_{21} + \mathbf{M}_{22}\mathbf{X}_k + \eta\mathbf{X}_k + \mathbf{X}_k\mathbf{M}_{11} \\ &\quad - \mathbf{X}_k\eta + \mathbf{X}_k\mathbf{M}_{12}\mathbf{X}_k] \\ &\quad \cdot\left[(1 + \mu' T)\{\mathbf{I} - T(\mathbf{M}_{11} + \mathbf{M}_{12}\mathbf{X}_k)\}\right]^{-1} \\ &= [\mathbf{M}_{21} + \mathbf{M}_{22}\mathbf{X}_k + \mathbf{X}_k\mathbf{M}_{11} + \mathbf{X}_k\mathbf{M}_{12}\mathbf{X}_k] \\ &\quad \cdot\{\mathbf{I} - T(\mathbf{M}_{11} + \mathbf{M}_{12}\mathbf{X}_k)\}^{-1}\end{aligned} \tag{21}$$

which is (2). [End of Proof]

Some observations on the proposed exact discretization are listed below:

1) Parameter μ, which affects the eigenvalues of the system matrix of the linearized system (6), disappears by subtraction in the differential Riccati equation (9). Furthermore, in the exact discrete-time Riccati equation, the parameter μ' disappears by division in (21).
2) The method presented in [3] is based on the forward-difference approximation of linearized (6) with $\mathbf{X}_k$ being square and $\mu = 0$. The original variable is then calculated from

$$\mathbf{X}_k = \mathbf{V}_k\mathbf{U}_k^{-1} \tag{22}$$

which corresponds to $\Xi_1 = \mathbf{0}_{p\times q}, \Xi_2 = \mathbf{I}_{q\times q}$ in (7). However, the linearized system is unstable when $\mu = 0$ and the computation of $\mathbf{U}_k^{-1}$ causes numerical problems in simulations as time elapses,

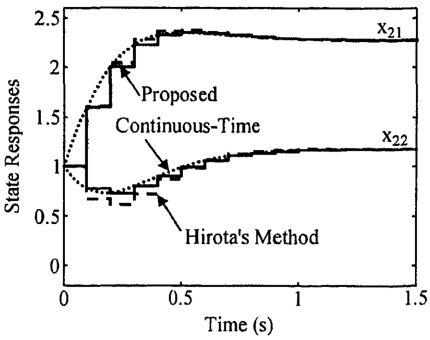

Fig. 1. Simulation results comparing the states x_{21} and x_{22} of continuous-time system and those of the proposed and Hirota's system for $T = 0.1$ s.

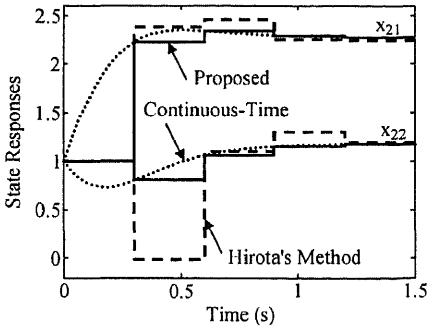

Fig. 2. Simulation results comparing the states x_{21} and x_{22} of continuous-time system and those of the proposed and Hirota's system for $T = 0.3$ s.

due to divergence of $\mathbf{U}_k$. The discrete-time model expressed directly in $\mathbf{X}_k$ can be obtained, along the same line as (18)–(21), as

$$\delta\mathbf{X}_k = \{\mathbf{X}_k A\mathbf{X}_k + B_1 X_k + \mathbf{X}_k\mathbf{B}_2 + \mathbf{C}\} \cdot \{\mathbf{I} - T(\mathbf{B}_2 + \mathbf{A}\mathbf{X}_k)\}^{-1}. \quad (23)$$

For comparison, the forward-difference discretization as applied directly to the differential Riccati equation is given by

$$\delta\mathbf{X}_k = \mathbf{X}_k\mathbf{A}\mathbf{X}_k + \mathbf{B}_1\mathbf{X}_k + \mathbf{X}_k\mathbf{B}_2 + \mathbf{C}. \quad (24)$$

The difference between the method of [3], (23), and the forward-difference discretization, (24), is only in the denominator and disappears as $T \to 0$.

3) When $\mu = 0, \mathbf{U}(0) = [\mathbf{0}_{p\times q}, \mathbf{I}_q]$, and Ξ_2 is chosen to be non-singular so that $\mathbf{V}(0)\Xi = \mathbf{X}(0)\Xi_2$, then (7) reduces to that given in [2], where the non-singularity of matrix $\mathbf{U}\Xi$ is guaranteed, as long as a continuous-time solution to the differential Riccati equation exists.
4) Since $\lim_{T\to 0}(e^{\mathbf{H}T} - \mathbf{I})/T = \mathbf{H}$, as $T \to 0$, it follows that:

$$\mathbf{M}_{11} \to \mathbf{B}_2, \mathbf{M}_{12} \to \mathbf{A}, \mathbf{M}_{21} \to \mathbf{C}, \mathbf{M}_{22} \to \mathbf{B}_1. \quad (25)$$

Thus, the form of discretization (2) approaches that of (1) as $T \to 0$.
5) When $\mathbf{A} = \mathbf{0}$, the right-hand side of the differential Riccati equation becomes a Lyapunov type equation, and its exact discretization can be written as

$$\delta\mathbf{X}_k = \{\beta_1\mathbf{X}_k + \mathbf{X}_k\beta_2 + \mathbf{C} + \mathrm{O}(T)\}\{\mathbf{I} + T\beta_2\}^{-1} \quad (26)$$

where

$$\beta_1 = \frac{e^{\mathbf{B}_1 T} - \mathbf{I}}{T} \to \mathbf{B}_1(T \to 0), \quad \beta_2 = \frac{e^{-\mathbf{B}_2 T} - \mathbf{I}}{-T} \to \mathbf{B}_2(T \to 0) \quad (27)$$

and

$$\mathrm{O}(T) = \sum_{j=2}^{\infty} \frac{T^{j-1}}{j!} \sum_{i=1}^{j} \mathbf{B}_1^{i-1}\mathbf{C}(-\mathbf{B}_2)^{j-i}. \quad (28)$$

6) When $\mathbf{A} = \mathbf{0}$ and $\mathbf{B}_2 = \mathbf{0}$, the differential Riccati equation reduces to a linear matrix form, for which the exact discretization becomes

$$\delta\mathbf{X}_k = \{\mathbf{M}_{22}\mathbf{X}_k + \mathbf{M}_{21}\}\{\mathbf{I} - T\mathbf{M}_{11}\}^{-1}. \quad (29)$$

More explicitly, since

$$\mathbf{H}^n = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_1^{n-1}\mathbf{C} & \mathbf{B}_1^n \end{bmatrix} \quad (30)$$

and, thus

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \frac{1}{T}\int_0^T e^{\mathbf{B}_1\tau}d\tau\mathbf{C} & \frac{e^{\mathbf{B}_1 T}-\mathbf{I}}{T} \end{bmatrix} \quad (31)$$

(29) yields a more familiar form as

$$\delta\mathbf{X}_k = \frac{e^{\mathbf{B}_1 T} - \mathbf{I}}{T}\mathbf{X}_k + \frac{1}{T}\int_0^T e^{\mathbf{B}_1\tau}d\tau \cdot \mathbf{C} \quad (32)$$

which is a matrix version of the well known vector case [4].

III. Simulation Example

Simulations are carried out for the following 3×2 matrix differential Riccati equation:

$$\begin{bmatrix} \dot{x}_{11} & \dot{x}_{12} \\ \dot{x}_{21} & \dot{x}_{22} \\ \dot{x}_{31} & \dot{x}_{32} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ -4 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} + \begin{bmatrix} -2 & 1 & 5 \\ 4 & -2 & 1 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} + \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 4 & -4 \\ 1 & -2 \end{bmatrix} \quad (33)$$

with the initial condition given by

$$x(0) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}. \quad (34)$$

Figs. 1 and 2 show, respectively, for $T = 0.1$ and 0.3 seconds, the initial-state responses of the continuous-time system evaluated using the Runge-Kutta method in Simulink, the proposed discrete-time model, and Hirota's model. These plots show that the proposed model gives the discrete-time sequence that matches the continuous-time solution at discrete-time instants, while Hirota's model gives sequences that are non-diverging but inexact. Although not shown here, the forward-difference model applied directly to the given Riccati equation yields errors that are comparable to Hirota's for $T = 0.1$ s but are divergent for $T = 0.3$ s.

IV. Conclusion

A method has been proposed for discretizing a non-square matrix differential Riccati equation with constant coefficients such that there is no error in its solution at discrete-time instants for any discrete-time interval, as long as a continuous-time solution of the Riccati equation exists. Since the exact discrete-time equation for linear systems have been extremely valuable in the field of digital control [4], [6], the proposed method for the differential Riccati equation is believed to be an important first step towards its extension to nonlinear cases.

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