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ON THE GENERALIZED JOSEPHUS PROBLEM

Mar chuimhne air an *t*-ollamh Rob Alasdair Mac Fhraing nach mairean

By

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1. Introduction

The legendary problem of Josephus and the forty Jews and the problem of fifteen Christians and fifteen Tarks, and also some variants thereof, are widely well known (cf. [1], [6], [10], [11], [14]) and have been discussed and generalized mathematically by several authors (cf. e.g. [2], [3], [4], [5], [8]).

These problems, in a rather general form, may well be formulated thus: Let n and m be given positive integers; we arrange n distinct points, named $1, 2, \ldots, n$, in a circle in the natural order (the points adjacent to 1 being 2 and n if $n > 2$) and delete, starting from the point 1, every mth point in turn until all the points are removed. The problem is to determine the kth point $a_m(k, n)$ (sometimes called the kth Josephus number) to be deleted when n, m and k $(1 \leqq k \leqq n)$ are assigned. It is plain that

$$1 \leqq a_m(k, n) \leqq n$$

and

$$a_m(1, n) \equiv m \pmod{n}.$$

Consequently, if the validity is assumed of the congruence

$$(1) \qquad a_m(k+1, n+1) \equiv m + a_m(k, n) \pmod{n+1} \quad (1 \leqq k \leqq n),$$

then one can recursively determine all the numbers $a_m(k, n)$.

A simple proof of the congruence relation (1) which is due substantially to P. G. Tait [13], was given by R. A. Rankin [8] (see also [4]); however, it should be noted that the basic congruence (1) was practically known to Seki Takakazu (1642?–1708) in [12] and to Leonhard Euler (1707–1783) in [3] as well. On the basis of (1) Rankin [8] has established an algorithm for determining the last

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Josephus number $d_m(n) = a_m(n, n)$ for $n > m \geqq 2$ and obtained in the special case of $m = 2$ an explicit formula for $d_m(n)$, namely

$$(2) \qquad d_2(n) = 2n + 1 - 2^{i+1} \quad \text{for } 2^i \leqq n < 2^{i+1}.$$

Rankin [8] also gives in the case of $m = 3$ a second algorithm which he called a short form is in general not quite correct, and a few of the counterexamples found are: $n = 12, 13, 18, 19, 20, 27, 28, 29, 30, 31, 32, 45, 46, 47, 48, 49,$ and 50. It should be noticed here that A. M. Odlyzko and H. S. Wilf [7] have also treated the special case of $k = n$ and given a compact formula for $m = 3$ (cf. § 5 below) as well as the simple result (2) for $m = 2$. It might be noted further that general solutions $a_m(k, n)$ $(1 \leqq k \leqq n)$ for the Josephus problem in the extended form had already been found by H. Schubert [11] and by E. Busche [2]; their solutions which coincide with each other for $k = n$, may be described in terms of certain sequences of integers that are defined recursively by a recurrence relation (see § 3 below). Another recursive solution was given by F. Jakóbczyk [5] to the generalized Josephus problem, together with a solution to the problem converse to the original, that is, the problem to decide the number k $(1 \leqq k \leqq n)$ such that $a_m(k, n) = l$ when l $(1 \leqq l \leqq n)$ is specified in advance; we note that a simpler solution to this converse problem had also been provided by Busche [2] substantially on the basis of the congruence relation (1). Some other types of (modified) converse problems are discussed in W. W. Rouse Ball and H. S. M. Coxeter [10] and in W. J. Robinson [9]. Furthermore, still another solution based again upon the relation (1) to the extended Josephus problem has been furnished by L. Halbeisen and N. Hungerbühler [4] and, according to their claim, the result obtained by them is not a recursive one; however, in their solution, which depends partly on an unproved hypothesis, the Josephus numbers $a_m(k, n)$ involve a crucial constant α that depends on m, n and k and is defined, when $m > 2$, by an infinite series the coefficients of whose terms are to be determined obviously recursively.

Our principal aim in this article is twofold. Firstly, we shall provide an alternative proof for the classical results due to Schubert and to Busche mentioned above (§ 3); our proof is, we believe, simpler and more transparent than the original (cf. [2]). Secondly, without any unproven hypothesis, we describe in a somewhat modified form a new algorithm of Halbeisen and Hungerbühler's [4] to decide the Josephus numbers $a_m(k, n)$ $(1 \leqq k \leqq n)$, by reducing it to its apparently primitive order that has been found independently by the present writer (§ 4); we note that the underlying ideas in here go back to the idea due in substance to L. Euler [3] who had a clear conception of its importance.

In an appendix (§7, the final section) will be briefly reviewed the concern of Seki Takakazu's with the generalized Josephus problem.

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The writer expresses his thanks also to the referee for calling his attention to the paper [7] which otherwise would have remained unknown to him.

Meanwhile the writer learned with deep regret that Professor Rankin passed away on January 27, 2001, after a brave battle with cancer. And, to the memory of the late Professor Robert Alexander Rankin the writer would like to dedicate the present work of his on the Josephus problem which as a problem of Gaelic origin Professor Rankin loved with a profound knowledge.

2. Preliminaries

For the sake of completeness we here reproduce a proof of Tait's congruence relation (1), as given by Rankin [8]. Let m be a fixed positive integer and n an arbitrary positive integer. Let there be given at first n points, $1, 2, \dots, n$, in a circle, so that the neighbor of the points 2 and n is the point 1, if $n > 2$. We now place another point $n+1$ between n and 1, and we begin anew numbering the points with the point $b+1$, where b is determined so as to satisfy

$$b \equiv -m \pmod{n+1}, \quad 0 \leqq b < n+1.$$

Let $b(k, n+1)$ be the kth point to be removed in this novel situation. We find

$$b(1, n+1) = n+1, \quad b(k+1, n+1) = a_m(k, n) \quad (1 \leqq k \leqq n),$$

and it is easy to see that

$$b(k+1, n+1) - b \equiv a_m(k+1, n+1) \pmod{n+1},$$

and the relation (1) follows at once.

It will sometimes be convenient to define $a_m(0, n) = 0$. Thus the numbers $a_m(k, n)$ $(1 \leqq k \leqq n)$ are completely determined in principle by the congruence relation (1).

Now the simple solution by Busche to the converse Josephus problem can be described in the following manner. Let n and l $(1 \leqq l \leqq n)$ be given. We put

$A(0,n) = l$. Suppose $A(i, n-i)$ is defined for an i $(0 \leqq i < n)$. If $A(i, n-i) > 0$, then we determine $A(i+1, n-i-1)$ by the conditions

$$A(i+1, n-i-1) \equiv A(i, n-i) - m \pmod{n-i}$$

and

$$0 \leqq A(i+1, n-i-1) < n-i.$$

If $A(i+1, n-i-1) = 0$ then $k = i+1$ and $a_m(k,n) = l$, as a consequence of (1). By repeating this procedure if $A(i+1, n-i-1) > 0$, we can eventually find a unique value of k $(1 \leqq k \leqq n)$ for which $a_m(k,n) = l$.

EXAMPLE. For $m = 10$, $n = 30$, $l = 14$ we find $k = 15$:

i:	0	1	2	3	4	5	6	7	8	9
$n-i$:	30	29	28	27	26	25	24	23	22	21
$A(i,n-i)$:	14	4	23	13	3	19	9	23	13	3

i:	10	11	12	13	14	15
$n-i$:	20	19	18	17	16	15
$A(i,n-i)$:	14	4	13	3	10	0

Now, in order to describe classical solutions by Schubert and by Busche of the generalized Josephus problem, we have to introduce certain infinite sequences N_i, called modulation sequences, of integers. Let $m \geqq 2$ be again a fixed integer. A modulation sequence is the sequence of positive integers $N_i = N_i(m,t)$ $(i = 1, 2, \ldots)$ defined recursively by

$$N_1 > 0, \quad N_{i+1} = \left\lceil \frac{m(N_i + t)}{m-1} \right\rceil \quad (i \geqq 1), \tag{3}$$

where t is a fixed nonnegative integer. Here, and in what follows, we denote by $\lceil x \rceil$ (resp. by $\lfloor x \rfloor$) the least (resp. the greatest) integer not less than (resp. not greater than) the real number x. It is easily seen that $N_{i+1} > N_i$ for $i \geqq 1$. We have $N_i \not\equiv 1 \pmod m$ for all $i > 1$, since the inequality

$$\frac{m(N+t)}{m-1} \leqq mK + 1 < \frac{m(N+t)}{m-1} + 1$$

is equivalent to $0 < m(N+t) - m(m-1)K \leqq m-1$, but this is impossible, if N and K are integral.

The modulation sequence of Schubert's is the sequence $N_i = N_i(m,0)$ $(i =$

$1, 2, \ldots)$ with $N_1 = m(n-k)+1$, and the modulation sequence of Busche's is $N_i' = N_i(m, n-k)$ $(i = 1, 2, \ldots)$ with $N_1 = 1$. We shall show that

$$(4) \qquad N_i - N_i' = m(n-k) \quad \text{for all } i \geqq 1.$$

In fact, the assertion (4) is obvious for $i = 1$. Suppose now that (4) holds true for an $i \geqq 1$. We have then

$$\begin{aligned} N_{i+1}' + m(n-k) &= \left\lceil \frac{m(N_i' + n - k)}{m-1} \right\rceil + m(n-k) \\ &= \left\lceil \frac{m(N_i' + m(n-k))}{m-1} \right\rceil = N_{i+1}, \end{aligned}$$

which proves the relation (4) by induction.

The modulation sequences of Schubert's and of Busche's coincide with each other, if $k = n$.

Now, explicit upper and lower bounds for the terms N_i of Schubert's modulation sequence (so that for the terms N_i' of Busche's sequence also) can be found without difficulty. In fact, if we write actually, with $N_1 = m(n-k)+1$,

$$N_{i+1} = \frac{mN_i + \sigma_i}{m-1} \quad (i \geqq 1),$$

then $0 \leqq \sigma_i \leqq m-2$ and $\sigma_i = 0$ if and only if $m-1 \mid N_i$. Just as in [4] we define an analytic function of z

$$f(z) := N_1 z + \sum_{i=1}^{\infty} \frac{\sigma_i}{m-1} z^{i+1},$$

the convergence radius of the power series on the right being at least 1. We may apply *mutatis mutandis* the argument of [4; pp. 310–311], even in a manner much simpler than in there, to show that the limit

$$(5) \qquad \theta := \lim_{i \to \infty} N_i \cdot \left(1 - \frac{1}{m}\right)^i = f\left(1 - \frac{1}{m}\right)$$

exists, and further that

$$0 \leqq \theta \left(\frac{m}{m-1}\right)^i - N_i \leqq m-2 \quad (i \geqq 1),$$

where both of the inequality signs are strict if $m > 2$. Note that θ is a positive constant depending on m, n and k. Thus, if in particular $m = 2$, then $\theta = n - k + (1/2)$ and we have

$$N_i = (n-k)2^i + 2^{i-1} \quad (i \geqq 1),$$

which can also be shown easily by induction on i, and if $m = 3$ then we have

$$N_i = \left\lfloor \theta\left(\frac{3}{2}\right)^i \right\rfloor \quad (i \geqq 1).$$

We note that, if $m > 2$ then the sequence

$$\sigma_i = (m-1)N_{i+1} - mN_i \quad (i = 1, 2, \ldots)$$

is not (ultimately) periodic. Suppose the contrary; then there must exist positive integers i_0 and p such that $\sigma_{i+p} = \sigma_i$ for all $i \geqq i_0$. We have for $i \geqq i_0$

$$0 = \sigma_{i+p} - \sigma_i = (m-1)(N_{i+p+1} - N_{i+1}) - m(N_{i+p} - N_i)$$

and, therefore, for all $j \geqq 1$

$$N_{i+p+j} - N_{i+j} = \left(\frac{m}{m-1}\right)^j (N_{i+p} - N_i),$$

which is clearly impossible, since $N_{i+p} - N_i > 0$.

The constant θ is effectively computable.

Remark. It will be of some interest to observe that the sequence

$$w_i := \frac{\theta}{3}\left(\frac{3}{2}\right)^i \quad (i = 1, 2, \ldots),$$

where θ is defined by (5) with $m = 3$, $N_1 > 0$ being chosen arbitrarily, is *not* uniformly distributed modulo one, because otherwise the sequence of integers $N_i = \lfloor 3w_i \rfloor$ $(i = 1, 2, \ldots)$ would be uniformly distributed modulo 3 in the sense of I. Niven, but this is not the case, since we know that $N_i \not\equiv 1 \pmod 3$ for all $i > 1$, as has been noticed above. Compare: I. Niven, Uniform distribution of sequences of integers, Trans. Amer. Math. Soc., **98** (1961) 52–61.

3. Classical Solutions

Here we survey the leading traits of solutions by H. Schubert and by E. Busche of the generalized Josephus problem (cf. [2]). Let n and m be again two positive integers, $m \geqq 2$. Solutions will be furnished when formulas for the Josephus numbers $a_m(k, n)$ $(1 \leqq k \leqq n)$ are explicitly given.

(i) *H. Schubert's formula*. Let N_i $(i = 1, 2, \ldots)$ be the modulation sequence of Schubert's. Then, if $N_i < mn + 1 < N_{i+1}$, or equivalently if $N_{i-1} \leqq (m-1)n < N_i$ (with $N_0 = 0$), we have

$$a_m(k, n) = mn + 1 - N_i.$$

(ii) *E. Busche's formula*. Let N_i' $(i = 1, 2, \ldots)$ be the modulation sequence of Busche's. Then, if $N_i' < mk + 1 < N_{i+1}'$, we have

$$a_m(k, n) = mk + 1 - N_i'.$$

Computations needed for the determination of $a_m(k, n)$ are practically similar in these formulas, but the numerals that appear will be in genaral slightly shorter in Busche's method than in Schubert's. The formulas (i) and (ii) coincide for $k = n$, and Rankin's formula (2) for $d_2(n)$ is their special case of $m = 2$, $k = n$.

Since the formulas (i) and (ii) are mutually equivalent in view of (4), we shall give a proof only for (i).

In order to establish the validity of Schubert's solution (i) it will suffice to show that, if we define

$$f(k, n) := mn + 1 - N_i \quad (1 \leqq k \leqq n),$$

observing $0 < f(k, n) \leqq n$, where $N_i < mn + 1 < N_{i+1}$, then $f(k, n)$ satisfies the conditions

$$f(1, n) = a_m(1, n) \tag{6}$$

and

$$f(k+1, n+1) \equiv m + f(k, n) \pmod{n+1} \tag{7}$$

for all n and k $(1 \leqq k \leqq n)$.

Note that the modulation sequence N_i $(i = 1, 2, \ldots)$ depends only on the difference $n - k$ and not on the values of n and k separately, if m is once fixed; thus, the modulation sequence for $n+1$, $k+1$ is the same as the one for n, k. The largest integer n for which $mn + 1 - N_i \leqq n$ is given by $n = n_1 = \lfloor (N_i - 1)/(m-1) \rfloor$, whereas the largest integer n for which $mn + 1 < N_{i+1}$ is found to be $n = n_2 = \lfloor (N_{i+1} - 1)/m \rfloor$; here, we have $n_1 = n_2$, since the both sides are equal to

$$\begin{cases} \dfrac{N_i}{m-1} - 1 & \text{if } m-1 \mid N_i, \text{ and} \\ \left\lfloor \dfrac{N_i}{m-1} \right\rfloor & \text{if } m-1 \nmid N_i. \end{cases}$$

We first prove (7). Now, if

$$N_i < mn + 1 < m(n+1) + 1 < N_{i+1},$$

then

$$f(k+1, n+1) = m(n+1) + 1 - N_i = m + f(k, n)$$

and (7) is obvious. If

$$N_i < mn + 1 < N_{i+1} \leqq N_j < m(n+1) + 1 < N_{j+1} \quad (j \geqq i+1),$$

then we have $n = \lfloor (N_\nu - 1)/m \rfloor$ for each ν with $i + 1 \leqq \nu \leqq j$ and, therefore,

$$m - 1 \mid N_{\nu-1} \text{ implies } n = \left\lfloor \frac{mN_{\nu-1} - (m-1)}{m(m-1)} \right\rfloor = \frac{N_{\nu-1}}{m-1} - 1,$$

and

$$m - 1 \nmid N_{\nu-1} \text{ implies } n = \left\lfloor \frac{1}{m} \left\lfloor \frac{mN_{\nu-1}}{m-1} \right\rfloor \right\rfloor = \left\lfloor \frac{N_{\nu-1}}{m-1} \right\rfloor,$$

where use is made of the relation

$$N_\nu = \begin{cases} \dfrac{mN_{\nu-1}}{m-1} & \text{if } m-1 \mid N_{\nu-1}, \\ \left\lfloor \dfrac{mN_{\nu-1}}{m-1} \right\rfloor + 1 & \text{if } m-1 \nmid N_{\nu-1}. \end{cases}$$

It follows that we have

$$N_\nu - N_{\nu-1} = n + 1 \quad (i + 1 \leqq \nu \leqq j),$$

and hence

$$\begin{aligned} f(k+1, n+1) - f(k, n) &= m - (N_j - N_i) \\ &= m - \sum_{\nu=i+1}^{j} (N_\nu - N_{\nu-1}) \\ &= m - (j-i)(n+1), \end{aligned}$$

which proves (7).

We now proceed to prove (6) by distinguishing the cases according as $n \geqq m$ or $n < m$.

Case 1: $n \geqq m$. It is plain that $a_m(1, n) = m$. We have to show that for some (unique) $i \geqq 1$

$$N_i < mn + 1 < N_{i+1} \quad \text{and} \quad f(1, n) = mn + 1 - N_i = m.$$

We have $N_1 = m(n-1)+1$ by definition and

$$N_2 \geqq \frac{mN_1}{m-1} = m(n-1) + n + \frac{n}{m-1} > mn + 1,$$

so that $i = 1$ and $f(1,n) = mn + 1 - N_1 = m$.

Case 2: $n < m$. Write $m = \lambda n + \mu$ with integers λ, μ such that $\lambda \geqq 1$, $0 < \mu \leqq n$. We have then $a_m(1,n) = \mu$. We show that for some $i \geqq 1$ one has $N_i < mn + 1 < N_{i+1}$ and

$$mn + 1 - N_i = \mu, \quad \text{or} \quad N_i = m(n-1) + \lambda n + 1.$$

1) If $\lambda = \mu = 1$, then $m - 1 = n$, and we find

$$m - 1 \mid N_1 = (m-1)(n-1) + n,$$

$$N_2 = \frac{mN_1}{m-1} = m(n-1) + n + \frac{n}{m-1} = mn,$$

and

$$N_3 = \frac{mN_2}{m-1} = mn + n + 1.$$

Thus we have $i = 2$ and $f(1,n) = mn + 1 - N_2 = 1 = \mu$.

2) If $\lambda = 1$, $\mu > 1$, then $m = n + \mu$, $m - 1 \nmid N_1$ and

$$N_2 = \left\lceil \frac{mN_1}{m-1} \right\rceil = m(n-1) + n + 1,$$

$$N_3 \geqq mn + \frac{2n}{2n-1} > mn + 1,$$

so that $i = 2$ and $f(1,n) = mn + 1 - N_2 = \mu$.

3) If $\lambda \geqq 2$ then $m - 1 \geqq \lambda n$, $m - 1 \nmid N_1$, and

$$N_2 = \left\lceil \frac{mN_1}{m-1} \right\rceil = m(n-1) + n + 1 = (m-1)(n-1) + 2n.$$

Suppose now that for some v, $2 \leqq v < \lambda$, one has

$$N_v = m(n-1) + (v-1)n + 1 = (m-1)(n-1) + vn.$$

Then $m - 1 \nmid N_v$ and so

$$N_{v+1} = \left\lceil \frac{mN_v}{m-1} \right\rceil = m(n-1) + vn + 1$$

$$= (m-1)(n-1) + (v+1)n.$$

It follows that

$$N_\lambda = m(n-1) + (\lambda-1)n + 1 = (m-1)(n-1) + \lambda n.$$

Hence, if $\mu > 1$ then $m - 1 > \lambda n$ and

$$N_{\lambda+1} = m(n-1) + \lambda n + 1 = (m-1)(n-1) + (\lambda+1)n,$$

and if $\mu = 1$ then $m - 1 = \lambda n$ and

$$N_{\lambda+1} = \frac{mN_\lambda}{m-1} = m(n-1) + \lambda n + 1;$$

in either case one has

$$N_{\lambda+2} \geqq m(n-1) + (\lambda+1)n + \frac{(\lambda+1)n}{m-1}$$

$$> mn + n - \mu + 1 \geqq mn + 1,$$

whence $i = \lambda + 1$ and $f(1,n) = mn + 1 - N_{\lambda+1} = \mu$.

We thus have proved (6) and, in view of (7) and (1), our proof of Schubert's formula (i), and of Busche's formula (ii) as well, is now complete.

Example. For $m = 10$, $n = 30$, $k = 15$ we have $mn + 1 = 301$, $mk + 1 = 151$, and

i:	1	2	3	4	5	6	7	8
N_i:	151	168	187	208	232	258	287	319
N_i':	1	18	37	58	82	108	137	169

Thus $a_{10}(15,30) = 301 - 287 = 151 - 137 = 14$.

4. A New Solution

Our new algorithm for determining the Josephus numbers $a_m(k,n)$ $(1 \leqq k \leqq n)$, where $m \geqq 2$, will be formulated in terms of two sequences n_i and c_i $(i = 1, 2, \ldots)$, the definitions of which are as follows. We define three sequences of positive integers n_i, c_i and c_i^* $(i = 1, 2, \ldots)$ by taking n_1, c_1 and c_1^* that satisfy the conditions

$$n_1 > 0, \quad 0 < c_1 = c_1^* \leqq n_1 + 1,$$

and setting recursively for $i \geqq 1$

$$n_{i+1} = \left\lfloor \frac{m(n_i+1) - c_i}{m-1} \right\rfloor \tag{8}$$

(9) $$c^*_{i+1} = c_i + (m-1)(n_{i+1}+1) - m(n_i+1),$$

(10) $$c_{i+1} \equiv c^*_{i+1} \pmod{n_{i+1}+1}, \quad 0 < c_{i+1} \leqq n_{i+1}+1.$$

We have $n_{i+1} > n_i$ for all $i \geqq 1$. In fact, if $c_i = n_i + 1$ then $n_{i+1} = n_i + 1$ and, if $c_i \leqq n_i$ then

$$n_{i+1} > \frac{m(n_i+1) - n_i}{m-1} - 1 = n_i + \frac{1}{m-1}.$$

It follows from (8) and (9) that $0 < c^*_{i+1} \leqq m-1$; therefore, we have, by (10), $c_i = c^*_i$ for all i for which $n_i + 1 \geqq m-1$ or $n_i \geqq m-2$, apart from $i = 1$.

For the special case of $m = 2$ we have, with $n_1 = c_1 = 1$,

$$n_i = 2^i - 1, \quad c_i = 1 \quad \text{for all } i \geqq 1,$$

as can readily be shown by induction on i.

Let $m \geqq 2$ be arbitrary. It is easily seen that, if we define the sequence n_i with $n_1 = 1$ (so that $c_1 = 1$ or 2), then we have

$$n_i = i \quad \text{for } 1 \leqq i < m,$$

and $n_m = m$ provided $c_{m-1} > 1$.

Let $m \geqq 2$ be again an arbitrary fixed integer. Our formula for the numbers $a_m(k,n)$ $(1 \leqq k \leqq n)$ will now be described thus: let us construct the sequences n_i and c_i $(i = 1, 2, \ldots)$ with

$$n_1 = n - k, \quad c_1 = a_m(1, n_1+1) \quad \text{if } 1 \leqq k < n,$$

and

$$n_1 = 1, \quad c_1 = a_m(2,2) \quad \text{if } k = n,$$

and by relations (8), (9) and (10). We have then

(11) $$a_m(k,n) = c_i + m(n - n_i - 1)$$

if $n_i < n \leqq n_{i+1}$.

Note that

$$a_m(1, n_1+1) \equiv m \pmod{n_1+1}$$

and

$$a_m(2,2) = d_m(2) = 1 \text{ or } 2$$

according as m is even or odd.

The validity of our formula (11) follows from the validity of Schubert's formula (i) and from Lemmas 1 and 2 below.

Lemma 1. *To every* $i \geqq 1$ *there corresponds a unique* $j = j(i) \geqq 1$ *such that*

$$m(n_i + 1) + 1 - c_i = N_j. \tag{12}$$

Moreover, we always have $j(i+1) > j(i)$, *and* $j(i+1) = j(i) + 1$ *if* $c_{i+1} = c_{i+1}^*$, *or a fortiori if* $n_i \geqq m - 3$.

Proof. For $i = 1$ relation (12) is obvious from Schubert's formula (i), since $c_1 = a_m(k_1, n_1 + 1)$ with $k_1 = 1$ or 2 according as $1 \leqq k < n$ or $k = n$.

It is a matter of simple computations to see that, if $i \geqq 1$, $1 \leqq n \leqq m - 2$ and $c_i = a_m(k_i, n_i + 1)$, then we have $n_{i+1} = n_i + 1$ and

$$\begin{aligned} c_{i+1}^* &= c_i + (m-1)(n_{i+1} + 1) - m(n_i + 1) \\ &\equiv c_i + m \pmod{n_{i+1} + 1} \end{aligned}$$

and therefore, by (10) and (1), $c_{i+1} = a_m(k_{i+1}, n_{i+1} + 1)$, where $k_{i+1} = k_i + 1$. It follows from this that

$$m(n_{i+1} + 1) + 1 - c_{i+1} = N_j$$

with some unique $j = j(i+1)$.

Suppose now that $n_i \geqq m - 3$ and (12) holds true; then we have $c_{i+1} = c_{i+1}^*$ and, by (9) and (8),

$$m(n_{i+1} + 1) + 1 - c_{i+1} = N_j + n_{i+1} + 1 = N_{j+1},$$

where

$$n_{i+1} + 1 = \left\lfloor \frac{m(n_i + 1) - c_i}{m - 1} \right\rfloor + 1 = \left\lceil \frac{N_j}{m - 1} \right\rceil$$

Thus, we have $j(i+1) = j(i) + 1$ provided $n_i \geqq m - 3$. Existence of $j(i)$ for all $i \geqq 1$ now follows by induction.

We have, by writing $j = j(i)$ and $j' = j(i+1)$ for simplicity's sake,

$$N_{j'} - N_j = m(n_{i+1} - n_i) - (c_{i+1} - c_i) > 0,$$

since $n_{i+1} - n_i \geqq 1$ and $|c_{i+1} - c_i| \leqq m - 1$. This means that $j' > j$. If in here $c_{i+1} = c_{i+1}^*$ then we find

$$N_{j'} - N_j = n_{i+1} + 1 = \left\lceil \frac{N_j}{m - 1} \right\rceil = N_{j+1} - N_j.$$

Hence we have $j' = j + 1$, as asserted.

Lemma 2. *Suppose that the equality* (12) *holds true. Then, an integer n satisfies the inequality*

$$n_i < n \leqq n_{i+1} \tag{13}$$

if and only if it satisfies

$$N_j < mn + 1 < N_{j+1}. \tag{14}$$

Proof. Suppose that n satisfies (14). Since we have $j(i+1) \geqq j+1$ by Lemma 1, it follows from (14) that

$$n_i + \frac{m - c_i}{m} < n < n_{i+1} + \frac{m - c_{i+1}}{m}$$

where $0 < c_i \leqq m$ ($c_i = m$ may happen only for $i = 1$) and $0 < c_{i+1} \leqq m - 1$, and we have (13).

Conversely, suppose now that n satisfies (13). If $n_i \leqq m - 3$ then $n = n_{i+1} = n_i + 1$, and

$$N_j = m(n_i + 1) + 1 - c_i < mn + 1,$$

whereas, since $c_i \leqq n_i + 1$, we have

$$N_{j+1} = N_j + \left\lceil \frac{N_j}{m-1} \right\rceil = N_j + n_i + 2 > mn + 1;$$

and if $n_i \geqq m - 2$ then $j(i+1) = j+1$ by Lemma 1, and we obtain (14) from the inequality

$$mn_i + 1 < mn + 1 \leqq mn_{i+1} + 1,$$

since we have $mn_i + 1 + m \leqq mn + 1$ and $m - c_i < m$, $m - c_{i+1} > 0$.

Thus, our proof of the formula (11) is now complete.

Example. For $m = 10$, $n = 30$, $k = 15$, we have $n_1 = 15$, $c_1 = a_{10}(1, 16) = 10$ and:

i:	1	2	3	4	5	6	7	8
n_i:	15	16	18	20	23	25	28	31
c_i:	10	3	4	3	9	3	4	2

Hence $a_{10}(15, 30) = 4 + 10(30 - 28 - 1) = 14$.

A consequence of Lemmas 1 and 2 is that the number of iteration steps required in our algorithm to determine the value of $a_m(k, n)$ is in general slightly less than the number of corresponding steps in Schubert's or Busche's algorithm. For instance, for the case of $k = n$, $m \geqq 2$, we have

$$m(n_1+1)+1-c_1 = 2m-\varepsilon,$$

where $\varepsilon = 0$ or 1 according as m is even or odd, and it is not difficult to see that $N_i = i$ for $1 \leqq i \leqq m$. Moreover, we have $N_{m+j} = m+2j$ for $1 \leqq j \leqq (m-\varepsilon)/2$; indeed, if $N_{m+v} = m+2v$ for some v, $0 \leqq v \leqq (m-\varepsilon)/2 - 1$, then

$$\begin{aligned} N_{m+v+1} &= \left\lceil \frac{mN_{m+v}}{m-1} \right\rceil \\ &= \frac{m(m+2v)+m-1-\tau}{m-1} \\ &= m+2v+2, \end{aligned}$$

where $0 < \tau = 2v+1 \leqq m-1$. Thus we find $N_L = 2m-\varepsilon$, where $L = (3m-\varepsilon)/2$.

As a matter of course one may carry out, on the basis of the relation (1), a direct proof of (11) substantially in the same manner as, but in a way somewhat easier than, in the proof of Schubert's formula (i) for $a_m(k,n)$ $(1 \leqq k \leqq n)$, as given in §3 above, noticing that n_{i+1} is the largest positive integer n for which holds the inequality

$$c_i + m(n-n_i-1) \leqq n.$$

We omit the details, however.

5. A Special Case

Here we shall briefly review the Josephus problem for $k = n$, namely the problem to determine the last number to be removed, $d_m(n) = a_m(n,n)$, where we assume as before that $m \geqq 2$. Since $d_m(1) = 1$ for all m, we may suppose in what follows that $n > 1$.

Starting with $n_1 = 1$ and $c_1 = d_m(2)$, that is, $c_1 = 1$ or 2 according as m is even or odd, we construct the sequences n_i and c_i (and c_i^*) $(i = 1, 2, \ldots)$ using the relations (8), (9) and (10). It follows from the general formula (11), with $k = n$, that if $n_i < n \leqq n_{i+1}$ $(i \geqq 1)$ then

$$d_m(n) = c_i + m(n-n_i-1). \tag{15}$$

As was observed in §4, if $m = 2$ and $n_1 = c_1 = 1$, we have $n_i = 2^i - 1$ and $c_i = 1$ for all $i > 1$. Since then the inequality $n_i < n \leqq n_{i+1}$ is equivalent to $2^i \leqq n < 2^{i+1}$, we thus obtain again Rankin's formula (2).

It should be also noted that, for $m \geqq 2$, our construction shows that $n_i = i$ for $1 \leqq i < m$ and, therefore, $c_i = d_m(i+1)$ for those values of i. Since we have

$d_m(m) \geqq 2$ if $m \geqq 3$, it follows that $n_m = m$ and $c_{m-1} = d_m(m)$ for $m \geqq 3$. Hence, if $m \geqq 3$ and if the value of the number $d_m(m)$ is available beforehand, we may start just as in Rankin [8] the sequences n_i and c_i with $n_1 = m$ and $c_1 = d_m(m+1) = d_m(m) - 1$, in so far as we concern with the case of $n > m \geqq 3$.

The formula for $d_m(n)$ for $m = 3$ found in [7] is

$$d_3(n) = 3n + 1 - \left\lfloor C\left(\frac{3}{2}\right)^D \right\rfloor$$

where $C = 1.62227\ 05028\ 84767\ 31595\ 69509\ 82899\ 32411\ldots$ and $D = D(n) = \lceil \log_{3/2}((2n+1)/C) \rceil$; this is naturally a result of Schubert's (or equivalently of Busche's) type and, in view of our analysis in §2 above, a similar, concise formula can also be provided for any Josephus number $a_m(k,n)$ $(1 \leqq k \leqq n)$ in the case of $m = 3$.

Here is a short table of $d_m(m)$ $(1 \leqq m \leqq 60)$.

m:	1	2	3	4	5	6	7	8	9	10
$d_m(m)$:	1	1	2	2	2	4	5	4	8	8
m:	11	12	13	14	15	16	17	18	19	20
$d_m(m)$:	7	11	8	13	4	11	12	8	12	2
m:	21	22	23	24	25	26	27	28	29	30
$d_m(m)$:	13	7	22	2	8	13	26	4	26	29
m:	31	32	33	34	35	36	37	38	39	40
$d_m(m)$:	17	27	26	7	33	20	16	22	29	4
m:	41	42	43	44	45	46	47	48	49	50
$d_m(m)$:	13	22	25	14	22	37	18	46	42	46
m:	51	52	53	54	55	56	57	58	59	60
$d_m(m)$:	9	41	12	7	26	42	24	5	44	53

Remark. It will be of some interest to note that several entries of the above table of $d_m(m)$, namely the ones for $m = 2, 3, 4, 5$, and 6, are found among other numerals in the table given by L. Euler [3], who also discovered, moreover, the significance of our sequences n_i and c_i (or, of something equivalent to them); however, the sequences numerically presented by him in the special case of $m = 9$ would seem to contain, by contamination, errors from a certain point onwards, but the table of his as a whole is properly understandable.

6. Illustrations

We present in the following some numerical examples mostly treated by Rankin [8], by Jakóbczyk [5], by Busche [2] and by Halbeisen and Hungerbühler [4]. All the results obtained by newly applying our method agree, of course, with those of these authors'. The writer is much obliged to Dr H. Mikawa for the numerical computations relevant to examples 9) and 10) below.

1) To compute $d_3(41)$. Here $m = 3$, $n = 41$. We know $d_3(3) = 2$, so that $n_1 = 3$, $c_1 = 1$:

i:	1	2	3	4	5	6	7	8
n_i:	3	5	8	13	20	30	46	69
c_i:	1	1	1	2	2	1	2	1

Hence $d_3(41) = 1 + 3(41 - 30 - 1) = 31$.

2) To compute $d_3(53)$. Here $m = 3$, $n = 53$. Using the table given in 1) above, we find $d_3(53) = 2 + 3(53 - 46 - 1) = 20$.

3) To determine $d_6(117)$. Here $m = 6$, $n = 117$, and $d_6(6) = 4$; $n_1 = 6$, $c_1 = 3$. We have

i:	1	2	3	$\cdots$	15	16	17
n_i:	6	7	9	$\cdots$	88	106	127
c_i:	3	1	3	$\cdots$	2	3	1

and $d_6(117) = 3 + 6(117 - 106 - 1) = 63$.

4) To compute $a_6(46, 117)$. Here $m = 6$, $n = 117$, $k = 46$, and $n_1 = 71$, $c_1 = a_6(1, 72) = 6$. We find

i:	1	2	3	4
n_i:	71	85	102	123
c_i:	6	4	3	5

whence $a_6(46, 117) = 3 + 6(117 - 102 - 1) = 87$.

5) To compute $a_6(66, 117)$. Here $m = 6$, $n = 117$, $k = 66$; and so $n_1 = 51$, $c_1 = a_6(1, 52) = 6$:

i:	1	2	3	4	5	6
n_i:	51	61	73	88	106	127
c_i:	6	4	2	3	4	2

Thus $a_6(66, 117) = 4 + 6(117 - 106 - 1) = 64$.

6) To determine $a_6(116, 117)$. We have $m = 6$, $n = 117$, $k = 116$, so that $n_1 = 1$, $c_1 = a_6(1, 2) = 2$:

i:	1	2	3	$\cdots$	20	21	22
n_i:	1	2	3	$\cdots$	86	103	124
c_i:	2	2	4	$\cdots$	4	2	3

and we get $a_6(116, 117) = 2 + 6(117 - 103 - 1) = 80$.

7) To compute $a_9(10, 11)$. For $m = 9$, $n = 11$, $k = 10$ we have $n_1 = 1$, $c_1 = a_9(1, 2) = 1$, and we find $a_9(10, 11) = 4$ since

i:	1	2	3	4	5	6	7	8	9	10	11
n_i:	1	2	3	4	5	6	7	8	9	10	11
c_i:	1	1	2	1	4	6	7	7	6	4	1

8) To determine $a_9(9, 11)$. We are given $m = 9$, $n = 11$, $k = 9$; in here $n_1 = 2$, $c_1 = a_9(1, 3) = 3$:

i:	1	2	3	4	5	6	7	8	9
n_i:	2	3	4	5	6	7	8	9	11
c_i:	3	4	3	6	1	2	2	1	7

Hence $a_9(9, 11) = 1 + 9(11 - 9 - 1) = 10$.

9) To determine $a_m(k, n)$ for $m = 7$, $n = \mathrm{R}_{23}$, $k = n - 2001$, where, and in the next example 10), $\mathrm{R}_{23} = 11111111111111111111111$ is the so-called repunit prime of 23 digits. We have $n_1 = 2001$, $c_1 = a_7(1, 2002) = 7$, and eventually find $n_{280} < n < n_{281}$ and $c_{280} = 5$, where $n_{280} = 9538759184899654873314$. Thus

$$\begin{aligned} a_7(k, n) &= 5 + 7(n - n_{280} - 1) \\ &= 11006463483480193664577. \end{aligned}$$

10) To compute $d_m(n)$ for $m = 7$, $n = \mathrm{R}_{23}$. Here $n_1 = 7$, $c_1 = 4$, and we find $n_{316} < n < n_{317}$ and $c_{316} = 2$, where $n_{316} = 9711936891836718664167$, so that

$$\begin{aligned} d_7(n) &= 2 + 7(n - n_{316} - 1) \\ &= 9794219534920747128603. \end{aligned}$$

7. Appendix

Let us consider again the generalized Josephus problem with parameters $n \geqq 1$ and $m \geqq 2$ and, denoting by $a_m(k, n)$ $(k \geqq 1)$ as before the kth member to be removed, we write $d_m(n) = a_m(n, n)$. For a fixed m Seki Takakazu [12] called a natural number n a limitative number (正限数) if one has $d_m(n + 1) = 1$, that is, the number 1 is the last member to be deleted in this situation.

We define two sequences of integers N_i and σ_i by putting

$$N_1 = 1, \quad N_{i+1} = \left\lceil \frac{mN_i}{m-1} \right\rceil \quad (i \geqq 1)$$

and

$$\sigma_i = (m-1)N_{i+1} - mN_i, \quad 0 \leqq \sigma_i \leqq m-2 \quad (i \geqq 1).$$

If $N_i = (m-1)K_i - \sigma_i$ $(i \leqq j \leqq i+1)$ then $N_{i+1} = mK_i - \sigma_i$ and

$$\sigma_{i+1} - \sigma_i = (m-1)K_{i+1} - mK_i;$$

thus, $d_m(K_i) = 1$ when and only when $\sigma_i = 0$, so that if $K_i \geqq 2$ then $K_i - 1$ is a limitative number for m and *vice versa*.

A short table of limitative numbers for m $(2 \leqq m \leqq 10)$, with five entries for each of these m's, was given by Seki in [12], the computation of whom was carried out only by directly applying the fundamental congruence relation (1). Later on, Takebe Katahiro (1664–1739) of the Seki school slightly extended the table of limitative numbers of Seki's, but there seems to be some errors in his calculation.

It is almost apparent that Seki had an idea that the following hypothesis would be true:

Hypothesis. *For every fixed $m \geqq 2$ there exist infinitely many limitative numbers n.*

This hypothesis holds true for $m = 2$ and 3 at the least. In fact, for $m = 2$ we find $d_2(2^i) = 1$ $(i \geqq 1)$, and all of the numbers $2^i - 1$ $(i = 1, 2, 3, \ldots)$ are limitative, and any limitative number for $m = 2$ has this form, that is, a form of a power of two minus one. For the case of $m = 3$ we have

$$2N_{i+1} - 3N_i = \sigma_i, \quad \sigma_i = 0, 1 \ (i \geqq 1);$$

if $\sigma_i \neq 0$ for all sufficiently large i, then we would have for some $i_0 \geqq 1$ $\sigma_i = 1$ for all $i \geqq i_0$, which is impossible as was noticed before (cf. §2 above). Thus, there are infinitely many limitative numbers in this case, $m = 3$. For $m \geqq 4$ these simple arguments will fail to prove (or disprove) our hypothesis above. As a matter of fact, for odd $m > 3$ the existence even of a single limitative number is quite unclear, whereas for even m the number 1 is always a limitative number.

Here we shall give a table of limitative numbers for m up to 60, with ten entries for each of m.

Limitative Numbers

$m =$ 2	1	3	7	15	31	63	127	255	511	1023
3	3	5	8	30	69	104	354	798	1797	2696
4	1	4	8	11	15	217	516	1225	6889	12248
5	2	5	11	14	36	57	141	221	346	677
6	1	2	7	13	73	127	318	1143	1976	2846
7	22	49	92	234	319	2376	4403	5137	32672	60530
8	1	4	9	19	29	44	76	87	114	500
9	90	145	207	233	474	1083	1371	4455	5012	8029
10	1	15	21	70	226	527	1226	2850	5960	17096
11	2	6	13	16	24	105	170	206	366	865
12	1	2	3	6	15	171	1168	3044	12252	17353
13	23	25	35	38	894	1137	5208	13611	176328	308786
14	1	3	5	6	145	227	1688	24341	28230	115408
15	3	4	8	11	16	20	78	337	1440	3533
16	1	5	8	10	19	25	35	40	149	159
17	2	55	75	102	326	1319	3482	3931	8647	10372
18	1	2	4	5	10	27	41	55	186	197
19	4	89	94	117	704	923	1586	2876	6833	9452
20	1	8	9	17	20	80	515	777	1515	6375
21	5	6	12	36	44	108	205	1317	2609	6595
22	1	4	17	24	575	4462	15672	21705	39739	293749
23	2	5	15	28	37	51	181	207	296	370
24	1	2	3	6	7	9	12	21	24	41
25	8	14	16	32	371	913	2752	5510	6228	452865
26	1	3	37	114	139	373	511	598	1078	6304
27	3	11	13	16	49	55	98	155	161	416
28	1	10	15	19	24	29	44	64	120	134
29	2	4	10	16	47	65	75	111	115	1861
30	1	2	13	14	21	36	40	46	117	562
31	8	794	1635	2270	2856	5504	14722	17345	202881	223853
32	1	5	8	115	135	192	256	310	533	1384
33	4	61	67	1312	3305	21604	25985	41228	107025	530171
34	1	5	7	11	13	18	24	29	36	59
35	2	10	13	19	81	207	393	8039	18635	32325
36*										
37	4	7	12	30	41	70	74	180	472	2449
38	1	3	8	17	18	22	30	43	48	294
39	3	30	45	62	108	148	231	1547	2169	2470
40	1	36	41	255	590	6229	6893	14734	36659	37599
41	2	6	109	242	274	969	3096	5201	8955	84723
42	1	2	34	47	393	1756	1981	5862	6005	20527
43	10	106	364	570	12464	14696	15771	16531	22447	35102
44	1	8	11	77	95	166	373	780	3249	15879
45	20	22	26	37	50	125	140	14825	47702	308045
46	1	36	53	58	71	1288	1570	5148	9739	18423
47*										
48	1	2	3	37	56	68	606	1814	3272	10641

49	135	227	414	2396	3069	4358	71992	406910	3133496	3333448
50	1	3	4	22	25	39	58	63	330	1634
51	3	20	28	39	443	980	2858	3847	16993	21129
52	1	7	17	359	2068	2279	5679	16526	102546	231801
53	2	6	31	75	709	795	67401	114895	133808	139004
54	1	2	15	18	22	29	49	56	265	308
55	7	24	28	107	109	201	233	7383	10273	26676
56	1	11	14	36	172	9135	47940	59512	354260	1044344
57	155	182	192	2751	3525	8393	11141	103637	182598	217954
58	1	6	23	32	37	355	677	701	820	864
59	2	14	95	113	119	149	541	676	32265	234388
60	1	2	3	4	5	7	10	11	16	21

*The sequences of limitative numbers for $m = 36$ and 47 fairly rapidly increase and so will be given here separately:

$m = 36$: $1, 2, 3, 54519, 235911, 1694972, 2184101, 7981011, 31735572, 100730052$;

$m = 47$: $2, 4, 166, 1745, 2164, 273565, 468341, 675061, 1402505, 1936444$.

REMARK. It is a matter of simple calculation to verify that we have

$d_m(3) = 1$ if and only if $m \equiv 0$ or $5 \pmod 6$;

$d_m(4) = 1$ if and only if $m \equiv 0, 2$ or $3 \pmod{12}$;

$d_m(5) = 1$ if and only if $m \equiv 0, 4, 8, 15, 18, 19, 22, 29, 33, 37, 47$ or $50 \pmod{60}$;

and

$d_m(6) = 1$ if and only if $m \equiv 0, 3, 5, 14, 16, 18, 21, 23, 32$ or $34 \pmod{60}$.

Thus, we may conclude that 19/30 (ca. 63.3%) of odd integers $m \geqq 3$ admit at least one limitative number n satisfying $2 \leqq n \leqq 5$ and 7/10 (70%) of even integers $m \geqq 2$ admit one or more limitative numbers n with $2 \leqq n \leqq 5$.

As is observed in Rankin [8] it will be worth noticing that we have for every fixed $n \geqq 1$

$$d_r(n) = d_s(n)$$

whenever $r \equiv s \pmod M$, M being the least common multiple of the integers $1, 2, \ldots, n$.

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